# THE ERROR OF A PROJECTION METHOD FOR A NONLINEAR MODEL OF THE SYMMETRICAL STATE OF THE TIMOSHENKO STATIC SHELL

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**Abstract**

We consider the boundary value problem for a nonlinear system of ordinary differential equations that describes the symmetrical state of a Timoshenko static shell. To get an approximate solution a Green functions and the Galerkin method are used. The error of the projection method is estimated.

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## 1 Introduction

The nonlinear system of Timoshenko shell equations is important from the theoretical and applied standpoints. I. Vorovich [7] attributed the topic of the solvability of the system of Timoshenko equations and construction for it of approximate algorithms to the range of unsolved problems of the mathematical theory of plates and shells. It seems to us that the study of one-dimensional variants of the Timoshenko system will help get a better insight into the nature of nonlinearity inherent in these models and will make it easier to proceed to the investigation of two-dimensional cases.

Using the densifying operator theory and the principle of the construction mapping, the existence of a solution of a Timoshenko some two-dimensional problems are shown in [4] and [5].

## 2 Statement of problem

If in the system of Timoshenko equations for a shell given in [6, p. 42] we preserve the terms with cubic nonlinearity and discard the variables $t$ and $y$, then we obtain a one-dimensional system of equations which characterizes the static symmetrical state of the shell. It has the form

$$\begin{aligned} &N' = 0, \\ &Q' + kN + (Nw')' + f = 0, \\ &M' - Q = 0, \end{aligned} \tag{1}$$

where

$$\begin{aligned} &N = \frac{Eh}{1-\nu^2}\left(u' - kw + \frac{1}{2}\,w'^2\right), \\ &Q = k_0^2\,\frac{Eh}{2(1+\nu)}\,(\psi + w'), \\ &M = D\psi'. \end{aligned} \tag{2}$$

Here the displacements $u = u(x)$, $w = w(x)$ of the shell middle surface and the angle of rotation $\psi = \psi(x)$ of the normal to the shell middle surface are the functions we want to define, $x \in [0,1]$, the shell curvature $k = k(x)$ and the force $f = f(x)$ are the given functions, $x \in (0,1)$ and $E$, $0 < \nu < 0.5$ are respectively Young's modulus and Poisson's ratio, $D$ is the shell flexural rigidity, $k_0$ is the lateral shear coefficient, $h$ is the thickness. Take into account the equality

$$D = \frac{Eh^3}{12(1-\nu^2)}\ .$$

Note that system (1) for $k = 0$ can also be obtained from the system of Timoshenko equations for a plate presented in [2, p. 24].

Using (2) together with the formula for $D$, (1) can be rewritten as a system

$$\begin{aligned} &u'' - (kw)' + \frac{1}{2}\,(w'^2)' = 0, \\ &k_0^2\,\frac{Eh}{2(1+\nu)}\,(\psi' + w'') + k\,\frac{Eh}{1-\nu^2}\,\left(u' - kw + \frac{1}{2}\,w'^2\right) \\ &\qquad + \frac{Eh}{1-\nu^2}\,\left(\left(u' - kw + \frac{1}{2}\,w'^2\right)w'\right)' + f = 0, \\ &\frac{h^2}{6(1-\nu)}\,\psi'' - k_0^2(\psi + w') = 0. \end{aligned} \tag{3}$$

Suppose that the following boundary conditions are fulfilled

$$u(0) = u(1) = 0, \quad w(0) = w(1) = 0, \quad \psi'(0) = \psi'(1) = 0. \tag{4}$$

# 3 Reduction of problem

Using the first and the third equation from (3) and taking into account the respective boundary conditions from (4), the functions $u(x)$ and $\psi(x)$ can be expressed through the function $w(x)$ as follows

$$\begin{aligned} u(x) &= \int_0^1 G_u(x,\xi)\left(-2k(\xi)w(\xi)+w'^2(\xi)\right)d\xi, \\ \psi(x) &= \int_0^1 G_\psi(x,\xi)w'(\xi)\,d\xi, \end{aligned} \tag{5}$$

where the following Green functions are used

$$\begin{aligned} G_u(x,\xi) &= \begin{cases} \frac{1}{2}(x-1), & x>\xi, \\ \frac{1}{2}x, & x<\xi, \end{cases} \\ G_\psi(x,\xi) &= \begin{cases} -\dfrac{\sqrt{\sigma}}{\sinh\sqrt{\sigma}}\cosh\sqrt{\sigma}(x-1)\cosh\sqrt{\sigma}\xi, & x>\xi, \\ -\dfrac{\sqrt{\sigma}}{\sinh\sqrt{\sigma}}\cosh\sqrt{\sigma}\,x\cosh\sqrt{\sigma}(\xi-1), & x<\xi, \end{cases} \\ \sigma &= \frac{6(1-\nu)k_0^2}{h^2}. \end{aligned} \tag{6}$$

