ENTROPY OF A SET SPLIT INTO DUAL FUZZY SUBSETS

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Abstract

As is known the entropy of a set quantitatively describes the uncertainty index of any variable defined on it. Entropy is also an important property for fuzzy sets as well. The work deals with the operation of splitting a crisp indicator in the dual fuzzy subsets. More exactly, the information measure of a split set – entropy is discussed. A comparison between the split set entropy and classical fuzzy set entropy is discussed. The information measure of the Cartesian product of two separated sets is presented as the main determinant of the fuzzy entropy. The analytical form of a split dual fuzzy sets' entropy is obtained. It is proved that the Shannon entropy of a set can be represented as the additive sum of the entropies of its split dual fuzzy sets. Two types of entropy are considered: entropy of dual fuzzy sets of a split set and entropy of dual fuzzy sets obtained by point splitting (by its elements). It is proved that the entropy of the dual fuzzy sets obtained by splitting the set is the additive sum of the Shannon entropy of this set and the entropies of the dual fuzzy sets obtained by the point splitting of the same set.

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1 Introduction

1.1. On the duality of fuzzy sets

In the investigation of problems related to the analysis and synthesis of complex events, the application of L.A. Zadeh's fuzzy sets theory [1] holds significant relevance in today's context. This is especially crucial as the challenges in semantically representing expert qualitative information have intensified due to the intricate nature of the subjects under investigation. Present methods for assessing the precision of studying objects no longer meet the expectations of contemporary researchers. Consequently, there is a growing independence of the levels of object compatibility and incompatibility in recent research [2-15]. This independence arises from the dual representation of evaluation, which has become a crucial aspect in dealing with incomplete information. Exploring imprecisions and uncertainties in modeling complex events is a topic of great interest, and the prevailing direction of research involves portraying the dual nature of information evaluation through independent degrees of belonging and non-belonging. This concept was initially introduced by Atanasov [2].

Atanasov's Intuitionistic Fuzzy Sets (IFS) theory [2] serves as an extension of Zadeh's Fuzzy Sets (FS) theory [1]. IFS assign each element (μ, ν) a membership degree (μ), a non-membership degree (ν), and a hesitancy degree $(1-\mu, 1-\nu)$ through Intuitionistic Fuzzy Numbers (IFN), making it more adept at handling vagueness than FS. The extensive application of IFS theory across various areas has been documented [5, 6], with definitions of fundamental arithmetic operations on IFN provided in [2]. The theory finds significant utility in decision-making research. After investigation of existing materials, the authors of [16] presented a review on IFS studies. Despite its usefulness, IFN (μ, ν) encounters a limitation – the sum of membership and non-membership degrees must be equal to or less than 1. But sometimes for the data provided by DM for certain attribute, the same sum is greater than 1 $(\mu+\nu) > 1$. Yager introduced the concept of Pythagorean Fuzzy Sets (PFS) (μ, ν) [3] as a generalization of IFS, relaxing this constraint to the sum of squared degrees $\mu^2 + \nu^2 \leq 1$. However, expert orthopair assessments often defy full description using PFNs or IFNs due to the intricacies of decision makers' psychological evaluations and the challenging expression of attribute information. Yager addressed this issue by proposing q-rung Orthopair Fuzzy Sets (q-ROFS) [5], where $q \geq 1$, and the sum of the q-th power of membership and non-membership degrees cannot exceed 1.

For a q-rung orthopair fuzzy number (q-ROFN), the condition holds that $(\mu^q + \nu^q \leq 1)$. It is evident that q-ROFSs generalize IFSs and PFSs, where IFSs and PFSs are specific instances of q-ROFSs when is equal to 1 and 2, respectively. Consequently, q-ROFNs seem more fitting and proficient in conveying decision maker's (DM) assessment information. Ongoing research efforts by the authors of this paper focus on the development of aggregations of experts' q-rung orthopair fuzzy evaluations, specifically addressing multi-criteria decision-making problems [6-14, 17].

A distinct approach to the dual representation of a fuzzy set is presented in [18]. Additionally, in their works [17, 18], the authors introduce the innovative concept of lower α-level sets for fuzzy sets, considered as a dual counterpart to upper q-level sets for fuzzy sets. This work also introduces the concept of dual fuzzy subsets and establishes a dual decomposition theorem. The study delves into the dual arithmetic of fuzzy sets in and yields compelling results based on both upper and lower α-level sets.

