VISCOELASTIC ROD VIBRATION PROBLEM WHEN CONSTITUTIVE RELATIONSHIP CONTAIN FRACTIONAL DERIVATIVE IN CAPUTO SENSE

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Abstract

The equation of motion of a viscoelastic rod of density ρ is considered when the constitutive relationship contains fractional derivatives (in the Caputo sense) of order $\beta,\ 0 < \beta < 1$. Analytical form received the solution of the task. .

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1 Introduction

Let us Consider the following equations of motion of a homogeneous viscoelastic, infinite (in both directions) rod of density ρ, which performs longitudinal oscillations under the actions of external load $f(t,x)$ (expressed in terms of units of volume)

$$\frac{\partial\sigma(t,x)}{\partial x} + f(t,x) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$\varepsilon(t,x) = \frac{\partial u(t,x)}{\partial x}. \tag{2}$$

Here x is the rod point coordinate, t the time, σ the stress, ε the strain, u the displacement of a material element of the rod. We close the system of equations of motions by the stess–strain relation

$$\sigma(t,x) = E\eta^\beta D_C^\beta \varepsilon(t,x),\ 0 < \beta < 1. \tag{3}$$

Here D_C^β is a fractional derivative in the Caputo sense.

Assume that $f(t,x)$ depends only on a spatial variable x

$$f(t,x) = f(x).$$

Differentiating (3) with respect to the variable and taking into account (2), we obtain

$$\frac{\partial\sigma(t,x)}{\partial x} = E\eta^{\beta} D^{\beta} \frac{\partial^2 u}{\partial x^2}. \tag{4}$$

Upon substituting (4) in (1), we have

$$\frac{\partial^2 u)}{\partial t^2} = a^2 D^{\beta} \frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho} f(x), \tag{5}$$

Where $a^2 = \frac{E\eta^{\beta}}{\rho}$. Note that a^2 has a dimension $L^2 T^{-2+\beta}$.

By using tension and compression we can always manage that the coefficient of $D^{\beta}\frac{\partial^2 u}{\partial x^2}$ in equation (5) become equal to one.

In an analogous manner we can get an equation of motion in terms of stress and strain.

2 Preliminaries

The Fractional calculus is a generalization of classical calculus concerned with operations of integration and differentiation of non–integer order. The concept of fractional operators has been introduced almost simultaneously with the Classical ones. The Applications of Fractional calculus are very wide nowadays. It safe to say that almost no discipline of modern engineering and science in general, remains untouched by the tools and techniques of fractional calculus. For example, wide and fruitful applications can be found in rheology, viscoelasticity, acoustic, optics, chemical and statistical physics, electrical and mechanical engineering, bioengineering, etc.

Fractional Calculus is an Extension of ordinary calculus with more 300 years of History.

There are many different form of fractional operators In use today. Environments of them are self popular Riemann – Liouville, Grunvald – Letnikov and Caputo derivatives and Integrals (see [1]).

Definition 1. Suppose that $\beta > 0,\ t > a,\ \beta,\ t,\ a \in R$. The fractional operator

$$D_C^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\beta)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\beta+1-n}}, & n-1 < \beta < n \in N, \\ \dfrac{d^n}{dt^n} f(t), \quad \beta = n \in N \end{cases} \tag{6}$$

Is called the Caputo fractional derivative or Caputo fractional differential operator of order β. This operator is introduced by the Italian mathematician Caputo in 1967 (see [2]).

The fractional derivative Caputo has the following basic properties

1. Let $n-1<\alpha<n$, ??? and $f(t)$ be such that $D_C^{\beta}f(t)$ exist, then

$$D_C^{\alpha}f(t)=I^{n-\alpha}D^{n}f(t).$$

This means that the Caputo fractional operator is equivalent to $(n-\alpha)$ fold integration after n^{th} order differentiation.

2. Let $n-1<\alpha<n$, ??? and $f(t)$ be such that $D_C^{\beta}f(s)$ exist. Then following properties hold for the Caputo operator

$$\lim_{\alpha\to n}D_C^{\alpha}f(t)=f^{(n)}(t),$$

$$\lim_{\alpha\to n-1}D_C^{\alpha}f(t)=f^{(n-1)}(t)-f^{(n-1)}(0).$$

3. Let $n-1<\alpha<n$, ??? and functions $f(t)$ and $g(t)$ be such that both $D_C^{\alpha}f(t)$ and $D_C^{\alpha}g(t)$ exist. The Caputo fractional derivative is a linear operators

$$D_C^{\alpha}(\lambda f(t)+g(t))=\lambda D_C^{\alpha}f(t)+D_C^{\alpha}g(t).$$

4. Let $n-1<\alpha<n$, ??? and $f(t)$ be such that $D_C^{\beta}f(t)$ exist. Then in general

$$D_C^{\alpha}D^{m}f(t)=D_C^{\alpha+m}f(t)\neq D^{m}D_C^{\beta}f(t).$$

The benefit of using the Caputo definitions is that it not only allows for the consideration of easily interpreted initial conditions, but it is also bounded, meaning that the derivative of a constant is equal to 0. When applying a fractional Caputo derivative, need standard initial conditions in terms of derivatives of integer order. These initial conditions have clear physical interpretation as an initial position $y(a)$ a point a, the initial velocity $y'(a)$, initial acceleration $y''(a)$ and so on.

