ON THE TEST RESULTS OF THE METHOD OF SOLUTION OF THE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION OF A DYNAMIC BEAM

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Abstract

We consider the initial boundary value problem for a nonlinear integro-differential equation that describes the vibration of a beam. We constructed an algorithm to numerically solve the problem. We solved test example to demonstrate the effectiveness of constructed algorithm. the results are presented in the form of tables and graphs.

Keywords and phrases: J. Ball equation, Galerkin method, difference scheme, iteration method

AMS subject classification (2010): 35L20, 65H10, 65M60.

1 Introduction

Consider the initial boundary value problem

$$\begin{aligned} &u_{tt}(x,t) + \delta u_t(x,t) + \gamma u_{xxxxt}(x,t) + \alpha u_{xxxx}(x,t) \\ &\quad - \left(\beta + \rho \int_0^L u_x^2(x,t)dx\right) u_{xx}(x,t) \\ &\quad - \sigma \left(\int_0^L u_x(x,t) u_{xt}(x,t)dx\right) u_{xx}(x,t) = f(x,t), \\ &\quad 0 < x < L,\ 0 < t \le T, \end{aligned} \tag{1}$$

with the initial boundary conditions

$$\begin{aligned} &u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \\ &u(0,t) = u(L,t) = 0, \quad u_{xx}(0,t) = u_{xx}(L,t) = 0. \end{aligned} \tag{2}$$

here $\alpha, \beta, \gamma, \delta, \sigma$ and ρ are given constants, where α, γ, σ and ρ are positive numbers. Additionally, $u^0(x) \in W_2^2(0, L)$ and $u^1(x) \in L_2(0, L)$ are given functions that satisfy the conditions

$$u^0(0) = u^0(L) = u^1(0) = u^1(L) = 0.$$

Furthermore, if $\delta < 0$, then $|\delta| < \gamma \left(\frac{\pi}{L}\right)^4$, and if $\beta < 0$, then $\alpha \left(\frac{\pi}{L}\right)^2 > |\beta|$.

The equation (1) was derived by J. Ball [1] using the Timoshenko theory, and it describes the vibration of a beam.

To numerically solve the homogeneous equation corresponding to problem (1), (2), an algorithm was proposed in [2]. In [3], an algorithm for the numerical solution of this problem is proposed, which, like the algorithm built in [2], consists of three components. An implicit symmetric difference scheme is used as the second component. The algorithm proposed in [3] is generalized in [4] for problem (1), (2). A test example is solved. Results are presented as tables for exact and approximate solutions. The algorithm proposed in [4] is used in [5] to solve five test examples. The results are presented in the form of an error table. In [6], using the algorithm developed in [2], the homogeneous equation corresponding to equation (1) with initial boundary conditions for an iron beam is solved. Calculations are performed for three different values of the parameter σ. The results are presented in the form of tables.

We have generalized the algorithm proposed in [2] for the problem (1), (2). Below, we briefly describe the algorithm and demonstrate its effectiveness using a test example.

2 Algorithm

2.1 Galerkin method

The solution of problem (1), (2) is sought in the form of a finite sum

$$u_n(x, t) = \sum_{i=1}^{n} u_{ni}(t) \sin \frac{i\pi x}{L} .$$

The coefficients u_{ni} are found using the Galerkin method from the system of ordinary differential equations

$$\begin{aligned} u''_{ni}(t) &+ \left(\delta + \gamma\left(\frac{i\pi}{L}\right)^4\right) u'_{ni}(t) \\ &+ \left[\alpha\left(\frac{i\pi}{L}\right)^4 + \left(\frac{i\pi}{L}\right)^2 \left(\beta + \rho\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}^2(t)\right.\right. \\ &+ \left.\left. \sigma\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}(t)u'_{nj}(t)\right)\right] u_{ni}(t) = f_i(t), \\ & i = 1,2,\ldots,n, \end{aligned} \tag{3}$$

with the initial conditions

$$u_{ni}(0) = a_i^0, \quad u'_{ni}(0) = a_i^1, \quad i = 1,2,\ldots,n,$$

where

$$a_i^p = \frac{2}{L}\int_0^L u^p(x)\sin\frac{i\pi x}{L}dx, \quad f_i(t) = \frac{2}{L}\int_0^L f(x,t)\sin\frac{i\pi x}{L}dx,$$

$$p = 0,1, \quad i = 1,2,\ldots,n.$$

2.2 Difference scheme

let's introduce the notation

$$y_{ni}(t) = u'_{ni}(t), \quad z_{ni}(t) = \frac{i\pi}{L}u_{ni}(t), \quad i = 1,2,\ldots,n.$$