Let us briefly discuss the novel concept: In practical scenarios, there are instances where experts face challenges in determining compatibility levels for all objects. These levels are essentially depicted through a specific sampling of the universe, and experts may vary in their chosen samplings. Even the samplings for fuzzy terms in linguistic variables can differ. Despite these variations, there is still a need for aggregating this information, acknowledging that the universe may not be entirely represented. In such situations, both Zadeh's fuzzy set analysis and the presented dual forms in the form of q-rung orthogonal fuzzy sets may not fulfill the requirements for aggregations.To elaborate, for any expert within a particular universe $X\{x_1, ..., x_n\}$, there exists a specific sampling of items $A = \{x_{i_1}, ..., x_{i_l}\}$, available for evaluation. The compatibility levels generated by an expert can be represented as a function, $f_A(x) : A \to [0, 1]$ with values known only for elements within the set $A \subset X$. This data may differ for other evaluations by the same expert or for evaluations by different experts. This new type of information source diverges from Zadeh's perspective on determining compatibility levels. Here, the information source is presented in pairs $\langle A, f_A\rangle$, involving a different nature of both the source and the data.To address this, the article proposes a means of semantically representing such information through the concept of a split fuzzy set, which is based on Zadeh's fuzzy set concept. Specifically, the operation of splitting a crisp subset into dual fuzzy sets is introduced. This dual, split fuzzy sets lattice serves as a unified framework for aggregating expert evaluations from diverse samplings.

1.2. On finite set's entropy

Boltzmann introduced the concept of entropy in the late 19th century to quantify the irregularity of an ideal gas within a closed container. In contrast, information theory emerged in the 1940s while addressing telecommunications issues. The primary objective of information theory is to explore the principles governing the acquisition, transfer, processing, and storage of information. Due to the inherent randomness in information transfer, statistical methods have become essential for studying these processes. Shannon further developed information theory, aiming to elucidate both the quantity and regularity of information in a dataset [19].

The abundance and variety of information suggest a lower entropy in the data [20]. Entropy, alongside the concept of information, plays a crucial role. Information theory, as mentioned earlier, delves into the quantitative laws governing the acquisition, transfer, processing, and storage of information. Before defining information, it is pertinent to elaborate on entropy, also known as statistical entropy, which can be defined as a measure of diversity across probability distributions.

Entropy serves as a gauge of a system's uncertainty, with thermodynamics, statistical physics theory, and information theory providing prevalent definitions. This study exclusively adopts the information theory definition, disregarding other interpretations. The entropy of a system equals the information required to know all possible states. When a system shifts from an ordered, organized state to a disordered, unplanned one, its entropy increases. In this context, information is perceived as the inverse of entropy. Despite being commonly associated with thermodynamics, the connection between entropy and information theory relies on mathematical proofs rather than intuition. Although the two types of entropy differ, both are grounded in randomness. In thermodynamics, entropy is expressed by dividing energy by temperature, often in terms of Kelvin. Conversely, in communication engineering, Shannon entropy quantifies information in bits, devoid of dimensions. The distinction arises from the state of randomness within a system. When randomness is maximum or message probabilities are equal, the information source's entropy reaches its maximum.

Shannon's information theory considers information equivalent to fuzziness and is fundamentally a statistical theory focused on communication. Shannon's seminal work, "A Mathematical Theory of Communication" (1948) [19], defines information source as a person or device producing statistically characterized messages. Information is assessed in terms of unpredictability or information value for the receiver.

As was mentioned, in [21], the authors of the current work considered a concept of a split fuzzy set. Basically, this concept allows for the new dual representation or aggregation of expert evaluations. But the new concept is again and again based on the concept of Zadeh fuzzy set. In particular, the operation of splitting a crisp subset into dual fuzzy sets is introduced. In this work we consider information measurement problems of split set into dual fuzzy subsets. More exactly, we present a new definition of entropy for the split set, based on Shannon's [19] and Luka-Termini [22] entropies. The basic properties of a new definition of entropy are studied.

The 2^{nd} Section deals with the preliminary concepts. A brief review on the fuzzy entropy is developed. Several classical fuzzy entropy definitions are considered. In the second party of this section the information concept of a split crisp set into a pair of dual fuzzy sets is presented. In the 3rd Section a definition of Entropy of a split set is presented. The basic properties of a new entropy are studied.Basis conclusions, obtained results and future possible research directions are presented in the 4th Section.

2 Preliminary concepts

2.1. Basic fuzzy entropies

A. De Luca and S. Termini Entropy. The measure of a quantity of fuzzy information gained from a fuzzy set or fuzzy system is known as fuzzy entropy. The fuzzy entropy contains vagueness and ambiguity uncertainties, while Shannon entropy contains the randomness (probabilistic) uncertainty [23-25]. The concept of fuzzy entropy is established by incorporating the idea of a membership function. De Luca and Termini [22] formulated fuzzy entropy by building upon Shannon's function and delineated a set of properties that a fuzzy entropy should adhere to. The expression for the fuzzy entropy introduced by De Luca and Termini is depicted in equation (1). It is defined a fuzzy entropy on a fuzzy subset A of the finite set $X = \{x_1, x_2, ..., x_n\}$ based on the concept of membership function, where there are n values of the membership function A - $\{\mu_i \equiv \mu_A(x_i)\},\ i = 1, ..., n$

$$H_A = -K\sum_{i=1}^{n}\{\mu_i \log(\mu_i) + (1-\mu_i)\log(1-\mu_i)\}. \tag{1}$$

The four characteristics of fuzzy entropy are as follows:

P1: $H_A = 0$ if and only if the set A is a crisp set ($\mu_i = 0$ *or* $1\ \forall x_i \in A$);

P2: H_A is maximum if and only if $\mu_i = 0.5\ \forall x_i \in A$;

P3: $H \geq H^*$ where H^* is the entropy of a sharpened version of A, denoted as A^*;

P4: $H = \overline{H}$ where $\overline{H}$ is the entropy of the complement set $\overline{A}$.

