SOME PROPERTIES HAAR STATISTICAL STRUCTURES

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Abstract

In this paper we define Haar statistical structures. We prove necessary and sufficient conditions to be weakly separable and strongly separable Haar statistical structures in Banach space of measures.

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Let (E, S) be a measurable space, I the set indices, $\{\mu_i, i \in I\}$ the family of probability measures on S. The following definitions are taken from the works [1, 2, 3, 4].

Definition 1. An object $\{E, S, \mu_i, i \in I\}$ is called statistical structure.

Definition 2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal statistical structure if the family of probability measures $\{\mu_i, i \in I\}$ are pairwise singular measures.

Definition 3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable if there exists a family S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$\mu_i(X_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad \forall\, i, j \in I.$$

Definition 4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is strongly separable, it there exist pairwise disjoint S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$\mu_i(X_i) = 1, \ \ \forall\, i \in I.$$

Let M^σ be real linear space of all alternating finite measures on (E, S).

Definition 5. A linear subset $M_B \subset M^\sigma$ is called a Banach space of measures if:

(1) a norm can be defined on M_B so that with respect to this norm M_B will be a Banach space, and for any orthogonal measures, $\mu, \nu \in M_B$ and any real number $\lambda \neq 0$ the following inequality is fulfilled:

$$\|\mu + \lambda\nu\| \geq \|\mu\|;$$

(2) if $\nu \in M_B$ and $|f(x)| \leq 1$, then

$$\nu_f(t) = \int\limits_A f(x)\nu(dx) \in M_B,$$

where $f(x)$ is a S-measurable real function and $\|\nu_f\| \leq \|\nu\|$;

(3) if $\nu_n \in M_B$, $\nu_n > 0$, $\nu_n(E) < \infty$, $n = 1, 2, \ldots$ and $\nu_n \downarrow 0$, then for any linear functional $\ell^* \in M_B^*$:

$$\lim_{n\to\infty} \ell^*(\nu_n) = 0,$$

where M_B^* conjugate to linear space M_B.

Example. Let $\{\mu_i, i \in A\}$ be pairwise orthogonal Haar probability measures on $\{E, S\}$ and $g_i(x)$ be real S-measurable functions, M_B is the set of measures of the form $\nu(B) = \sum\limits_{i\in A_1} \int\limits_B g_i(x)\mu_i(dx)$, where $A_1 \subset A$ is a countable set form A and

$$\sum_{i\in A_1} \int\limits_E |g_i(x)|\mu_i(sx) < \infty.$$

Let

$$\|\nu\| = \sum_{i\in A_1} \int\limits_E |g_i(x)|\mu_i(dx).$$

Then M_B is Banach space of measures.

Definition 6. Let H be an arbitrary locally compact σ-compact topological group and $B(H)$ σ-algebra of subsets of H. We say that μ measure defined on $B(H)$ is Haar measures if μ is regular measure and

$$\mu(sX) = \mu(X), \ \forall\, s \in H, \ \forall X \in B(H).$$

Definition 7. An object $\{H, B(H), \mu_i, i \in I\}$ is called Haar statistical structure, where $\{\mu_i, i \in I\}$ the family of Haar measures on $(H, B(H))$.

Theorem 1.*[see [3]] Let M_B be a Banach space of measures. Then there exists a family of pairwise orthogonal Haar statistical structure* $\{H, B(H), \mu_i, i \in$

$I\}$ from this space, such that $M_B = \bigoplus_{i\in I} M_B(\mu_i)$, where $M_B(\mu_i)$ is a Banach space of elements ν of the form

$$\nu(B) = \int_B f(x)\mu_i(dx), \quad \int_H |f(x)|\mu_i(dx) < \infty$$

with the norm

$$\|\nu\|_{M_B(\mu_i)} = \int_H |f(x)\mu_i(x)(dx).$$

Theorem 2. *Let*

$$M_B = \bigoplus_{i\in I} M_B(\mu_i).$$

For an orthogonal Haar statistical structure $\{H, B(H), \mu_i, i \in I\}$ to be weakly separable it is necessary and sufficient that the correspondence

$$f \longleftrightarrow \ell_f$$

given by equality

$$\int_H f(x)\nu(dx) = \ell_f(\nu), \ \ \forall\, \nu \in M_B,$$

would be one-to-one. Here $\ell_f(\nu)$ is linear continuous functional on M_B, $f \in F(M_B)$. (Denote by $F(M_B)$ the set of those f, for which $\int_H f(x)\nu(dx)$ is defined for all $\nu \in M_B$.)