In this notation, the system (3) takes the following form

$$\begin{aligned} y'_{ni}(t) &+ \left(\delta + \gamma\left(\frac{i\pi}{L}\right)^4\right) y_{ni}(t) \\ &+ \left[\alpha\left(\frac{i\pi}{L}\right)^3 + \frac{i\pi}{L}\left(\beta + \rho\frac{L}{2}\sum_{j=1}^{n} z_{nj}^2(t)\right.\right. \\ &+ \left.\left. \sigma\frac{L}{2}\sum_{j=1}^{n}\frac{j\pi}{L} y_{nj}(t)z_{nj}(t)\right)\right] z_{ni}(t) = f_i(t), \\ z'_{ni}(t) &= \frac{i\pi}{L}y_{ni}(t), \quad i = 1,2,\ldots,n, \end{aligned} \tag{4}$$

$$y_{ni}(0) = a_i^1, \quad z_{ni}(0) = \frac{i\pi}{L} a_i^0, \quad i = 1, 2, \ldots, n. \tag{5}$$

Problem (4), (5) is solved using the difference method. On the time interval $[0, T]$ is introduced grid with step $\tau = \frac{T}{M}$ and nodes $t_m = m\tau$, $m = 0, 1, \ldots, M$. On the m-th layer the approximate values $y_{ni}(t)$ and $z_{ni}(t)$ are denoted by y_{ni}^m and z_{ni}^m.

Then is used a Crank-Nicolson type scheme

$$\begin{gathered}
\frac{y_{ni}^m - y_{ni}^{m-1}}{\tau} + \left(\delta + \gamma \left(\frac{i\pi}{L}\right)^4\right) \frac{y_{ni}^m + y_{ni}^{m-1}}{2} \\
+ \left[\alpha \left(\frac{i\pi}{L}\right)^3 + \frac{i\pi}{L} \left(\beta + \rho \frac{L}{2} \sum_{j=1}^{n} \frac{(z_{nj}^m)^2 + (z_{nj}^{m-1})^2}{2}\right.\right. \\
\left.\left. + \sigma \frac{L}{2} \sum_{j=1}^{n} \frac{j\pi}{L} \frac{(y_{nj}^m + y_{nj}^{m-1})(z_{nj}^m + z_{nj}^{m-1})}{4}\right)\right] \frac{z_{ni}^m + z_{ni}^{m-1}}{2} = f_i^{m-\frac{1}{2}}, \\
\frac{z_{ni}^m - z_{ni}^{m-1}}{\tau} = \frac{i\pi}{L} \frac{y_{ni}^m + y_{ni}^{m-1}}{2}, \\
m = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, n,
\end{gathered} \tag{6}$$

$$y_{ni}(0) = a_i^1, \quad z_{ni}(0) = \frac{i\pi}{L} a_i^0, \quad i = 1, 2, \ldots, n. \tag{7}$$

Here

$$f_i^{m-\frac{1}{2}} = \frac{1}{2}(f_i(t_m) + f_i(t_{m-1})).$$

2.3 Iteration method

System (6), (7) is solved layer-by-layer. For fixed m the counting is carried out by the formula

