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AN APPROXIMATE METHOD FOR A NONLINEAR BEAM EQUATION

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Abstract

An approximate method is proposed for solving the initial boundary value problem for an integro-differential equation describing the behavior of a beam. The exactness of the proposed method is studied.

Let us consider the nonlinear Kirchhoff-type equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}-\left(\alpha+\beta\int_0^l\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \qquad (1)$$

$$0<x<l, \qquad 0<t\le T,$$

with the conditions

$$u(0,t)=u(l,t)=0,\ \frac{\partial^2 u}{\partial x^2}(0,t)=\frac{\partial^2 u}{\partial x^2}(l,t)=0, \qquad (2)$$

$$u(x,0)=u^{(0)}(x), \qquad \frac{\partial u}{\partial t}(x,0)=u^{(1)}(x). \qquad (3)$$

$$0\le x\le l, \qquad 0\le t\le T,$$

Here $u(x,t)$ is the sought beam deflection function at the point x at the time moment t, l is the beam length, α and β are the given values, $u^{(0)}(x)$, $u^{(1)}(x)$ are the known functions, and

$$u^{(p)}(x)=\sum_{i=1}^{\infty}a_i^{(p)}\sin\frac{i\pi x}{l}, \qquad (4)$$

$$\frac{d^{8-2p}u^{(p)}}{dx^{8-2p}}(x)\in L_2(0,l), \quad p=0,1.$$

Remark 1. *The second requirement in (4) can be replaced by*

$$\left|a_i^{(0)}\right|\le\frac{c}{i^{8,5+\varepsilon}}\ , \qquad \left|a_i^{(1)}\right|\le\frac{c}{i^{6,5+\varepsilon}}\ ,$$

where $i\ge i_0$, $\varepsilon>0$, $c=const>0$, i_0 *is some natural number.*

Equation (1) was proposed by Woinowsky-Kriger [1] in 1950. Subsequently, the investigation of equations of type (1) and with the same kind of nonlinearity evoked interest on the part of various researchers (see, e.g., [2 – 4]), who were mainly concerned with questions of the existence of a solution. In [5], a numerical method of solving Timoshenko's wave system, where nonlinearities are of the same kind as (1), is studied.

As far as we know, the question of construction and substantiation of approximate algorithms for equation (1) has not been so far investigated.

A solution of problem (1)-(3) will be sought in the form

$$u_n(x,t)=\sum_{i=1}^{n}u_{ni}(t)\sin\frac{i\pi x}{l} .$$

The coefficients $u_{ni}(t)$ are defined by means of Galerkin's method from the system of ordinary differential equations

$$u''_{ni}(t)+\left(\frac{i\pi}{l}\right)^4 u_{ni}(t)+\left(\frac{i\pi}{l}\right)^2\left(\alpha+\beta\frac{\pi^2}{2l}\sum_{j=1}^{n} j^2 u_{nj}(t)\right)u_{ni}(t)=0\,,\quad i=1,2,...,n\,, \tag{5}$$

with the initial condition

$$\frac{d^p u_{ni}}{dt^p}(0)=a_i^{(p)}\qquad p=0,1,\quad i=1,2,...,n\,. \tag{6}$$

Let us define the matrix $Q_n=\frac{\pi}{l}diag(1,2,...,n)$ and the vectors $\vec{u}_n(t)=\left(u_{ni}(t)\right)_{i=1}^{n}$, $\vec{a}_n^{(p)}=\left(a_i^{(p)}\right)_{i=1}^{n}$, $p=0,1$. We introduce the scalar product $(\lambda,\mu)_n=\sum_{i=1}^{n}\lambda_i\mu_i$, the norm $\|\lambda\|_n=(\lambda,\lambda)_n^{1/2}$ and the energy norm $\|\lambda\|_{Q_n^{2p}}=(Q_n^{2p}\lambda,\lambda)_n^{1/2}$, $p=1,2,$ for the vectors $\lambda,\mu\in R^n$, $\lambda=(\lambda_i)_{i=1}^{n}$, $\mu=(\mu_i)_{i=1}^{n}$.