Applying (5) and (6), from the second equation of system (3) we obtain the integro-differential equation with respect to $w(x)$

$$\begin{aligned} k_0^2\,\frac{Eh}{2(1+\nu)}\,w''(x) &+ \frac{Eh}{1-\nu^2}\int_0^1\left(-k(\xi)w(\xi)+\frac{1}{2}\,w'^2(\xi)\right)d\xi\,\big(k(x)+w''(x)\big) \\ &- \frac{3(1-\nu)Ek_0^4}{(1+\nu)h\sinh\sqrt{\sigma}}\left(\sinh\sqrt{\sigma}\,(x-1)\int_0^x\cosh\sqrt{\sigma}\,\xi w'(\xi)\,d\xi\right. \\ &\left.+\sinh\sqrt{\sigma}\,x\int_x^1\cosh\sqrt{\sigma}\,(\xi-1)w'(\xi)\,d\xi\right)+f(x)=0, \end{aligned} \tag{7}$$

which we complement with the corresponding boundary condition

$$w(0)=w(1)=0. \tag{8}$$

Thus problem (3), (4) reduces to problem (7), (8) for the function $w(x)$. After solving the latter problem, we construct the functions $u(x)$ and $\psi(x)$ by explicit formulas of form (5).

Now let us consider the question of approximate solution of problem (7), (8) with respect to the argument $x$.

# 4 Assumptions

Assume that for each $i = 1, 2, \dots$ there exist integrals

$$f_i = 4\int_0^1 f(x)\sin i\pi x\,dx, \quad k_i = \int_0^1 k(x)\sin i\pi x\,dx, \tag{9}$$

the series $\sum\limits_{i=1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2$ and $\sum\limits_{i=1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2$ converge.

Let us assume that the following inequality is fulfilled

$$\max(\tau_1,\tau_2)\,\tau_2 \le \frac{1}{2}\,\varepsilon(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(h\pi)^2}}\,, \tag{10}$$

where $\varepsilon$ is some value from the interval $(0,1)$, while $\tau_1$ and $\tau_2$ are defined by the formulas

$$\tau_1 = \left(\frac{8(1-\nu^2)}{Eh}\right)^{\frac{1}{3}}\left(\sum_{i=1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\right)^{\frac{1}{6}}, \quad \tau_2 = \left(\sum_{i=1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac{1}{2}}. \tag{11}$$

Suppose there exists a solution of problem (7), (8) representable as a series

$$w(x) = \sum_{i=1}^{\infty} w_i \sin i\pi x, \tag{12}$$

the coefficients of which satisfy the system of equations

$$2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}\,i\pi w_i + \sum_{j=1}^{\infty}\left(-4k_jw_j + (j\pi w_j)^2\right)\left(-2\,\frac{k_i}{i\pi} + i\pi w_i\right) - \frac{2(1-\nu^2)}{Ehi\pi}\,f_i = 0, \quad i = 1, 2, \dots\,. \tag{13}$$

Note that the $i$-th equation of system (13) is a result of the substitution of (12) into (7) followed by the multiplication of the obtained equation by $\sin i\pi x$ and its integration over $x$ from 0 to 1 and also by the use of the formulas [1]

$$\begin{aligned} &\int_0^1 \sin i\pi x \sin j\pi x\,dx = \begin{cases} 0, & i \ne j,\\ \frac{1}{2}\,, & i = j, \end{cases}\\ &\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\sin bx - b\cos bx),\\ &\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\cos bx + b\sin bx). \end{aligned} \tag{14}$$