L. Zadeh introduced Probability Entropy, an extension of Shannon entropy, to function as a fuzzy entropy on a fuzzy subset A of the finite set $X = \{x_1, x_2, ..., x_n\}$ based on the probability distribution $P = \{p_1, p_2, ..., p_n\}$ on X. This entropy is mathematically expressed as [26]:

$$H_A = -\sum_{i=1}^{n}\mu_A(x_i)p_i \log(p_i). \tag{2}$$

Here μ_A represents the membership function of A and p_i is the probability of the occurrence of x_i. It is evident that this scenario encompasses three types of uncertainties: randomness, ambiguity, and vagueness; encapsulating both randomness and fuzziness.

In the context of fuzzy sets, specifically when the intersection between a fuzzy set and its complement is not an empty set $\emptyset$, Yager [27, 28] introduced a fuzzy entropy applicable to a fuzzy subset A of the finite set $X = \{x_1, x_2, ..., x_n\}$. This serves as a fitting measure of fuzziness for the set A:

$$H_A = \frac{1 - \sum\limits_{x \in X} (|\mu_A(x) - (1 - \mu_A(x))|)}{card(X)}, \tag{3}$$

where

$$card(X) = \frac{1}{2} \sum_{x \in X} (1 - |\mu_A(x) - (1 - \mu_A(x))|). \tag{4}$$

The summation term $\sum\limits_{x \in X} (|\mu_A(x) - (1 - \mu_A(x))|)$ in equation (3) can be understood as the Hamming distance, denoted as $\varphi_1(A, A^c)$, between the sets A and A^c. Consequently, H_A can be expressed as

$$H_A = \frac{1 - \varphi_1(A, A^c)}{card(X)}. \tag{5}$$

(5) Alternatively, an equivalent expression can be obtained by using the Euclidean distance, denoted as, $\varphi_2(A, A^c)$, instead of the Hamming distance to obtain an equivalent H_A. The Euclidean distance is defined as:

$$\varphi_2(A, A^c) = \sqrt{\sum_{x \in X} (|\mu_A(x) - (1 - \mu_A(x))|)^2}. \tag{6}$$

R. L. Kaufmann Entropy. Kaufmann [29] introduced an entropy to measure the fuzziness of the fuzzy set as follows:

$$H_A = -\frac{1}{\ln(n)} \sum_{x \in X} \pi_A(x) - \mu_A(x). \tag{7}$$

where

$$\pi_A(x) = \frac{\mu_A(x)}{\sum\limits_{i=1}^{n} \mu_A(x_i)}. \tag{8}$$

Kaufmann noticed that this method of computing the entropy for a fuzzy set does not depend on accounting for the effective values of μ, and instead it does for their relative values. The relative values come from the function given in equation (8). This is a drawback for this measure because the relative values lead to the same entropy for different fuzzy sets and ordinary sets.

2.2. Representation of a crisp set as a pair of dual fuzzy subsets

Consider the information source previously introduced in the introduction, pertaining to expert evaluations. Assume that, for any expert within a particular universe $X = \{x_1, x_2, ..., x_n\}$, there exists a specific sampling of elements available for assessment. Suppose the expert's compatibility levels are denoted by a certain function $f_A(x) : A \longrightarrow [0, 1]$, with values known solely for elements within the set $A \subset X$. This information may vary for the expert's other evaluations and for assessments by different experts. This new type of information source diverges from Zadeh's perspective on determining compatibility levels [1]. In this instance, the information source is represented by pairs $\langle A, f_A \rangle$. Let $A \subset X$ and $I_A \in \{0, 1\}^X$ be its indicators. Express it as follows:

$$I_A(x) = f(x)I_A(x) + (1 - f(x))I_A(x),\ x \in X \tag{9}$$

here $f(x) : X \longrightarrow [0, 1]$ is an extension of the function

$$f_A(x) : A \longrightarrow [0, 1] \tag{10}$$

on the universe X ($f(x) = f_A(x),\ x \in A$).

Definition 1 [21]. We designate the representation (9)-(10) as the splitting of the indicator I_A concerning the function f.

