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GENERALIZED ROBIN'S PROBLEMS FOR THE EQUATION OF STATICS OF THE ELASTIC MIXTURES

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Abstract

Using potentials with complex densities, solution Robin's plane generalized problem for the equation statics of the elastic mixtures, we reduce solution of Fredholm linear integral equation of second kind, which have only one solution.

The homogeneous equation of statics of the theory of elastic mixture in the complex form is written as [4]

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + K \frac{\partial^2 \bar{U}}{\partial \bar{z}^2} = 0, \tag{1}$$

where $U = (u_1 + iu_2, u_3 + iu_4)^T, u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements,

$$K = -\frac{1}{2} l m^{-1}, \quad l = \begin{pmatrix} l_4 & l_5 \\ l_5 & l_6 \end{pmatrix}, \quad m = \tfrac{1}{\delta_0} \begin{pmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{pmatrix}, \tag{2}$$

$$\begin{aligned}
& m_k = l_k + \frac{1}{2} l_{3+k}, k = 1,2,3, l_1 = \frac{a_2}{d_2}, l_2 = -\frac{c}{d_2}, l_3 = \frac{a_1}{d_2}, \ \delta_0 = m_1 m_3 - m_2^2, \\
& a_1 = \mu_1 - \lambda_5, a_2 = \mu_2 - \lambda_5, c = \mu_3 - \lambda_5, d_2 = a_2 a_1 - c^2, l_1 + l_4 = \frac{b}{d_1}, \\
& l_2 + l_5 = \frac{-c_0}{d_1}, l_3 + l_6 = \frac{a}{d_1}, a = a_1 + b_1, b = a_2 + b_2, \\
& c_0 = c + d, b_q = \mu_q - \lambda_q + \lambda_5 + (-1)^q \rho_{3-q} \frac{\alpha_2}{\rho}, q = 1,2, d_1 = ab - c_0^2, \\
& \rho = \rho_1 + \rho_2, \alpha_2 = \lambda_3 - \lambda_4, d = \mu_3 + \lambda_3 - \lambda_5 - \rho_1 \tfrac{\alpha_2}{\rho} = \mu_3 + \lambda_4 - \lambda_5 + \rho_2 \frac{\alpha_2}{\rho.}
\end{aligned} \tag{3}$$

Here $\mu_1, \mu_2, \mu_3, \lambda_p, p = 1,..,5$ are elastic constants, $\rho_1 and \rho_2$ are partial densities .The above constants satisfy the inequalities guaranteeing the positive definiteness of the potential energy [3].

In [1] M.Basheleishvili obtained the representations:

$$U = m\varphi(z) + \frac{1}{2} l z \overline{\varphi'(z)} + \overline{\psi}(z), \tag{4}$$

$$TU = ((TU)_2 - i(TU)_1, (TU)_4 - i(TU)_3))^T = \frac{\partial}{\partial s(x)} [-2\varphi(z) + 2\mu U(x)] \tag{5}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector functions, $(TU)_k$, $k = 1,..,4$ are components of the stress vector [2], $\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}$ $n(n_1, n_2)$ is an unit vector,

$$\mu = \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix}. \tag{6}$$

To investigate Robin's problems, the use will be made of the following vectors [4]

$$V = i[-m\varphi(x) + \frac{1}{2}lz\overline{\varphi'}(x) + \overline{\psi}(x)], \;\; TV = i\frac{\partial}{\partial s(x)}[2\varphi(z) - 2i\mu V(z)], \tag{7}$$

$$NU = \frac{\partial}{\partial s(x)}[-2\varphi(x) + m^{-1}U(x)], \;\; NV = i\frac{\partial}{\partial s(x)}[2\varphi(z) - im^{-1}V(z)]. \tag{8}$$

It is not difficalt to prove that $V(x)$ satisfies (1), morevere

$$U(x) + iV(x) = 2m\varphi(x), \;\; NU = -im^{-1}\frac{\partial V}{\partial s(x)}, NV = im^{-1}\frac{\partial U}{\partial s(x)}. \tag{9}$$

Let $D^+(D^-)$ be a finite (infinite) two-dimensional domain bounded by the contours $S \in C^{2\beta}, 0 < \beta < 1, \overline{D^+} = D \cup S, D^- = R^2 \backslash \overline{D^+}, \overline{D^-} = \bar{D}^{\cup} S$.

The Robin's plane generalized problems are formulated as follows:Find in the domain $D^+(D^-)$ a vector function $U(x)$, which belongs to the class $C^2(D^+) \cap C^{1,\alpha}(\overline{D^+})$, $(C^2(D^-) \cap C^{1,\alpha}(\overline{D^-}))$, is a solution of equation (1) and satisfies the boundary condition

$$(TU(t) - i\sigma_0 U(t))^{\mp} = f(t), \quad (R)^{\mp}_{0,f} \tag{10}$$

where $f = (f_1, f_2) \in C^{1,\alpha}(S), 0 < \alpha < \beta \leq 1$ is a given vector-function on the boundary, $\sigma_0 \neq 0$ is a parameter. Note that for the infinite domain $\overline{D^-}$ the vector U additionaly satisfies following condition an infinite $U = O(1), \; \dfrac{\partial U}{\partial x_k} = O(x^{-2}), k = 1, 2, \mid x \mid^2 = x_1^2 + x_2^2$.

