Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 23, 2009

ON ASYMPTOTIC BEHAVIOR OF SOLUTION OF ONE NONLINEAR ONE-DIMENSIONAL INTEGRO-DIFFERENTIAL ANALOGUE OF MAXWELL'S SYSTEM

Aptsiauri M., Jangveladze T., Kiguradze Z.

Abstract. Large time behavior of solutions of a system of nonlinear integro-differential equations associated with the penetration of a magnetic field into a substance is studied. Initial-boundary value problem with Dirichlet boundary conditions is considered. Exponential stabilization of solution is established. Corresponding finite difference scheme is considered as well.

Keywords and phrases: Nonlinear integro-differential system, asymptotic behavior as $t \to \infty$, finite difference scheme.

AMS subject classification: 45K05; 65M06; 35K55.

A great variety of applied problems are modeled by the systems of nonlinear integro-differential equations. Such systems arise for instance for mathematical modeling of the process of penetrating of magnetic field in the substance. Corresponding Maxwell's system [1] can be rewritten in the following form [2]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2\, d\tau\right) rotH\right], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

If the magnetic field has the form $H = (0, U, V)$ and $U = U(x,t)$, $V = V(x,t)$, then we obtain the following system of nonlinear integro-differential equations:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left[a\left(\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial U}{\partial x}\right], \\ \frac{\partial V}{\partial t} &= \frac{\partial}{\partial x}\left[a\left(\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial V}{\partial x}\right]. \end{aligned} \tag{2}$$

Many scientific works are devoted to uniqueness and solvability of various problems for (2) type models (see, for example, [2]-[8] and references therein). The existence of the global solutions for initial-boundary value problems of such models have been proved in [2]-[5],[8] by using the modified Galerkin and compactness methods [9],[10]. For solvability and uniqueness properties of such type models see also [6],[7] and number of other works as well. In [6] some generalization of equations of type (1) is proposed. In this case analogous of the model type (2) has the following form:

$$\begin{aligned} \frac{\partial U}{\partial t} &= a\left(\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dxd\tau\right)\frac{\partial^2 U}{\partial x^2},\\ \frac{\partial V}{\partial t} &= a\left(\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dxd\tau\right)\frac{\partial^2 V}{\partial x^2}. \end{aligned} \tag{3}$$

The asymptotic behavior of the solutions of (2) and (3) type models have been object of intensive research (see, for example, [8],[11]-[19]).

The purpose of this note is to study the asymptotic behavior of solutions of the initial-boundary value problem for the system (3) in the case $a(S) = 1 + S$ and confirmation of theoretical results by numerical experiments.

In the domain $[0,\ 1] \times [0,\ \infty)$ for the system (3) with $a(S) = 1 + S$ let us consider the following initial-boundary value problem:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \left\{1+\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dxd\tau\right\}\frac{\partial^2 U}{\partial x^2},\\ \frac{\partial V}{\partial t} &= \left\{1+\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dxd\tau\right\}\frac{\partial^2 V}{\partial x^2}, \end{aligned} \tag{4}$$

$$U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \tag{5}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \tag{6}$$

where $\psi_1 = Const \geq 0$, $\psi_2 = Const \geq 0$; $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are given functions.

The existence and uniqueness of the solution of such type problems for one equation in suitable classes are proved in [8]. One must note that asymptotic behavior of solution for problem (4)-(6) with homogeneous as well as non-homogenous boundary conditions is studied in [17]. For problem with homogeneous boundary condition exponential stabilization while for problem with nonhomogeneous boundary condition on one side of lateral boundary power-like stabilization is obtained.

We use usual Sobolev spaces $H^k(0,1)$ and $H_0^k(0,1)$.

Theorem 1. *If $U_0, V_0 \in H_0^1(0,1)$, then for the solution of problem (4)-(6) the following estimate is true*

$$\|U\| + \left\|\frac{\partial U}{\partial x}\right\| + \|V\| + \left\|\frac{\partial V}{\partial x}\right\| \leq C\exp\left(-\frac{t}{2}\right).$$

Note that here and below C denotes positive constants independent of t and $\|\cdot\|$ denotes norm in the space $L_2(0,1)$.