## 5 Galerkin method and its error

1. method

Let us write an approximate solution of problem (7), (8) in the form

$$w_n(x) = \sum_{i=1}^{n} w_{ni} \sin i\pi x, \tag{15}$$

where the coefficients $w_{ni}$ are found according to the Galerkin method from the system of nonlinear equations

$$2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}\,i\pi w_{ni} + \sum_{j=1}^{n}\left(-4k_j w_{nj} + (j\pi w_{nj})^2\right)\left(-2\,\frac{k_i}{i\pi} + i\pi w_{ni}\right)$$
$$-\frac{2(1-\nu^2)}{Ehi\pi}\,f_i = 0, \quad i = 1, 2, \ldots, n, \tag{16}$$

which is obtained by means of formulas (14) and can be considered as a system of nonlinear equations with respect to $i\pi w_{ni}$, $i = 1, 2, \ldots, n$ .

2. defining the method error of the function $w(x)$

Let us compare the approximate solution (15) with the $n$-th truncation of the exact solution (12)

$$p_n w(x) = \sum_{i=1}^{n} w_i \sin i\pi x. \tag{17}$$

This means that the approximation error of the function $w(x)$ is defined as a difference

$$\Delta w_n(x) = p_n w(x) - w_n(x). \tag{18}$$

By $\|\ \|$ will be denoted the norm in the space $L_2(0,1)$. We set the task of estimating the norm of $\Delta w_n(x)$ .

Let us expand $\Delta w_n(x)$ into a series. Taking (17) and (15) into account we write

$$\Delta w_n(x) = \sum_{i=1}^{n} \Delta w_{ni} \sin i\pi x, \tag{19}$$

where

$$\Delta w_{ni} = w_i - w_{ni}, \quad i = 1, 2, \ldots, n. \tag{20}$$

(19) implies

$$\left\|\frac{d^l}{dx^l}\,\Delta w_n(x)\right\| \le \left(\frac{1}{2}\sum_{i=1}^{n}(i\pi)^{2l}\Delta w_{ni}^2\right)^{\frac{1}{2}}, \quad l = 0, 1. \tag{21}$$

3. the corresponding system of equations

Equation (13) can be rewriten in the form

$$2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}\,i\pi w_i+\left(\sum_{j=1}^{n}\left(-4k_jw_j+(j\pi w_j)^2\right)\right.$$
$$\left.+\sum_{j=n+1}^{\infty}\left(-4k_jw_j+(j\pi w_j)^2\right)\right)\left(-2\,\frac{k_i}{i\pi}+i\pi w_i\right)-\frac{2(1-\nu^2)}{Ehi\pi}\,f_i=0.$$

Subtracting equation (16) from this relation and taking (20) into consideration together with the equality

$$ab-cd=\frac{1}{2}\left((a-c)(b+d)+(a+c)(b-d)\right),$$

we obtain the equations for the error

$$2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}\,i\pi\Delta w_{ni}+\frac{1}{2}\Bigg(\sum_{j=1}^{n}\Big(-4\,\frac{k_j}{j\pi}+j\pi(w_j+w_{nj})\Big)j\pi\Delta w_{nj}$$
$$\times\Big(-4\,\frac{ki}{i\pi}+i\pi(w_i+w_{ni})\Big)+\sum_{j=1}^{n}\left(-4k_j(w_j+w_{nj})+(j\pi)^2(w_j^2+w_{nj}^2)\right)i\pi\Delta w_{ni}\Bigg)$$
$$+\sum_{j=n+1}^{\infty}\left(-4k_jw_j+(j\pi w_j)^2\right)\left(-2\,\frac{k_i}{i\pi}+i\pi w_i\right)=0,\quad i=1,2,\ldots,n.$$

Multiply it by $i\pi\Delta w_{ni}$ and sum the obtained expression over $i=1,2,\ldots,n$. The result will be as follows

$$2(1-\nu)\sum_{i=1}^{n}\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}(i\pi\Delta w_{ni})^2\le 2\left|\sum_{j=1}^{n}k_j(w_j+w_{nj})\right|\sum_{i=1}^{n}(i\pi\Delta w_{ni})^2$$
$$+\left(4\left(\sum_{j=n+1}^{\infty}\left(\frac{k_j}{j\pi}\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=n+1}^{\infty}(j\pi w_j)^2\right)^{\frac{1}{2}}+\sum_{j=n+1}^{\infty}(j\pi w_j)^2\right)$$
$$\times\left(2\left(\sum_{i=1}^{n}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n}(i\pi w_i)^2\right)^{\frac{1}{2}}\right)\left(\sum_{i=1}^{n}(i\pi\Delta w_{ni})^2\right)^{\frac{1}{2}}.\qquad(22)$$

Let us estimate the right-hand side of inequality (22). For this, we have to obtain some a priori estimates for the coefficients from expansions (12) and (15).