$$\begin{gathered}
\frac{y_{ni,k+1}^m - y_{ni}^{m-1}}{\tau} + \left(\delta + \gamma \left(\frac{i\pi}{L}\right)^4\right) \frac{y_{ni,k+1}^m + y_{ni}^{m-1}}{2} + \left[\alpha \left(\frac{i\pi}{L}\right)^3\right. \\
+ \frac{i\pi}{L} \left(\beta + \rho \frac{L}{2} \frac{(z_{ni,k+1}^m)^2 + (z_{ni}^{m-1})^2}{2} + \rho \frac{L}{2} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(z_{nj,k}^m)^2 + (z_{nj}^{m-1})^2}{2}\right. \\
+ \sigma \frac{L}{2} \frac{i\pi}{L} \frac{(y_{ni,k+1}^m + y_{ni}^{m-1})(z_{ni,k+1}^m + z_{ni}^{m-1})}{4} \\
\left.\left. + \sigma \frac{L}{2} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{j\pi}{L} \frac{(y_{nj,k}^m + y_{nj}^{m-1})(z_{nj,k}^m + z_{nj}^{m-1})}{4}\right)\right] \frac{z_{ni,k+1}^m + z_{ni}^{m-1}}{2} = f_i^{m-\frac{1}{2}},
\end{gathered} \tag{8}$$

$$\frac{z^m_{ni,k+1} - z^{m-1}_{ni}}{\tau} = \frac{i\pi}{L}\frac{y^m_{ni,k+1} + y^{m-1}_{ni}}{2},$$
$$m = 1, 2, \dots, M, \quad i = 1, 2, \dots, n, \tag{9}$$

where $y^m_{ni,k+p}$ and $z^m_{ni,k+p}$ are respectively the $(k+p)$-th iteration approximations for y^m_{ni} and z^m_{ni}, $i = 1, 2, \dots, n$, $p = 0, 1$. y^{m-1}_{ni} and z^{m-1}_{ni}, $i = 1, 2, \dots, n$, are the known values and

$$y^0_{ni} = a^1_i, \quad z^0_{ni} = \frac{j\pi}{L}a^0_i, \quad i = 1, 2, \dots, n.$$

From (9) we have

$$y^m_{ni,k+1} = -y^{m-1}_{ni} + 2\frac{L}{i\pi}\frac{z^m_{ni,k+1} - z^{m-1}_{ni}}{\tau}. \tag{10}$$

Substituting this expression for $y^m_{ni,k+1}$ in (8) we obtain

$$\begin{aligned}
&\frac{1}{\tau}\left(-y^{m-1}_{ni} + 2\frac{L}{i\pi}\frac{z^m_{ni,k+1} - z^{m-1}_{ni}}{\tau}\right) - \frac{y^{m-1}_{ni}}{\tau} \\
&+ \left(\delta + \gamma\left(\frac{i\pi}{L}\right)^4\right)\frac{L}{i\pi}\frac{z^m_{ni,k+1} - z^{m-1}_{ni}}{\tau} + \left[\alpha\left(\frac{i\pi}{L}\right)^3\right. \\
&+ \frac{i\pi}{L}\left(\beta + \rho\frac{L}{2}\frac{(z^m_{ni,k+1})^2 + (z^{m-1}_{ni})^2}{2} + \rho\frac{L}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(z^m_{nj,k})^2 + (z^{m-1}_{nj})^2}{2}\right. \\
&+ \sigma\frac{L}{4}\frac{(z^m_{ni,k+1} - z^{m-1}_{ni})(z^m_{ni,k+1} + z^{m-1}_{ni})}{\tau} \\
&\left.\left. + \sigma\frac{L}{4}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(z^m_{nj,k} - z^{m-1}_{nj})(z^m_{nj,k} + z^{m-1}_{nj})}{\tau}\right)\right]\frac{z^m_{ni,k+1} + z^{m-1}_{ni}}{2} = f^{m-\frac{1}{2}}_i,
\end{aligned} \tag{11}$$

From (11) after some transformation we get cubic equations with respect to $z^m_{ni,k+1}$ for each i. By solving these equations using the Cardano's formula we obtain

$$z^m_{ni,k+1} = -\frac{z^{m-1}_{ni}}{3} + \sum_{p=1}^{2}(-1)^{p+1}\sigma_{i,p}$$
$$k = 0, 1, \dots, \; i = 1, 2, \dots, n, \tag{12}$$

where

$$\sigma_{i,p} = \left((-1)^p\frac{s_i}{2} + \left(\frac{s_i^2}{4} + \frac{r_i^3}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} \tag{13}$$