Proof. *Sufficiently.* For $f \in F(M_B)$ we define the linear continuous functional ℓ_f by the equality

$$\int_H f(x)\nu(dx) = \ell_f(\nu).$$

Denote by I_f the countable subset I, for which

$$\int_H f(x)\mu_i(dx) = 0 \ \text{ for } \ i \notin I_f.$$

Let us consider the functional ℓ_{f_φ} on $M_B(\mu_i)$, to which there corresponds f_φ. Then for $\psi_1, \psi_2 \in M_B(\mu_i)$:

$$\int_H f_{\psi_1}(x)\psi_2(dx) = \ell_{f_{\psi_1}}(\psi_2) = \int_H f_1(x)f_2(x)\mu_i(dx) = \int_E f_{\psi_1}(x)f_2(x)\mu_i(dx).$$

Therefore $f_{\psi_1}(x) = f_1$ a.e. with respect to the measure μ_i. Let $f_i(x) > 0$ with respect to the measure μ_i and

$$\int\limits_H f_i(x)\mu_i(dx) < \infty, \quad \mu_i^*(C) = \int\limits_C f_i(x)\mu_i(dx),$$

then

$$\int\limits_H f_{\mu_i^*}(x)\mu_j(dx) = \ell_{f_{\mu_i^*}}(\mu_j) = 0, \ \ \forall\, j \neq i.$$

Denote $C_i = \{x: \ f_{\mu_i^*}(x) > 0\}$, then $\int\limits_H f_{\mu_i^*}(x)\mu_j(dx) = \ell_{f_{\mu_i^*}}(\mu_j) = 0, \forall\, j \neq i$. Hence it follows that $\mu_j(C_i) = 0$, $\forall\, j \neq i$.

On the other hand

$$\mu_i^*(H - C_i) = \int\limits_{H-C_i} f_{\mu_i}(x)\mu_i(dx)$$
$$= \int\limits_H f_{\mu_i}(x)I_{(H-C_i)}(x)\mu_i(dx) = \int\limits_H f_{\mu_i^*}(x)I_{(H-C_i)}(x)\mu_i(dx) = 0.$$

Since $f_{\mu_i^*}(x) = f_{\mu_i}(x)$ a.e. with respect to the measure μ_i and $f_{\mu_i^*}(x)I_{(H-C_i)}(x) \equiv 0$. The sufficiency is proved.

Necessary. Since the Haar statistical structure $\{H, B(H), \mu_i, i \in I\}$ is weakly divisible, there exist S-measurable sets C_i such that $\mu_i(H - C_i) = 0$ and $\mu_j(C_i) = 0$, $\forall\, j \neq i$. We put the linear continuous functional ℓ_{C_i} into correspondence to a function $I_{C_i}(x) \in F(M_B)$ by the formula

$$\int\limits_H I_{C_i}(x)\mu_i(dx) = \ell_{C_i}(\mu_i) = \|\mu_i\|_{M_B(\mu_i)}.$$

We put the linear continuous functional $\ell_{f_{\psi_1}}$ into correspondence to the function $f_{\psi_1}(x) = f_1(x)I_{C_i}(x) \in F(M_B)$. Then for any $\psi_i \in M_B(\mu_i)$,

$$\int\limits_H f_{\psi_1}(x)\psi_2(dx) = \int\limits_H f_1(x)I_{C_i}(x)\psi_2(dx)$$
$$= \int\limits_H f(x)f_1(x)I_{C_i}(x)\mu_i(dx) = \ell_{f_{\psi_1}}(\psi_2) = \|\psi_2\|_{M_B(\mu_i)}.$$

Let $\mathcal{E}$ be the collection of extensions of functionals ℓ satisfying the condition $\ell_f \leq p(x)$ on those subspaces where they are defined. Let us introduce, on $\mathcal{E}$, a partial ordering, having assumed $\ell_{f_1} < \ell_{f_2}$, if ℓ_{f_2} is defined on a larger set that ℓ_{f_1}, and $\ell_{f_2}(x) = \ell_{f_1}(x)$ there where both of them are defined, Let $\{\ell_{f_i}\}_{i\in I}$ be a linear ordered subset in $\mathcal{E}$. Let $M_B(\mu_i)$ be the subspace on which ℓ_{f_i} is defined. Define ℓ_f on $\bigcup\limits_{i\in I} M_B(\mu_i)$, having assumed $\ell_f(x) = \ell_{f_i}(x)$ if $x \in M_B(\mu_i)$. It is obvious that $\ell_{f_i} < \ell_f$. Since any linearly ordered subset in $\mathcal{E}$ has an upper bound, by virtue of Chorn's lemma $\mathcal{E}$ contains a maximal element Λ defined on some set X' and satisfying the condition $\Lambda(x) \leq p(x)$ for $x \in X'$. But X' must coincide with the entire space M_B because otherwise we could extend Λ to a wider space by adding, as above, one more dimension. This contradicts the maximality of Λ. Hence $X' = M_B$. Therefore the extension of the functional is defined everywhere.