## 4. auxilliary inequalities

Multiplying equation (13) by $i\pi w_i$ and summing the resulting equality over $i = 1, 2, \dots,$ we obtain

**Lemma 1.** *The estimate*

$$\left\| w'(x) \right\| \le \frac{1}{\sqrt{2}} \tau_1 \tag{23}$$

*is valid.*

By analogy with the above reasoning, using equation (16) and manipulating with finite sums, we obtain

**Lemma 2.** *The inequality*

$$\left\| w_n'(x) \right\| \le \frac{1}{\sqrt{2}} \tau_{1n}, \tag{24}$$

*where*

$$\tau_{1n} = \left( \frac{8(1-\nu^2)}{Eh} \right)^{\frac{1}{3}} \left( \sum_{i=1}^{n} \left( \frac{f_i}{i\pi} \right)^2 \right)^{\frac{1}{6}} \le \tau_1, \tag{25}$$

*is fulfilled.*

We multiply equation (13) by $i\pi w_i$ and sum the obtained equality over $i = n+1, n+2, \dots .$ As a result of some transformations we find that **Lemma 3.** *The relation*

$$\begin{aligned} &\left\| (w(x) - p_n w(x))' \right\| \\ &\quad \le \frac{1}{\sqrt{2}} \Bigg( c_1 \Bigg( \sum_{i=n+1}^{\infty} \left( \frac{k_i}{i\pi} \right)^2 \Bigg)^{\frac{1}{2}} + c_2 \Bigg( \sum_{i=n+1}^{\infty} \left( \frac{f_i}{i\pi} \right)^2 \Bigg)^{\frac{1}{2}} \Bigg)^{\frac{1}{3}}, \end{aligned} \tag{26}$$

*where*

$$c_1 = 2\tau_1(\tau_1 + 4\tau_2), \quad c_2 = \frac{2(1-\nu^2)}{Eh}, \tag{27}$$

*holds.*

## 5. estimation of the method error of the function $w(x)$

Taking (22)–(25) and relations (10), (11) into account, we get the inequality

$$\begin{aligned} \sum_{i=1}^{n} (i\pi \Delta w_{ni})^2 \le \tau\sqrt{2} \Bigg( \Bigg( c_0 \sum_{j=n+1}^{\infty} \left( \frac{k_j}{j\pi} \right)^2 \Bigg)^{\frac{1}{2}} \Bigg( \sum_{j=n+1}^{\infty} (j\pi w_j)^2 \Bigg)^{\frac{1}{2}} \\ + \sum_{j=n+1}^{\infty} (j\pi w_j)^2 \Bigg) \Bigg( \sum_{i=1}^{n} (i\pi \Delta w_{ni})^2 \Bigg)^{\frac{1}{2}}, \end{aligned}$$

where

$$c_0 = 16, \quad \tau = \frac{1}{2\sqrt{2}(1-\varepsilon)(1-\nu)}(\tau_1 + 2\tau_2)\left(\frac{1}{k_0^2} + \frac{6(1-\nu)}{(h\pi)^2}\right). \tag{28}$$

If this inequality is used in (21) together with (26), then we obtain the desired estimate

$$\left\|\frac{d^l}{dx^l}\,\Delta w_n(x)\right\| \leq \frac{1}{\pi^{1-l}}\,\tau \sum_{p=0}^{1}\left(c_0 \sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac{1}{2}(1-p)}$$
$$\times\left(c_1\Bigg(\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\Bigg)^{\frac{1}{2}} + c_2\Bigg(\sum_{i=n+1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\Bigg)^{\frac{1}{2}}\right)^{\frac{1}{3}(1+p)}, \quad l = 0,1. \tag{29}$$

The obtained result summarized as follows

**Theorem 1.** *The method error of the function $w(x)$ is estimated by inequality* (29)*, where the constants $c_0$, $c_1$, $c_2$ and $\tau$ are calculated by formulas* (28) *and* (27)*.*