and

$$r_i = \frac{8}{L\left(\rho + \frac{\sigma}{\tau}\right)} \left[\left(\frac{2}{\tau} + \delta + \gamma \left(\frac{i\pi}{L} \right)^4 \right) \left(\frac{L}{i\pi} \right)^2 \frac{1}{\tau} + \frac{1}{2}\alpha \left(\frac{i\pi}{L} \right)^2 + \frac{1}{2}\beta \right] + \sum_{\substack{j=1 \\ j \neq i}}^{n} (z_{nj,k}^m)^2 - \frac{1}{3}(z_{ni}^{m-1})^2 + \frac{\rho - \frac{\sigma}{\tau}}{\rho + \frac{\sigma}{\tau}} \sum_{j=1}^{n} (z_{nj}^{m-1})^2, \tag{14}$$

$$s_i = \frac{2(z_{ni}^{m-1})^3}{27} - \frac{16 y_{ni}^{m-1}}{i\pi(\sigma + \tau\rho)} + \frac{8 z_{ni}^{m-1}}{3L\left(\rho + \frac{\sigma}{\tau}\right)} \left[-8 \left(\frac{L}{i\pi} \right)^2 \frac{1}{\tau^2} - 4 \left(\delta + \gamma \left(\frac{i\pi}{L} \right)^4 \right) \left(\frac{L}{i\pi} \right)^2 \frac{1}{\tau} + \alpha \left(\frac{i\pi}{L} \right)^2 + \beta \right] + \frac{2}{3}(z_{ni}^{m-1})^2 \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} (z_{nj,k}^m)^2 + \frac{\rho - \frac{\sigma}{\tau}}{\rho + \frac{\sigma}{\tau}} \sum_{j=1}^{n} (z_{nj}^{m-1})^2 \right) - \frac{8}{i\pi\left(\rho + \frac{\sigma}{\tau}\right)} f_i^{m-\frac{1}{2}}. \tag{15}$$

After finding the coefficients $z_{ni,k+1}^m$ using formulas (12)-(15) and (10), the approximate solution of problem (1), (2) for $t = t_m$ is written as the sum

$$u_{n,k}^m(x) = \sum_{i=1}^{n} \frac{L}{i\pi} z_{ni,k}^m \sin \frac{i\pi x}{L}.$$

3 Test Example

Let $L = 1, T = 1, \alpha = 1.0, \beta = -1.0, \gamma = 1.0, \delta = -1.0, \rho = 1.0, \sigma = 1.0$ and

$$f(x,t) = -(3x^5 - 5x^4 + 2x)(\delta \sin t + \cos t) - 120(3x - 1)(\gamma \sin t - \alpha \cos t) - 60x^2(x - 1) \left[\beta + \frac{22}{7}(\rho \cos t - \sigma \sin t) \cos t \right] \cos t;$$

$$u^0(x) = 3x^5 - 5x^4 + 2x, \qquad u^1(x) = 0.$$

The exact solution is the function $u(x) = (3x^5 - 5x^4 + 2x)\cos t$. The algorithm is applied with $n = 3$ and $M = 5$. The algorithm error for $t = t_m$ is calculated using the formula

$$e_{n,k}^m = \|u(x, t_m) - u_{n,k}^m(x)\|_{L^2(0,1)}. \tag{16}$$

The calculations were performed using GNU Octave software. The results are presented below in tables and graphs.

Table 1: Iterations number

t_m	0.4	0.6	0.8	1.0
k	6	6	6	6

Table 2: Values of the algorithm error

t_m	0.4	0.6	0.8	1.0
$e^m_{n,k}$	0.0015555	0.0015291	0.0015921	0.0017111

Table 1 contains the numbers of iterations on each layer $t = t_m$.

Table 2 contains the values of the algorithm error for $t = t_m$, calculated using formula (16).

Table 3 contains the values of the exact solution of problem (1), (2) at chosen points.

Table 3: Values of the exact solution

tx	0.0	0.25	0.5	0.75	1.0
0.0	0.0	0.483398438	0.781250000	0.629882813	0.0
0.4	0.0	0.473762652	0.765677014	1.178686040	0.0
0.6	0.0	0.445239445	0.719578902	0.580160489	0.0
0.8	0.0	0.398965947	0.644793449	0.519864718	0.0
1.0	0.0	0.261181290	0.422111177	0.340327136	0.0

Table 4 contains the values of the approximate solution of problem (1), (2) at chosen points.