Using these notations, system (5) and condition (6) can be rewritten as

$$\vec{u}''_n(t)+Q_n^4\vec{u}_n(t)+\left(\alpha+\frac{\beta l}{2}\|\vec{u}_n(t)\|^2_{Q_n^2}\right)Q_n^2\vec{u}_n(t)=0,\qquad 0<t\le T, \tag{7}$$

$$\frac{d^p\vec{u}_n}{dt^p}(0)=\vec{a}_n^{(p)},\qquad p=0,1. \tag{8}$$

Consider the vector functions

$$\vec{u}_n(t),\quad \vec{v}_n(t),\quad \vec{w}_n(t), \tag{9}$$

where the last two are the new ones defined by the formulas

$$\vec{v}_n(t)=Q_n^2\vec{u}_n(t),\qquad \vec{w}_n(t)=\vec{u}'_n(t).$$

Then problem (7),(8) leads to the following problem

$$\begin{gathered}\vec{u}'_n(t)=\vec{w}_n(t),\qquad \vec{v}'_n(t)=Q_n^2\vec{w}_n(t),\\ \vec{w}'_n(t)+Q_n^2\vec{v}_n(t)+\left(\alpha+\frac{\beta l}{2}\|\vec{u}_n\|^2_{Q_n^2}\right)\vec{v}_n(t)=0,\end{gathered} \tag{10}$$

$$\vec{u}_n(0) = \vec{a}_n^{(0)}, \quad \vec{v}_n(0) = Q_n^2 \vec{a}_n^{(0)}, \quad \vec{w}_n(0) = \vec{a}_n^{(1)}. \tag{11}$$

To solve problem (10), (11) we will use the difference method. On the time interval [0, T] we introduce the grid $\{ t_m \,|\, 0 = t_0 < t_1 < \ldots < t_M = T \}$ with a variable pitch $\tau_m = t_m - t_{m-1}$, $m = 1,2,\ldots,M$. Approximate values of vectors (9) on the m-th time layer, i.e., for $t = t_m$, $m = 0,1,\ldots,M$, are denoted by $\vec{u}_n^m$, $\vec{v}_n^m$, $\vec{w}_n^m$. They will be sought by using the modified Crank-Nicolson scheme

$$\frac{\vec{u}_n^m - \vec{u}_n^{m-1}}{\tau_m} = \frac{\vec{w}_n^m + \vec{w}_n^{m-1}}{2}, \qquad \frac{\vec{v}_n^m - \vec{v}_n^{m-1}}{\tau_m} = Q_n^2 \frac{\vec{w}_n^m + \vec{w}_n^{m-1}}{2},$$

$$\frac{\vec{w}_n^m - \vec{w}_n^{m-1}}{\tau_m} + Q_n^2 \frac{\vec{v}_n^m + \vec{v}_n^{m-1}}{2} + \left(\alpha + \frac{\beta l}{2} \frac{\left\| \vec{u}_n^m \right\|_{Q_n^2}^2 + \left\| \vec{u}_n^{m-1} \right\|_{Q_n^2}^2}{2} \right) \frac{\vec{v}_n^m + \vec{v}_n^{m-1}}{2} = 0, \tag{12}$$

where

$$\vec{u}_n^0 = \vec{a}_n^{(0)}, \quad \vec{v}_n^0 = Q_n^2 \vec{a}_n^{(0)}, \quad \vec{w}_n^{(0)} = \vec{a}_n^{(1)}. \tag{13}$$

System (12),(13) will be solved layerwise by an iteration method. Assuming that $\vec{u}_n^{m-1}$, $\vec{v}_n^{m-1}$, $\vec{w}_n^{m-1}$ have already been calculated, the problem can be reduced to finding $\vec{u}_n^m$, $\vec{v}_n^m$, $\vec{w}_n^m$. For this, we use a Picard-type iteration method

$$\frac{\vec{u}_n^{m,k} - \vec{u}_n^{m-1}}{\tau_m} = \frac{\vec{w}_n^{m,k-1} + \vec{w}_n^{m-1}}{2}, \qquad \frac{\vec{v}_n^{m,k} - \vec{v}_n^{m-1}}{\tau_m} = Q_n^2 \frac{\vec{w}_n^{m,k-1} + \vec{w}_n^{m-1}}{2},$$