Note that Theorem 1 gives exponential stabilization of the solution of the problem (4)-(6) in the norm of the space $H^1(0,1)$. The stabilization is also achieved in the norm of the space $C^1(0,1)$. In particular, the following statement takes place.

Theorem 2. *If* $U_0, V_0 \in H^3(0,1) \cap H_0^1(0,1)$*, then for the solution of problem (4)-(6) the following relations hold:*

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \le C\exp\left(-\frac{\alpha t}{2}\right), \quad \left|\frac{\partial V(x,t)}{\partial x}\right| \le C\exp\left(-\frac{\alpha t}{2}\right),$$

$$\left|\frac{\partial U(x,t)}{\partial t}\right| \le C\exp\left(-\frac{\beta t}{2}\right), \quad \left|\frac{\partial V(x,t)}{\partial t}\right| \le C\exp\left(-\frac{\beta t}{2}\right),$$

where $0 < \alpha = Const < 1$, $0 < \beta = Const < \alpha$.

Following lemma is necessary to prove Theorem 2.

Lemma 1. *For the solution of problem (4)-(6) the following estimate is true*

$$\left\|\frac{\partial U(x,t)}{\partial t}\right\| + \left\|\frac{\partial V(x,t)}{\partial t}\right\| \le C\exp\left(-\frac{t}{2}\right).$$

In the rectangle $[0,1]\times[0,T]$, where T is a positive constant, we consider problem (4)-(6). We assume that $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are sufficiently smooth given functions of their arguments.

Using usual notations [20] we correspond to the problem (4)-(6) the difference scheme:

$$\begin{aligned}
\frac{u_i^{j+1}-u_i^j}{\tau} &= \left\{1+\tau h\sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(u_{\bar{x},l}^k)^2+(v_{\bar{x},l}^k)^2\right]\right\}u_{\bar{x}x,i}^{j+1},\\
\frac{v_i^{j+1}-v_i^j}{\tau} &= \left\{1+\tau h\sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(u_{\bar{x},l}^k)^2+(v_{\bar{x},l}^k)^2\right]\right\}v_{\bar{x}x,i}^{j+1},\\
i &= 1,2,...,M-1; \quad j=0,1,...,N-1,
\end{aligned} \tag{7}$$

$$u_0^j = v_0^j = 0, \quad u_M^j = \psi_1, \quad v_M^j = \psi_2, \quad j=0,1,...,N, \tag{8}$$

$$u_i^0 = U_{0,i}, \quad v_i^0 = V_{0,i} \quad i=0,1,...,M. \tag{9}$$

Many scientific works are devoted to the construction of descrete analogues for (2) and (3) type models (see, for example, [14],[17],[18],[21],[22] and references therein).

It is not difficult to obtain for (7)-(9) the following estimation:

$$\|u^n\|_h^2 + \sum_{j=1}^{n}\|u_{\bar{x}}^j]|_h^2\tau \le C, \quad \|v^n\|_h^2 + \sum_{j=1}^{n}\|v_{\bar{x}}^j]|_h^2\tau \le C, \quad n=1,2,...,N. \tag{10}$$

The a-priori estimate (10) guarantees the stability of the scheme (7)-(9).

Theorem 3. *If the problem (4)-(6) has a sufficiently smooth unique solution* $U = U(x,t)$*,* $V = V(x,t)$*, then exists the unique solution* $u^j = (u_1^j, u_2^j, \ldots, u_{M-1}^j)$*,* $v^j = (v_1^j, v_2^j, \ldots, v_{M-1}^j)$*,* $j = 1,2,\ldots,N$ *of the finite difference scheme (7)-(9) which tends to the* $U^j = (U_1^j, U_2^j, \ldots, U_{M-1}^j)$*,* $V^j = (V_1^j, V_2^j, \ldots, V_{M-1}^j)$ *for* $j = 1,2,\ldots,N$ *as* $\tau \to 0$*,* $h \to 0$ *and the following estimates are true*

$$\|u^j - U^j\|_h \le C(\tau+h), \quad \|v^j - V^j\|_h \le C(\tau+h), \quad j=1,2,\ldots,N.$$