## 6. defining the method errors of the functions $u(x)$ and $\psi(x)$

Let us turn to formulas (5). Using $p_n w(x)$ and $w_n(x)$, we construct the $n$-th truncation of the functions $u(x)$ and $\psi(x)$

$$\begin{aligned} p_n u(x) &= \int_0^1 G_u(x,\xi)\left(-2k(\xi)p_n w(\xi) + (p_n w(\xi))'^2\right)\, d\xi, \\ p_n\psi(x) &= \int_0^1 G_\psi(x,\xi)(p_n w(\xi))' d\xi \end{aligned} \tag{30}$$

and the approximation of these functions

$$\begin{aligned} u_n(x) &= \int_0^1 G_u(x,\xi)\left(-2k(\xi)w_n(\xi) + (w_n'(\xi))^2\right)\, d\xi, \\ \psi_n(x) &= \int_0^1 G_\psi(x,\xi) w_n'(\xi)\, d\xi\,. \end{aligned} \tag{31}$$

By analogy with (18), we define the approximation errors of the functions $u(x)$ and $\psi(x)$ through the differences $\Delta u_n(x) = p_n u(x) - u_n(x)$ and $\Delta\psi_n(x) = p_n\psi(x) - \psi_n(x)$ and estimate the $L_2(0,1)$-norm of either of them. From (30) and (31) we obtain

$$\Delta u_n(x) = p_n u(x) - u_n(x) = \int_0^1 G_u(x,\xi)\Big(-2k(\xi)\Delta w_n(\xi)$$

$$+\left((p_n w(\xi))'\right)^2-(w_n'(\xi))^2\Big)d\xi \quad (32)$$

and

$$\Delta\psi_n(x)=p_n\psi(x)-\psi_n(x)=\int_0^1 G_\psi(x,\xi)\,(\Delta w_n(\xi))'\,d\xi. \quad (33)$$

7. estimations of the method errors of the functions $u(x)$ and $\psi(x)$

From (32) and (6) we get

$$(\Delta u_n(x))^2=\frac{1}{4}\Bigg((x-1)^2\left(\int_0^x H(\xi)\,d\xi\right)^2$$
$$+2x(x-1)\int_0^x H(\xi)\,d\xi\int_x^1 H(\xi)\,d\xi+x^2\left(\int_x^1 H(\xi)\,d\xi\right)^2\Bigg),$$

where

$$H(\xi)=-2k(\xi)\Delta w_n(\xi)+\left((p_n w(\xi))'\right)^2-\left(w_n'(\xi)\right)^2.$$

Consequently,

$$(\Delta u_n(x))^2\le\left(x-\frac{1}{2}\right)^2\left(\int_0^1|H(\xi)|\,d\xi\right)^2. \quad (34)$$

Let us estimate $\int\limits_0^1|H(\xi)|\,d\xi$. We have

$$\int_0^1|H(\xi)|\,d\xi\le 2\|k(x)\|\|\Delta w_n(x)\|+\left\|(\Delta w_n(x))'\right\|$$
$$\left(2\left\|(p_n w(x))'\right\|+\left\|(\Delta w_n(x))'\right\|\right).$$

Applying (15) and (17) with (23), we obtain

$$\int_0^1|H(\xi)|\,d\xi\le\sum_{l=1}^{2}c_{l+2}\left\|(\Delta w_n(x))'\right\|^l,$$

where

$$c_3=\tau_1\sqrt{2}+\frac{2}{\pi}\left(\int_0^1 k^2(x)\,dx\right)^{\frac{1}{2}},\quad c_4=1. \quad (35)$$

Therefore by (34) and (29) we have

$$\|\Delta u_n(x)\|\le\frac{1}{2\sqrt{3}}\sum_{l=1}^{2}c_{l+2}\left(\left(\tau\sum_{p=0}^{1}\left(c_0\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac{1}{2}(1-p)}\right.\right.$$

$$\times\left(c_1\Big(\sum_{i=n+1}^{\infty}\Big(\frac{k_i}{i\pi}\Big)^2\Big)^{\frac{1}{2}}+c_2\Big(\sum_{i=n+1}^{\infty}\Big(\frac{f_i}{i\pi}\Big)^2\Big)^{\frac{1}{2}}\Big)\Big)^{\frac{1}{3}(1+p)}\right)^l. \quad (36)$$