Let's introduce the following notations:

$$I_{\tilde{A}}(x) \equiv f(x)I_A(x) \text{ and } I_{\tilde{A}^D}(x) \equiv (1 - f(x))I_A(x). \tag{11}$$

Indicators $I_{\tilde{A}},\ I_{\tilde{A}^D} \in [0, 1]^X$ belonging to two fuzzy subsets $\tilde{A}, \tilde{A}^D \subset X$ are termed as the splitting of an indicator I_A associated with a subset $A \subset X$ and

$$I_A = I_{\tilde{A}} + I_{\tilde{A}^D}. \tag{12}$$

Definition 2 [21]. Indicators $I_{\tilde{A}},\ I_{\tilde{A}^D} \in [0, 1]^\Omega$ and fuzzy subsets $\tilde{A}, \tilde{A}^D \subset X$ are referred to as dual entities, respectively.

As per L. Zadeh [1] $I_{\tilde{A}}$ represents an indicator or membership function (compatibility function) of a fuzzy subset $\tilde{A}$. It is evident that the process of splitting remains unaffected by the extension of the function $f_A(x)$: $A \longrightarrow [0, 1]$. More precisely, the pair $\langle A, f_A \rangle$ results in a pair of splitting fuzzy sets $(\tilde{A}, \tilde{A}^D)$.

Example 1. Consider a set of digits $X = \{0,1,2,3,4,5,6,7,8,9\}$. Additionally, let the universe's sampling be a certain subset $A \subset X$. For instance, assume A represents the set of odd digits - $A = \{1,3,5,7,9\}$ and the expert's evaluation exclusively on this sampling is expressed by the function $f_A(x) : A \longrightarrow [0,1]$, $f_A(x) = \frac{1}{x+1}$, $x \in A$. Let $f(x) : A \longrightarrow [0,1]$ be any extension of the function $f_A(x)$ on X. In this case, the splitting of the I_A of the $A \subset X$, associated with the subset within the universe X into two dual fuzzy sets (or their indicators/membership functions) appears as follows:

$$\tilde{A} = \{0/0, 1/(1/2), 2/0, 3/(1/4), 4/0, 5/(1/6), 7/(1/8), 8/0, 9/(1/10)\}$$
and
$$\tilde{A}^D = \{0/0, 1/(1/2), 2/0, 3/(3/4), 4/0, 5/(5/6), 6/0, 7/(7/8), 8/0, 9/(9/10)\}. \tag{13}$$

In practical terms, dual splitting fuzzy subsets $\langle \tilde{A}, \tilde{A}^D \rangle$ are created as fuzzy subsets within the universe X. The practical interpretation is as follows: when describing an uncertain term related to a linguistic variable over the elements of a universe, a membership function is typically constructed. However, to extend the information contained in the membership function only to specific elements of a concrete crisp subset, the set is split into dual split fuzzy subsets. Therefore, the extended information is encapsulated within dual fuzzy sets. The duality of this extension implies that both fuzzy sets convey the same information, but encoded in different manners.

Note that the generation of splitting dual fuzzy subsets $\tilde{A}$ and $\tilde{A}^D$ within X is influenced by the subset A, $A \subset X$ and a particular function $f_A(x) : A \longrightarrow [0,1]$. As previously discussed, the application of the split operation can be relevant in various scenarios. Here's one example: let's explore the application of splitting a set into dual fuzzy sets in the context of multi-attribute decision making (MADM).

Imagine a MADM model involving 5 attributes $X = \{x_1, x_2, ..., x_5\}$ and 3 alternatives $D = \{d_1, d_2, d_3\}$. Assume that a decision-making matrix represents normed ratings in the range $[0,1]$, and certain ratings are unspecified:

As evident from this matrix, each alternative has attributes for which the rating evaluations are absent. Such instances can occur in practical scenarios for various reasons. One such situation arises when there is a considerable number of attributes due to the detailed nature of the task, making it challenging for experts to assess all attributes. Such occurrences are common, especially when constructing recommendation models in collaborative filtering problems. Filling out these empty elements is important. If there is a substantial amount of prehistorical data, a machine learning approach can effectively handle this issue. However, when objective data is lacking,

D/X	x_1	x_2	x_3	x_4	x_5
d_1	0.2	-	0.7	0.6	-
d_2	-	0.4	-	0.3	0.8
d_3	0.3	0.5	0.6	-	-

and expert evaluations are sparse, the splitting operation introduced here provides a viable solution.

Observing that alternative d_1 is evaluated on a subset of attributes $A_1 \equiv \{x_1, x_3, x_4\}$, alternative d_2 - on a different subset $A_2 \equiv \{x_2, x_4, x_5\}$, and alternative d_3 - on yet another subset of attributes $A_3 \equiv \{x_1, x_2, x_3\}$, we can employ the split operation on these sets to create dual fuzzy sets. Let us split these sets into dual fuzzy sets. Consequently, the decision matrix can be expressed as:

D/X	x_1	x_2	x_3	x_4	x_5
$\tilde{A}_{d_1}$	0.2	0.0	0.7	0.6	0.0
$\tilde{A}^D_{d_1}$	0.8	0.0	0.3	0.4	0.0
$\tilde{A}_{d_2}$	0.0	0.4	0.0	0.3	0.8
$\tilde{A}^D_{d_2}$	0.0	0.6	0.0	0.7	0.2
$\tilde{A}_{d_3}$	0.3	0.5	0.6	0.0	0.0
$\tilde{A}^D_{d_3}$	0.7	0.5	0.4	0.0	0.0