Table 4: Values of the approximate solution

tx	0.0	0.25	0.5	0.75	1.0
0.0	0.0	0.483315329	0.781035854	0.630364693	0.0
0.4	0.0	0.473761274	0.765588763	0.617891951	0.0
0.6	0.0	0.445473066	0.719852146	0.580961999	0.0
0.8	0.0	0.399544655	0.645598300	0.521008456	0.0
1.0	0.0	0.262489255	0.424054209	0.342157753	0.0

Fig. 1 presents the graphs of exact and approximate solutions on layer $t = 0.4$.

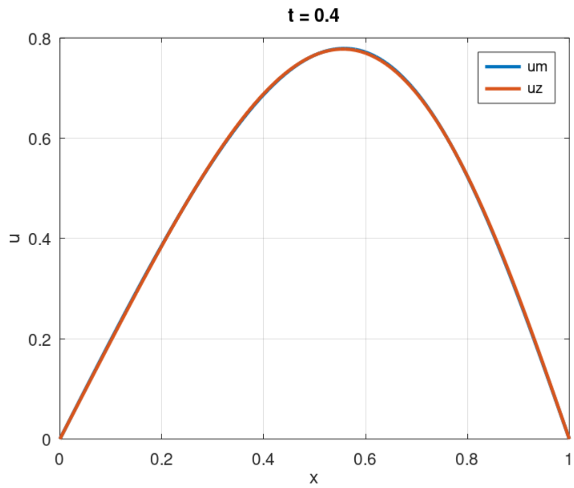

Figure 1: Graphs of exact and approximate solutions

Fig. 2 presents the graphs of exact and approximate solutions on layer $t = 1.0$.

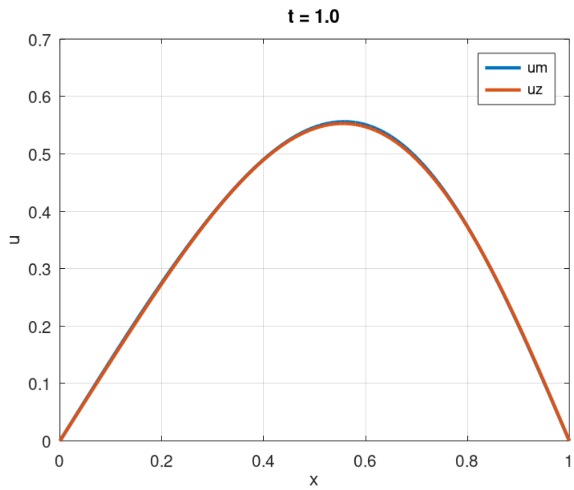

Figure 2: Graphs of exact and approximate solutions

Fig. **??** presents the graph of approximate solution of problem (1), (2).

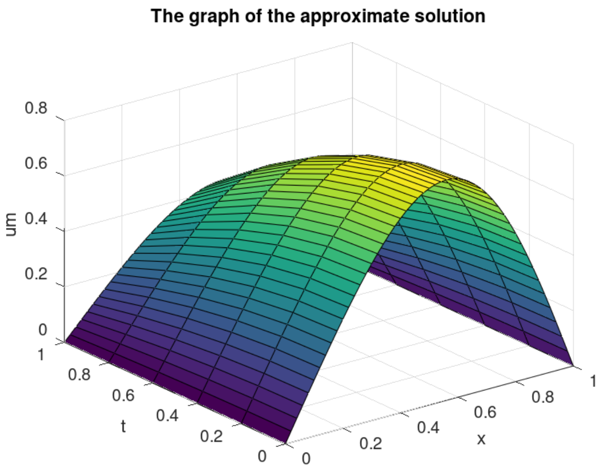

Figure 3: Graph of approximate solution

4 Conclusion

Based on the obtained results, it can be concluded that the numerical algorithm for solving problem (1), (2) is highly effective.

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