Further, (33) and (6) imply

$$(\Delta\psi_n(x))^2=\frac{\sigma}{\sinh^2\sqrt{\sigma}}\bigg(\cosh\sqrt{\sigma}\,(x-1)\int_0^x\cosh\sqrt{\sigma}\,\xi\,(\Delta w_n(\xi))'\,d\xi$$
$$+\cosh\sqrt{\sigma}\,x\int_x^1\cosh\sqrt{\sigma}\,(\xi-1)(\Delta w_n(\xi))'d\xi\bigg)^2.$$

So,

$$\|\Delta\psi_n(x))\|\le c_5\left\|(\Delta w_n(x))'\right\|, \quad (37)$$

where

$$c_5=\left(\int_0^1F(x)\,dx\right)^{\frac{1}{2}} \quad (38)$$

and

$$F(x)=\frac{2\sigma}{\sinh^2\sqrt{\sigma}}\bigg(\cosh^2\sqrt{\sigma}\,(x-1)\int_0^x\cosh^2\sqrt{\sigma}\,\xi\,d\xi$$
$$+\cosh^2\sqrt{\sigma}\,x\int_x^1\cosh^2\sqrt{\sigma}\,(\xi-1)\,d\xi\bigg).$$

After calculating the integrals we get

$$F(x)=\frac{\sigma}{2\sinh^2\sqrt{\sigma}}\bigg(1+\frac{1}{2\sqrt{\sigma}}\sinh2\sqrt{\sigma}+\cosh2\sqrt{\sigma}x$$
$$+\sinh\sqrt{\sigma}\Big(\frac{1}{\sqrt{\sigma}}\cosh\sqrt{\sigma}\,(2x-1)-2x\sinh\sqrt{\sigma}\,(2x-1)\Big)\bigg).$$

Substituting this expression into (38), once more performing integration and applying in particular the formula

$$\int x\sinh x\,dx=x\cosh x-\sinh x,$$

we find

$$c_5=\frac{1}{\sinh\sqrt{\sigma}}\left(\frac{\sigma}{2}+\frac{\sqrt{\sigma}}{4}\sinh2\sqrt{\sigma}+\sinh^2\sqrt{\sigma}\right)^{\frac{1}{2}}. \quad (39)$$

By (37) and (29) we have

$$\|\Delta\psi_n(x)\|\le c_5\bigg(\tau\sum_{p=0}^{1}\bigg(\Big(c_0\sum_{i=n+1}^{\infty}\Big(\frac{k_i}{i\pi}\Big)^2\Big)^{\frac{1}{2}(1-p)}$$

$$\times\left(c_1\left(\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac{1}{2}}+c_2\left(\sum_{i=n+1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\right)^{\frac{1}{2}}\right)\Bigg)^{\frac{1}{3}(1+p)}\Bigg). \quad (40)$$

The obtained result is formulated as follows

**Theorem 2.** *The method errors of the functions* $u(x)$ *and* $\psi(x)$ *are estimated by inequalities* (36) *and* (40)*, respectively, where the coefficients* $c_i$ $i=0,1,\dots,5$*, and* $\tau$ *defined by formulas* (28)*,* (27)*,* (35) *and* (39)*.*

Imposing some restrictions on the change velocity of the values $f_i$ and $k_i$ from (9) and ignoring, to some extent, the accuracy, the infinite series in relations (11), (29), (36) and (40) can be eliminated. The approach is based on the following well known fact. If, for example, the inequality $|f_i|\le\frac{\omega}{i^m}$, $i=1,2,\dots$ , is fulfilled for numbers $f_i$, where $m=const>0$, $\omega=const>0$, then by virtue of the integral test for convergence the series $\sum\limits_{i=1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\le\left(1+\frac{1}{2m+1}\right)\frac{\omega^2}{\pi^2}$, $\sum\limits_{i=n+1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\le\frac{1}{(2m+1)n^{2m+1}}\frac{\omega^2}{\pi^2}$ As a result the infinite sums are replaced by the corresponding upper bounds. In an analogous way we can also handle infinite series containing the values $k_i$.

An iteration method for one-dimensional Timoshenko plate system, analogous to the one considered here, is studied in [3].

# 6 Conclusion

We here considered the Galerkin method for an axially symmetrical problem in Timoshenko's nonlinear shell theory. The error estimates (29), (36) and (40) of this method are established.

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