Thus, the alternative d_i $i = 1, 2, 3$ is characterized by dual split fuzzy subsets $\langle \tilde{A}_{d_i}, \tilde{A}^D_{d_i} \rangle$ across the entire attribute's universe $X = \{x_1, x_2, ..., x_5\}$. The development of an aggregation tool and the methodologies for constructing ranking relations can explore various directions, utilizing the definitions and findings outlined in the subsequent sections regarding the operations of dual split sets. One straightforward approach is to amalgamate the elements of split dual fuzzy sets into pairwise intuitionistic fuzzy numbers through a simple concatenation.

It is important to note the symbolic intuitionistic fuzzy number $(0.0, 0.0)$, where both the attribution and non-attribution values are 0.0, indicating that the evaluation has not been conducted. In practical terms, if we assign a formulaic quantitative value to unrated assessments, it is natural to substitute it with the zero intuitionistic fuzzy numerical rating - $(0.0, 1.0)$. Consequently, the decision-making matrix assumes the following structure:

D/X	x_1	x_2	x_3	x_4	x_5
d_1	(0.2,0.8)	(0.0,1.0)	(0.7,0.3)	(0.6,0.4)	(0.0,1.0)
d_2	(0.0,1.0)	(0.4,0.6)	(0.0,1.0)	(0.7,0.3)	(0.8,0.2)
d_3	(0.3,0.7)	(0.5,0.5)	(0.6,0.4)	(0.0,1.0)	(0.0,1.0)

Assume the attribute weight vector in this model is represented by $W = \{w_1, w_2, ..., w_5\} = \{0.1, 0.2, 0.4, 0.1, 0.2\}$. To rank alternatives, we will employ the Intuitionistic Fuzzy Weighted Averaging (IFWA) operator:

$$d_i \sim IFWA(a_1, ..., a_5) = (w_1 \odot a_1)\hat{+} \cdots \hat{+}(w_5 \odot a_5)$$

As an illustration, for d_1 the calculation would be:

$$d_1 \sim 0.1 \odot (0.2, 0.8)\hat{+}0.2 \odot (0.0, 1.0)\hat{+}0.4 \odot (0.7, 0.3)$$

$$\hat{+}0.1 \odot (0.6, 0.4)\hat{+}0.2 \odot (0.0, 1.0).$$

Here, $\hat{+}$ and $\odot$ refer to the addition and multiplication operations on intuitionistic fuzzy numbers, respectively [2].

3 Entropy of a split set

Let $X = \{x_1, ..., x_n\}$ be a finite set of elementary random events with probabilities $p_1, ..., p_n$. As was mentioned, a dual element (dual subset) in [21] is defined basing on the procedure of representation of a crisp set as a pair of dual fuzzy sets, which is called a procedure of splitting a crisp set. Accordingly, let us split the set $X : X = \tilde{X} \oplus \tilde{X}^D$. Let this splitting is done "point-by-point" (pointwise splitting) [21]. The amount of information corresponding to an unsplit point depends on the corresponding probability $I = I(p(x))$. For a split point, it is natural to assume that the information depends on both the probability and the corresponding value of the membership function of the split subset:

$$\tilde{I} = I(\mu_{\tilde{X}}(x), p(x)). \tag{14}$$

Similarly, for a dual fuzzy point:

$$\tilde{I}^D = I(\mu_{\tilde{X}^D}(x), p(x)) = I(1 - \mu_{\tilde{X}}(x), p(x)). \tag{15}$$

These functions, as we shall see below, are defined in such a way that

$$I(p(x)) = I(\mu_{\tilde{X}}(x), p(x)) + I(\mu_{\tilde{X}^D}(x), p(x)). \tag{16}$$

In order to establish the form of the function $\tilde{I}$, let us consider its three properties (without proof).

1) $\tilde{I}$ as a function of p is continuous on $(0;1]$.

2) Let be given some finite set $Y = \{y_1, ..., y_m\}$. Let be given splitting: $X = \tilde{X} \oplus \tilde{X}^D$ and $Y = \tilde{Y} \oplus \tilde{Y}^D$. Splitting of $X \times Y$ is carried out according to the rules given in [21]. If for some $x \in X$ and $y \in Y$ the values of the membership functions are the same $\mu_{\tilde{X}}(x) = \mu_{\tilde{Y}}(y) = \mu$, then the value of the membership function of $(x, y) \subset X \times Y$ will be the same. We assume that $\tilde{I}$ has the following property

$$I(\mu, pq) = I(\mu, p) + I(\mu, q) \; for \; all \; 0 \leq \mu, \; p, q \leq 1. \tag{17}$$

3) An increase in the value of the membership function entails the same increase in the information of the corresponding fuzzy event:

$$I(\lambda\mu, p) = \lambda I(\mu, p). \tag{18}$$

for any $0 \leq \mu, \; p \leq 1$ and nonnegative λ.