If we put the linear continuous functional ℓ_f into correspondence to the function

$$f(x) = \sum_{i\in I} g_i(x) I_{C_i}(x) \in F(M_b),$$

than we obtain

$$\int\limits_H f(x)\nu(dx) = \|\nu\| = \sum_{i\in I_0} \|\mu_i\|_{M_B(\mu_i)},$$

where

$$\nu = \sum_{i\in I_0} \int\limits_H g_i(x)\mu_i(dx).$$

Remark 1. From the proven theorem it follows that the above-indicated correspondence puts some function $f \in F(M_B)$ into correspondence to each linear continuous functional ℓ_f. If in $F(M_B)$ we identify the functions coinciding with respect to the measure $\{\mu_i, i \in I\}$, then the correspondence will be bijective.

It is also well known that in the (ZFC) & (CH) theory exists a continual weakly separable structure that is not strongly separable. Here and in the sequel we denote by (MA) Martin's axiom (see [4]).

Theorem 3. *Let*

$$M_B = \bigoplus_{i\in I} M_B(\mu_i),$$

H be total metric space, and $\{\mu_i, i \in I\}$ be the family of pairwise orthogonal Haar Borel probability measures on the space H. Let $Card\, I < 2^{\aleph_0}$. Then in the (ZFC) & (MA) *theory, for an orthogonal Haar Borel statistical structure $\{H, B(H), \mu_i, i \in I\}$ to be strongly separable it is necessary and sufficient that the correspondence*

$$f \longleftrightarrow \ell_f$$

given by the equality

$$\int_H f(x)\nu(dx) = \ell_f(\nu), \ \ \forall\, \nu \in M_B,$$

would be one-to-one and $\ell_f(\nu)$ would be a linear continuous functional. (Denote $F = F((M_H)$ the set real functions for which $\int_H f(x)\nu(dx)$ is defined $\forall\, \nu \in M_B$.)

Proof. The *necessity* is proved in the same manner as the necessity in Theorem 2. We will show the *sufficiently.*

According to Theorem 2, a Haar Borel orthogonal statistical structure $\{H, B(H), \mu_i, i \in I\}$, $Card\, I < 2^{\chi_0}$, is weakly separable. We represent $\{\mu_i, i \in I\}$ as an inductive sequence $\{\mu_i, i < w_\alpha\}$ where W_α denotes the first ordinal number of the power of the set I. Since $\{H, B(H), \mu_i, i \in I\}$ is weakly separable, there exists a family of measurable parts $\{X_i\}_{i<w_\alpha}$ of the space H, such that the following relation is fulfilled:

$$(\forall\, i)(\forall\, j)(i \in [0, w_\alpha)\,\&\, j \in [0, w_\alpha) \Longrightarrow \mu_i(X_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad \forall\, i, j \in I.$$

We define the w_α-sequence of parts of the space H so that the following relations are fulfilled:

(1) $(\forall\, i)(i < w_\alpha \Longrightarrow B_i$ is a Borel subset $H)$;

(2) $(\forall\, i)(i < w_\alpha \Longrightarrow B_i \subset X_i)$;

(3) $(\forall\, i_1)(\forall\, i_2)(i_1 < w_\alpha)\,\&\,(i_2 < w_\alpha)\,\&\,(i_1 \neq i_2) \Longrightarrow B_{i_1} \cap B_{i_2} = \varnothing$;

(4) $(\forall\, i)(i < w_\alpha \Longrightarrow \mu_i(B_i) = 1)$.

Assume that $B_0 = X_0$. Let further the partial sequence $(B_j)_{j<i}$ be already defined for $i < w_\alpha$. It is clear that $\mu^*(\bigcup_{j<i} B_j) = 0$. Thus there exists a Borel subset Y_i of the space H such that the following relations ar valid:

$$\bigcup_{j<i} B_j \subset Y_i \ \text{ and } \ \mu_i(Y_i) = 0.$$

Assume $B_i = X_i - Y_i$.

Thereby the w_α-sequence of $(B_i)_{i<w_\alpha}$-disjunctive measurable subsets of the space E is constructed. Therefore

$$(\forall\, i)\big(i < w_\alpha \Longrightarrow \mu_i(B_i) = 1\big).$$

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