$$\frac{\vec{w}_n^{m,k} - \vec{w}_n^{m-1}}{\tau_m} + Q_n^2 \frac{\vec{v}_n^{m,k-1} + \vec{v}_n^{m-1}}{2} + \left(\alpha + \frac{\beta l}{2} \frac{\left\| \vec{u}_n^{m,k-1} \right\|_{Q_n^2}^2 + \left\| \vec{u}_n^{m-1} \right\|_{Q_n^2}^2}{2} \right) \frac{\vec{v}_n^{m,k-1} + \vec{v}_n^{m-1}}{2} = 0,$$

$$k = 1,2,\ldots,$$

where $\vec{u}_n^{m,k-p}$, $\vec{v}_n^{m,k-p}$, $\vec{w}_n^{m,k-p}$ is the $(k-p)$-th iteration approximation for $\vec{u}_n^m$, $\vec{v}_n^m$, $\vec{w}_n^m$, $p = 0,1$.

In [6], the system of equations obtained by discretization of problem (1)-(3) is solved by means of Jacobi's nonlinear iteration method and the exactness of this method is studied.

Below we give the results of investigation of the solution exactness of systems (10),(11) and (12),(13).

It is proved that there exists a strong solution of problem (1)-(3), which is expanded into a series

$$u(x,t) = \sum_{i=1}^{\infty} u_i(t) \sin \frac{i\pi x}{l} \tag{14}$$

where the coefficients $u_i(t)$ satisfy an infinite system of equations

$$u_i''(t)+\left(\frac{i\pi}{l}\right)^4 u_i(t)+\left(\frac{i\pi}{l}\right)^2\left(\alpha+\beta\frac{\pi^2}{2l}\sum_{j=1}^{\infty} j^2u_j^2(t)\right)u_i(t)=0,$$
$$i=1,2,..., \qquad 0<t\le T,$$

provided that

$$\frac{d^p u_i}{dt^p}(0)=a_i^{(p)}, \qquad p=0,1.$$

Using n first coefficients of series (14), we form the vector $p_n\vec{u}(t)=(u_i(t))_{i=1}^n$. Let us consider the difference of vectors $\Delta\vec{u}_n(t)=\vec{u}_n(t)-p_n\vec{u}(t)$. It satisfies the equation

$$\Delta\vec{u}_n''(t)+Q_n^4\Delta\vec{u}_n(t)+\left(\alpha+\frac{\beta l}{2}\|\vec{u}_n(t)\|^2_{Q_n^2}\right)Q_n^2\vec{u}_n(t)-$$
$$-\left(\alpha+\frac{\beta l}{2}\|p_n\vec{u}(t)\|^2_{Q_n^2}\right)Q_n^2 p_n\vec{u}(t)=\vec{\xi}_n(t) \qquad (15)$$

with the initial condition

$$\frac{d^p\Delta\vec{u}_n}{dt^p}(0)=0, \qquad p=0,1.$$

In (15), $\vec{\xi}_n(t)=\frac{\beta\pi^2}{2l}\left(\sum_{i=n+1}^{\infty} i^2u_i^2(t)\right)Q_n^2 p_n\vec{u}(t)$.

Equation (15) is scalarly multiplied by $2\Delta\vec{u}_n'(t)$. After that, using the a priori estimates for $\|p_n\vec{u}(t)\|_{Q_n^2}$, $\|\vec{u}_n'(t)\|_n$, $\|\vec{u}_n(t)\|_{Q_n^{2j}}$, $j=1,2$, $\|\vec{\xi}_n(t)\|_n$ and the Gronwall inequality, we prove

Theorem 1: *For an error of Galerkin's method, the following estimate is valid*

$$\|\vec{u}_n'(t)-p_n\vec{u}_n'(t)\|_n+\|\vec{u}_n(t)-p_n\vec{u}(t)\|_{Q_n^2}\le\frac{C_1}{n^5},$$

where C_1 is some positive value not depending on n and t.

Remark 2: *Theorem 1 holds true for a weaker restriction on $u^{(p)}(x)$, $p=0,1$, than the second restriction in (4), namely,* $\frac{d^{4-2p}u^{(p)}}{dx^{4-2p}}(x)\in L_2(0,l)$, $p=0,1$.