Proposition 1. Let the function $\tilde{I} = I(\mu, p)$ satisfy conditions 1). - 3). Then $\tilde{I}$ has the form

$$\tilde{I} = I(\mu, p) = -k\mu \log p. \tag{19}$$

Proof. Relation (18) entails:

$$I(\mu, p) = \mu I(1, p). \tag{20}$$

for any $0 \leq \mu, \; p \leq 1$, while (20) for unsplit sets ($\mu = 1$)

$$I(1, p, q) = I(1, p) + I(1, q). \tag{21}$$

The only continuous solution of this functional equation, as is known [30], is as follows

$$I(1, p) = -k \log p, \tag{22}$$

where k ($k > 0$) is constant. Since the amount of information is considered to be a non-negative value, then Proposition 1 is proved. $\square$

Let us go back to the total set of random elementary events. We have shown that the amount of information corresponding to a split event $\tilde{x}_i$ is determined by the formula

$$\tilde{I}_i = I(\mu_i, p_i) = -k\mu_i \log p_i. \tag{23}$$

Therefore, the average amount of information corresponding to the split set $\tilde{X}$ is computed using the formula:

$$Z(\tilde{X}) = \sum_{i=1}^{n} p_i \tilde{I}_i = -k \sum_{i=1}^{n} \mu_i p_i \log p_i. \tag{24}$$

This is the entropy of a fuzzy subset according to Zadeh [26]. Analogously,

$$Z(\tilde{X}^D) = \sum_{i=1}^{n} p_i \tilde{I}_i^D = -k \sum_{i=1}^{n} \mu_i^D p_i \log p_i. \tag{25}$$

as $\mu_i + \mu_i^D = 1,\ i = 1, ..., n$. The following proposition becomes clear:

Proposition 2. The Shannon entropy of a set is represented as the additive sum of the entropies of its split dual fuzzy sets

$$Z(\tilde{X}) + Z(\tilde{X}^D) = H(X), \tag{26}$$

where $H(X)$ is the Shannon entropy.

Let's make one remark. According to [21], the probability of a split point is considered to be the value $\mu_i p_i$, and the conditional probability of $\tilde{x}_i$ in the case when $\tilde{x}_i \in \tilde{X}$ is determined by the formula

$$p(\tilde{x}_i/\tilde{X}) = \frac{\mu_i p_i}{\tilde{\mathbb{P}}}, \tag{27}$$

where $\tilde{\mathbb{P}}$ is the probability normalized value. The corresponding Shannon amount of information has the following form

$$H(\tilde{X}) = -k \sum_{i=1}^{n} \frac{\mu_i p_i}{\tilde{\mathbb{P}}} \log \frac{\mu_i p_i}{\tilde{\mathbb{P}}}. \tag{28}$$

In connection with this formula, let us cite the following two inequalities [31]. First inequality:

$$-k \sum_{i=1}^{n} \frac{\mu_i p_i}{\tilde{\mathbb{P}}} \log \frac{\mu_i p_i}{\tilde{\mathbb{P}}} \leq -k \sum_{i=1}^{n} \frac{\mu_i p_i}{\tilde{\mathbb{P}}} \log p_i \tag{29}$$

besides, equality takes place when all μ_i are equal. This inequality can be rewritten as follows

$$-k \sum_{i=1}^{n} \frac{\mu_i p_i}{\tilde{\mathbb{P}}} \log \frac{\mu_i}{\tilde{\mathbb{P}}} \leq 0 \tag{30}$$

i.e., at least one

$$\mu_i \geq \tilde{\mathbb{P}}. \tag{31}$$

The use of normalized probabilities on $\tilde{X}$ [21] entails the need for a certain agreement between ordinary probabilities and membership functions. The last inequality must be considered as a condition for such agreement. The second inequality is

$$H(\tilde{X}) \leq \tilde{\mathbb{P}}^{-1} Z(\tilde{X}) \tag{32}$$

which is a direct consequence of the former.

We will introduce two more entropy measures, the entropy of the split set and the entropy of splitting, as a result of the pointwise splitting of the universal set.

Based on the well-known property of the Shannon entropy function [31], we can write

$$\begin{aligned} H(\tilde{x}_1 \oplus \tilde{x}_1^D, ..., \tilde{x}_n \oplus \tilde{x}_n^D) &= H(\mu_1 p_1, \mu_1^D p_1, ..., \mu_n p_n, \mu_n^D p_n) \\ &= H(p_1, ..., p_n) + \sum_{i=1}^{n} p_i H(\mu_i, \mu_i^D). \end{aligned} \tag{33}$$

The obtained expression

$$S(\tilde{X}, \tilde{X}^D) = H(\tilde{X} \oplus \tilde{X}^D) \equiv H(\tilde{x}_1 \oplus \tilde{x}_1^D, ..., \tilde{x}_n \oplus \tilde{x}_n \oplus \tilde{x}_n^D). \tag{34}$$

we consider as the entropy of a split set $\tilde{X} \oplus \tilde{X}^D$, and the expression

$$L(\tilde{X} \oplus \tilde{X}^D) = \sum_{i=1}^{n} p_i H(\mu_i, \mu_i^D) \tag{35}$$

as the entropy of splitting.