We now comment on the numerical implementation of the discrete problem (7)-(9). Note that (7) can be rewritten as:

$$\frac{u_i^{j+1}-u_i^j}{\tau}-A^{j+1}\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{h^2}=0,$$
$$\frac{v_i^{j+1}-v_i^j}{\tau}-A^{j+1}\frac{v_{i+1}^{j+1}-2v_i^{j+1}+v_{i-1}^{j+1}}{h^2}=0,$$
$$i=1,2,...,M-1,\quad j=0,1,...,N-1,$$

where

$$A^j=1+\tau h\sum_{\ell=1}^{M}\sum_{k=1}^{j}\left[\left(\frac{u_\ell^k-u_{\ell-1}^k}{h}\right)^2+\left(\frac{v_\ell^k-v_{\ell-1}^k}{h}\right)^2\right].$$

In order to rewrite this in matrix form, we define the vectors $\mathbf{u}^j=\left[u_1^j,\ u_2^j,\ \ldots,\ u_{M-1}^j\right]^T$ and similarly $\mathbf{v}^j$. We also define the symmetric tridiagonal $(M-1)\times(M-1)$ matrix $\mathbf{T}$ as follows

$$\mathbf{T}_{rs}^{j+1}=\begin{cases}-\dfrac{1}{h^2}A^{j+1}, & s=r-1,\\[2ex] \dfrac{2}{h^2}A^{j+1}, & s=r,\\[2ex] -\dfrac{1}{h^2}A^{j+1}, & s=r+1,\\ 0, & \text{otherwise.}\end{cases}$$

Thus the system (7) becomes

$$\frac{1}{\tau}\begin{bmatrix}\mathbf{u}^{j+1}\\ \\ \mathbf{v}^{j+1}\end{bmatrix}-\frac{1}{\tau}\begin{bmatrix}\mathbf{u}^{j}\\ \\ \mathbf{v}^{j}\end{bmatrix}+\begin{bmatrix}\mathbf{T}^{j+1} & 0\\ & \\ 0 & \mathbf{T}^{j+1}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{j+1}\\ \\ \mathbf{v}^{j+1}\end{bmatrix}=0. \tag{11}$$

We use Newton's method to solve the nonlinear system (11). Let $\mathbf{P}^j=[\mathbf{u}^j,\ \ \mathbf{v}^j]^T$ and define

$$\mathbf{H}(\mathbf{P}^{j+1})=\frac{1}{\tau}\mathbf{P}^{j+1}-\frac{1}{\tau}\mathbf{P}^j+\hat{\mathbf{T}}^{j+1}\mathbf{P}^{j+1}, \tag{12}$$

where $\hat{\mathbf{T}}^{j+1}$ is the 2 by 2 block diagonal matrix with $\mathbf{T}^{j+1}$ on diagonal. Newton's method for the system (12) is given by

$$\nabla\mathbf{H}\left(\mathbf{P}^{j+1}\right)\Big|^{(n)}\left(\mathbf{P}^{j+1}\Big|^{(n+1)}-\mathbf{P}^{j+1}\Big|^{(n)}\right)=-\mathbf{H}\left(\mathbf{P}^{j+1}\right)\Big|^{(n)}.$$

It is well known that if H_i are three times continuously differentiable in a region containing the solution and the Jacobian does not vanish in that region, then Newton's method converges at least quadratically (see, for example, [23]). In our case the Jacobian is the matrix ∇H in which the term $\dfrac{1}{\tau}$ on diagonal ensures that it doesn't vanish. The differentiability is guaranteed, since ∇H is quadratic. So, we obtain convergence of the considered iterative method.

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Received 30.04.2009; accepted 10.09.2009.

Authors' addresses:

M. Aptsiauri
Ilia State University
Faculty of Physics and Mathematics
32, Chavchavadze Av., Tbilisi 0179
Georgia
E-mail: maiaptsiauri@yahoo.com

T. Jangveladze and Z. Kiguradze
I. Vekua Institute of Applied Mathematics
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia

Ilia State University
Faculty of Physics and Mathematics
32, Chavchavadze Av., Tbilisi 0179
Georgia
E-mail: tjangv@yahoo.com
zkigur@yahoo.com