The following theorems are valid

Theorem 1. *If $\sigma_0 > 0(\sigma_0 < 0)$, then the homogeneous problem $[(R)^-_{0,0}]$ has only the trivial solution in the class of regular vectors.*

Consider the problem $(R)^+_{0,f}$, and we look the analytic vector functions $\phi(x)$ and $\psi(x)$ in the form

$$\varphi(z) = \frac{(A - 2E)^{-1}}{2\pi i}\int_s \ln(1 - \frac{z}{\zeta})g(y)ds + m^{-1}\int_s g(y)ds, \tag{11}$$

$$\overline{\psi}(x) = -\frac{(2\mu)^{-1}}{2\pi i}\int_s \ln(1 - \frac{\bar{z}}{\bar{\zeta}})g(y)ds + \frac{l(A - 2E)^{-1}}{4\pi i}\int_s \left(\frac{\zeta}{\bar{z} - \bar{\zeta}} - \frac{\zeta}{\bar{\zeta}}\right)\overline{g}(y)ds, \tag{12}$$

where $z = x_1 + ix_2$. $\zeta = y_1 + iy_2, g(y)$ is the complex vector we seek for matrices l, m and μ are definid by (2) and (6), $A = 2\mu m, E$ is the unit matrix.

Substitution of (11) and (12) into (4) and (5) leads to the equalities

$$\begin{aligned} U(x, g) = &\frac{m(A - 2E)^{-1}}{2\pi i}\int_s \ln(1 - \frac{z}{\zeta})g(y)ds - \frac{\mu^{-1}}{4\pi i}\int_s \ln\left(1 - \frac{\bar{z}}{\bar{\zeta}}\right)g(y)ds \\ &- \frac{l(A - 2E)^{-1}}{4\pi i}\int_s \left(\frac{z - \zeta}{\bar{z} - \bar{\zeta}} - \frac{\zeta}{\bar{\zeta}}\right)\overline{g}(y)ds + \int_s g(y)ds, \end{aligned} \tag{13}$$

$$TU(x, g) = \frac{1}{\pi}\int_s \frac{\partial}{\partial n}\ln(z - \zeta)g(y)ds - \frac{H}{2\pi i}\int_s \frac{\partial}{\partial s(x)}\frac{z - \zeta}{\bar{z} - \bar{\zeta}}\overline{g}(y)ds, \tag{14}$$

where $H = B(A-2E)^{-1}$, $B = \mu l$, $\dfrac{\partial}{\partial n(x)} = n_1 \dfrac{\partial}{\partial x_1} + n_2 \dfrac{\partial}{\partial x_2}$.

From (13),(14) and $(10)^+_{0,f}$ we obtain the following equation of Fredholm type for the unknown vector g

$$\begin{aligned} -g(t) &+ \frac{1}{\pi}\int_s \frac{\partial}{\partial n(t)} \ln(t-\zeta) g(y)ds - \frac{H}{2\pi i}\int_s \frac{\partial}{\partial s(t)} \frac{t-\zeta}{\bar{t}-\bar{\zeta}} \bar{g}(y)ds \\ &- \frac{\sigma_0 m(A-2E)^{-1}}{2\pi}\int_s \ln(1-\frac{t}{\zeta}) g(y)ds - \frac{\sigma_0\mu^{-1}}{4\pi}\int_s \ln(1-\frac{\bar{t}}{\bar{\zeta}}) g(y)ds \\ &+ \frac{\sigma_0 l(A-2E)^{-1}}{4\pi}\int_s (\frac{t-\zeta}{\bar{t}-\bar{\zeta}} - \frac{\zeta}{\bar{\zeta}}) \bar{g}(y)ds - i\sigma_0 \int_s g(y)ds = f(t), t \in S. \end{aligned} \tag{15}$$

Let's use prove that the homogeneous version of the equation (15) has only the trivial solution. Indeed g_0 some solution to it. Then for the function $U_0(x) = U(x, g_0)$ we have $(TU_0(t) - i\sigma_0 U_0(t))^+ = 0, t \in S$ and virtue Theorem 1 we obtain $U_0(x) = 0, x \in D^+, U_0(0) = 0$, whence we get $\int_s g_0 ds = 0$, which implies that $U_0(x)$ is bounded in the domain D^-.

From $U_0(x) = 0, x \in D^+$, it follows (see (9)) that $NU_0 = -im^{-1}\dfrac{\partial}{\partial s(x)} V_0 = 0, x \in D^+$, and $V_0(x) = const, x \in D^+$. V_0 is defined from (11), (12) and (7), when $g = g_0$.On the basis of $\int_s g_0 ds = 0, V_0$ is bounded in the domain D^-. By $V_0(x) = const, x \in D^+$ we obtain

$$TV_0(x) = -\frac{1}{\pi}\int_s \frac{\partial}{\partial s} \ln(z-\zeta) g(y)ds - \frac{H}{2\pi}\int_s \frac{\partial}{\partial s(x)} \frac{z-\zeta}{\bar{z}-\bar{\zeta}} \bar{g}(y)ds = 0, x \in D^+. \tag{16}$$

In (16) we have $(TV_0(t))^+ = (TV_0(t))^- = 0, t \in S$. On the basis of the uniqueness theorem for the second external problem of statics of elastic mixtures [4] we can conclude that $V_0 = const, x \in D^-$. From $V_0 = const, x \in D^-$, for the assocaite to $V_0(x)$ vector $U_0(x)$ in the domain D^- we can write $U_0(x) = const, x \in D^-$.

Taking into account $U_0(x) = 0, x \in D^+$ and $U_0(x) = const, x \in D^-$ for $g = g_0$, we get $(TU_0(t))^- - (TU_0(t))^+ = 2g_0(t) = 0, t \in S$. Thus our assumption is not valid. Equation (15) has a solution for an arbitrary right -hand part. Consequently we have shown that the $R^+_{o,f}$ problem always has solution which is represented in the form (10).

Analogous can be considered problem $R^-_{o,f}$.

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