As a result, we receive the following proposition:

Proposition 3. The entropy of the dual fuzzy sets obtained by splitting a set is the additive sum of the Shannon entropy of this set and the entropy of the dual fuzzy sets obtained by the pointwise splitting of the same set:

$$S(\tilde{X}, \tilde{X}^D) = H(X) + L(\tilde{X}, \tilde{X}^D). \tag{36}$$

For the dual pair $u^{\sim}$ [21] we assume:

$$p_{u^{\sim}} = p_{\tilde{X}} + p_{\tilde{X}^D} = p. \tag{37}$$

Therefore, the conditional probabilities [31] have the following form

$$p(\tilde{x}/u^{\sim}) = \frac{p_{\tilde{x}}}{p_{u^{\sim}}} = \mu, \;\; p(\tilde{x}^D/u^{\sim}) = \frac{p_{\tilde{x}^D}}{p_{u^{\sim}}} = 1 - \mu. \tag{38}$$

and the conditional entropy equals

$$X(\tilde{X}, \tilde{X}^D/u^{\sim}) = H(\mu_{\tilde{X}}(x), 1 - \mu_{\tilde{X}}(x)). \tag{39}$$

(33) can be rewritten as follows

$$H(\tilde{x}_1 \oplus \tilde{x}_1^D, ..., \tilde{x}_n \oplus \tilde{x}_n^D) = H(x_1, ..., x_n) + \sum_{i=1}^{n} p_i H(\tilde{x}_i, x_i^D / u^\sim). \ \square \quad (40)$$

Finally, we present a proposition.

Proposition 4. Let the set of dual pairs of a split set be ordered according to the levels of fuzziness [32]. Then if $u^\sim \preceq v^\sim$, we have

$$H(\tilde{x}, \tilde{x}^D / u^\sim) \leq H(\tilde{y}, \tilde{y}^D / v^\sim). \quad (41)$$

Indeed,

$$u^\sim \preceq v^\sim \ \Leftrightarrow \ \min(\mu_{\tilde{x}}, \mu_{\tilde{x}^D}) \leq (\mu_{\tilde{y}}, \mu_{\tilde{y}^D}) \leq \max(\mu_{\tilde{x}}, \mu_{\tilde{x}^D}) \ \Rightarrow$$

$$|1/2 - \mu_{\tilde{x}}|, |1/2 - \mu_{\tilde{x}^D}| \geq |1/2 - \mu_{\tilde{y}}|, |1/2 - \mu_{\tilde{y}^D}| \ \Rightarrow$$

$$H(\tilde{x}, \tilde{x}^D / u^\sim) \leq H(\tilde{y}, \tilde{y}^D / v^\sim). \ \square$$

4 Conclusions

The phenomenon of operation of splitting a set into two dual fuzzy sets is presented. The information measure of this operation - entropy is studied. The comparison between the entropy of a split set and the entropy of classical fuzzy sets is discussed. It is proved that the Shannon entropy of a set represents an additive sum of the entropies of its split dual fuzzy sets. The second type of splitting operation is also considered. The entropy of dual fuzzy sets obtained by pointwise splitting of the set is studied. It is argued that the entropy of the dual fuzzy sets obtained by splitting a set is the additive sum of the Shannon entropy of this set and the entropies of dual fuzzy sets obtained by the pointwise split of the same set. The obtained results will play an important role in the multi-attribute/criteria decision-making problems with use of the split operation and with their entropy involvement. In future studies, the analysis of the split dual sets will be developed for different fuzzy environment.

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References

1. Zadeh, L.A., Fuzzy set. *Information and Control*, **8** (1965), 338-353.

2. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Set Syst*, **20**, (1986), 87–96.

3. Yager, R.R. Pythagorean Membership Grades in Multicriteria Decision Making. *IEEE T Fuzzy Syst*, **22**, 4 (2014), 958–965.

4. Yager, R.R. Alajlan, N.; Bazi, Y. Aspects of Generalized Orthopair Fuzzy Sets. *Int J Intell Syst*, **33**, 11 (2018), 2154–2174.

5. Yager, R.R. Generalized Orthopair Fuzzy Sets. *IEEE T Fuzzy Syst*, **25**, 5 (2017), 1222–1230.

6. Sirbiladze, G., Khutsishvili, I., Badagadze, O., Tsulaia, G. Associated Probability Intuitionistic Fuzzy Weighted Operators in Business Start-up Decision Making. *Iranian Journal of Fuzzy Systems*, **15**, 5 (2018), 1–25.

7. Sirbiladze, G., Sikharulidze, A. Extentions of Probability Intuitionistic Fuzzy Aggregation Operators in Fuzzy Environmet. *International Journal of Information Technology and Decision Making*, **17**, 2 (2018), 621–655.