The solutions of systems (10), (11) and (12),(13) form respectively the vectors $\vec{\lambda}_n(t)=(\vec{u}_n(t),\vec{v}_n(t),\vec{w}_n(t))^T$ and $\vec{\lambda}_n^m=(\vec{u}_n^m,\vec{v}_n^m,\vec{w}_n^m)^T$, $0<t\le T$, $m=0,1,...,M$. We introduce the vector $\vec{z}_n^m=\vec{\lambda}_n(t_m)-\vec{\lambda}_n^m$, $m=0,1,...,M$. To define it we have the equation

$$\frac{\vec{z}_n^m-\vec{z}_n^{m-1}}{\tau_m}=L_n\frac{\vec{z}_n^m+\vec{z}_n^{m-1}}{2}+\frac{1}{2}\vec{\varphi}_n^{m,m-1}+\vec{\psi}_n^{m,m-1} \qquad (16)$$

and the condition

$$\vec{z}_n^0=0. \qquad (17)$$

Here

$$\vec{\varphi}_n^{m,m-1}=\left(K_n(\vec{\lambda}_n(t_m))+K_n(\vec{\lambda}_n(t_{m-1}))\right)\left(\vec{\lambda}_n(t_m)+\vec{\lambda}_n(t_{m-1})\right)-\left(K_n(\vec{\lambda}_n^m)+K_n(\vec{\lambda}_n^{m-1})\right)(\vec{\lambda}_n^m+\vec{\lambda}_n^{m-1}),$$

the approximation error is

$$\vec{\psi}_n^{m,m-1} = \frac{\vec{\lambda}_n(t_m) - \vec{\lambda}_n(t_{m-1})}{\tau_m} - L_n \frac{\vec{\lambda}_n(t_m) + \vec{\lambda}_n(t_{m-1})}{2} -$$

$$- \left(K_n(\vec{\lambda}_n(t_m) + K_n(\vec{\lambda}_n(t_{m-1}))\right)\frac{\vec{\lambda}_n(t_m) + \vec{\lambda}_n(t_{m-1})}{2}$$

and $K_n = (K_{nij})_{i,j=1}^3$, $L_n = (L_{nij})_{i,j=1}^3$ are the third order block matrices with nonzero blocks of the form

$$L_{n13} = I_n, \quad L_{n23} = Q_n^{2,}, \quad L_{n32} = -(\alpha I_n + Q_n^2), \quad K_{n32}(\lambda) = -\frac{\beta l \|u\|_{Q_n^2}^2}{4} I_n,$$

where I_n is the unit matrix of n-th order and the vector $\lambda = (u, v, w)$, $u, v, w \in R^n$.

It is proved that $\left\|\vec{\psi}_n^{m,m-1}\right\|_n \le C\tau_m^2$, where C is a positive value not depending on m and n. By virtue of this inequality and the a priori estimates for $\|\vec{u}_n(t)\|_{Q_n^{2j}}$, $\|\vec{u}_n^m\|_{Q_n^{2j}}$, $j = 1,2$, system (16),(17) gives rise to

Theorem 2: *If the pitch of the difference scheme (12),(13) satisfies the inequality*

$$\tau_m \le \frac{2(1-\omega_n)}{\gamma_n + s},$$

where ω_n *is an arbitrary number from the interval (0, 1),* $\gamma_n = \left(\frac{\pi}{l} n\right)^2 + \max(1, \alpha)$*, and* s *is some value expressed in terms of the norms* $L_2(0,l)$ *of the functions* $u^{(0)}(x)$, $u^{(1)}(x)$, *and their derivatives, then the following estimate holds for an error of the difference scheme (12),(13)*

$$\left\|\vec{u}_n(t_m) - \vec{u}_n^m\right\|_n + \left\|\vec{v}_n(t_m) - \vec{v}_n^m\right\|_n + \left\|\vec{w}_n(t_m) - \vec{w}_n^m\right\|_n \le C_2 \sum_{p=1}^m \tau_p^3,$$

where C_2 *is a positive value not depending on m, n and grid pitches.*

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