8. Sirbiladze, G., Khutsishvili, I., Midodashvili, B. Associated Immediate Probability Intuitionistic Fuzzy Aggregations in MCDM. *Computers and Industrial Engineering*, **123** (2018), 1-8.

9. Sirbiladze G. Associated Probabilities' Aggregations in Interactive MADM for q-Rung Orthopair Fuzzy Discrimination Environment. *Internationa Journal of Intelligent Systems*, **35**, 3 (2020), 335-372.

10. Sirbiladze, G., Sikharulidze, A., Matsaberidze, B., Khutsishvili, I., Ghvaberidze, B. TOPSIS Approach to Multi-Objective Emergency Service Facility Location Selection Problem under q-Rung Orthopair Fuzzy Information. *Transactions of A. Razmadze Mathematical Institute*, **173**, 3 (2019), 137-145.

11. Garg, H., Sirbiladze, G., Ali, Z., Mahmood, T. Hamy Mean Operators Based on Complex q-Rung Orthopair Fuzzy Setting and Their Application in Multi-Attribute Decision Making. *Mathematics*, (2021), **9**, (18) 2312.

12. Sirbiladze, G. Associated Probabilities in Interactive MADM under Discrimination q-Rung Picture Linguistic Environment. *Mathematics*, **9**, (18) (2021), 2337.

13. Sirbiladze, G., Garg, H., Ghvaberidze, B., Matsaberidze, B., Khutsishvili, I., Midodashvili, B. Choquet Integral Based Possibilistic Approach in Multi-Objective Vehicle Routing Problem under Extreme Environment. Artificial Intelligent Review, (2022).

14. Sirbiladze, G., Garg, H., Khutsishvili, I., Ghvaberidze B., Midodashvili, B. Associated Probabilities Aggregations in Multistage Investment Decision-Making. *Kybernetes*, (2022).

15. Kacprzyk, J., Sirbiladze, G., Tsulaia, G. Associated fuzzy probabilities in MADM with interacting attributes. application in multi-objective facility location selection problem. *International Journal of Information Technology and Decision Making*, **1**, 4 (2022), 1155–1188.

16. Yu, D., Liao, H. Visualization and quantitative research on intuitionistic fuzzy studies. *J Intell Fuzzy Syst.*, **30**, 6 (2016), 3653–3663.

17. Ali, M.I. Another view on q-Rung orthopair fuzzy sets. *Int J Intell Syst.*, **33**, 11 (2018), 2139–2153.

18. Wu, H.-C. Duality in Fuzzy Sets and Dual Arithmetics of Fuzzy Sets. *Mathematics*, **7**, 11 (2019).

19. Shannon, C., A mathematical theory of communication. *Bell Syst., Tech. J.*, **27** (1948), 379–423.

20. Klir, G.J. and Yuan, B., *Fuzzy sets and fuzzy logic: theory and applications.* Prentice-Hall, Inc., 574pp., 1994.

21. Sirbiladze, G., Manjafarashvili,T., Midodashvili, B., Ghvaberidze,B., Mikadze, D., Representation of a crisp set as a pair of dual fuzzy sets. *Advances in Artificial Intelligence and Machine Learning Research*, **2**, 4 (2022), 477–500.

22. Luca, D.A., and Termini, S., A definition of non-probabilistic entropy in the setting of fuzzy set theory. *Information and Control*, **20** (1972), 301–312.

23. Criado, F. and Gachechiladze, T., Entropy of fuzzy events. *Fuzzy sets and systems*, **88** (1997), 99–106.

24. Herencia, J. and Lamata, L., Entropy measure associated with fuzzy basic probability assignment. *In IEEE int. Conf. On fuzzy systems*, **2** (1997), 863–868.

25. Reecan, B. and Markechova, D., The entropy of fuzzy dynamical systems, general scheme and generator. *fizzy sets and systems*, **96** (1998), 191–199.

26. Zadeh, L.A., Probability measures of fuzzy events. *J. Math. Anal. Appl.*, **23** (1968), 421–427.

27. Yager, R., On the measure of fuzziness and negation. part i: Membership in the unit interval. *International Journal on General systems*, **5** (1979), 221–229.

28. Yager, R., On the measure of fuzziness and negation. part ii: Lattice. *Information and control*, **44** (1980), 236–260.

29. Kaufmann, A. and Gupta, M., *Introduction to fuzzy arithmetic: theory and applications.* New York, Van Nostrand Reinhold Company, 1985.

30. Aczel, J., *Lectures on functional equations and their applications.* Acad. Press, N.Y., 509 pp., 1966.

31. Jelinek, F., *Probabilistic Information Theory: Discrete and Memoryless Models.* McGraw-Hill, 448 pp., 1968.

32. Yager, R., Inf. And Control, On the Measure of Fuzziness and Negation. *II. Lattices, Information and Control*, **44** (1980), 236–260.