In formula (6) Γ is Euler's gamma function. For complex argument with positive real part it is defined as

$$\Gamma(z)=\int_0^{\infty}e^{-x}x^{z-1}dx,\ \mathrm{Re}z>0. \tag{7}$$

By analytic continuation the function is expanded to whole complex plane except for the points $\{0,-1,-2,-3,\cdots\}$ where it has simple poles.

While the Gamma function is a generalization of factorial function, the Mittag–Leffler function is a generalization of exponential function as a one function by the series

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \alpha>0,\ \alpha,\beta\in \tag{8}$$

Later the two parameter generalization introduced by Agarwal

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \beta > 0, \ \alpha, \beta \in \tag{9}$$

The Laplace transform is a powerful means for solving differential equations.

Let $f(t)$ be a function of a variable t such that the function $e^{-st}f(t)$ is integrable in $[0, \infty)$ for some domain of values of s. Then Laplace transform of the function $f(t)$ is defined for above domain values of s and it is denoted by

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

The Laplace transform of function $f(t) = t^\alpha$ is given for α as non–integer order $n - 1 < \alpha \leq n$

$$f(t) = t^\alpha L\{t^\alpha\} = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}.$$

The laplace transform of Caputo fractional differential operator of order α is given by

$$L\{D_C^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

where

$$F(s) = L\{f(t)\},$$

which can also obtain in the form

$$L\{D_C^\alpha f(t)\} = \frac{s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}{s^{n-\alpha}}. \tag{10}$$

Let $\alpha, \ \beta, \ \lambda \in, \ \alpha, \beta > 0, \ m \in$. Then the Laplace transform of two–parameter function of Mittag–Leffler type is given by

$$L\{t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}(\pm \lambda t^\alpha)\} = \frac{m! s^{\alpha-\beta}}{(s^\alpha \mp \lambda)^{m+1}}, \ \ \mathrm{Re}s > |\lambda|^{\frac{1}{\alpha}}. \tag{11}$$

3 The Main Problem

Since we now consider the rod which is infinite in both direction, it is natural to seek for a solution in of (5) in the domain

$$\Omega = \{(t, x) : \ t > 0, \ -\infty < x < \infty\}.$$

Let boundary and initial conditions have the form

$$u(t,\infty) = u(t,-\infty) = 0,\ t > 0, \tag{12}$$

$$u(0+,x) = \frac{u(0+,x)}{\partial t} = 0,\ -\infty < x < \infty. \tag{13}$$

We observe that by virtue of (13) and (14) longitudinal oscilations of rod particles arise under the action of external force which is assumed to depend only on the spatial variable x.

So, we are to solve the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = D^\beta \dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{\rho} f(x),\ -\infty < x < \infty; \\ u(t,\infty) = u(t,-\infty) = 0,\ t > 0; \\ u(0+,x) = \dfrac{u(0+,x)}{\partial t} = 0,\ -\infty < x < \infty. \end{cases} \tag{14}$$

Applying the Fourier transform to problem (15) with respect to the variable x, we obtain

$$\begin{cases} \dfrac{\partial^2 \bar{u}(t,\omega)}{\partial t^2} + \omega^2 D^\beta \bar{u}(t,\omega) = \dfrac{1}{\rho}\bar{f}(\omega),\ t > 0; \\ \bar{u}(0+,\omega) = \dfrac{u(0+,x)}{\partial t} = 0, \end{cases} \tag{15}$$

where

$$\bar{u}(t,\omega) = \int_{-\infty}^{\infty} u(t,x) e^{i\omega x} dx. \tag{16}$$

On formula (11)

$$L\{D_C^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

where

$$F(s) = L\{f(t)\}.$$

Applying the Laplace transformation to problem (16) with respect to the variable t, we get

$$s^2 \bar{U}(s,\omega) + s^\beta \bar{U}(s,\omega) = \frac{\bar{f}(\omega)}{\rho} s^{-1} \Rightarrow$$

$$\left(s^{2-\beta} + \omega^2\right) \bar{U}(s,\omega) = \frac{\bar{f}(\omega)}{\rho} s^{-1-\beta} \Rightarrow$$

$$\bar{U}(s,\omega) = \frac{\bar{f}(\omega)}{\rho} \frac{s^{-1-\beta}}{s^{2-\beta} + \omega^2}. \tag{17}$$

If (18) is compared to (12), note that $m = 0,\ \alpha = 2 - \beta,\ \beta = 3$.

From (18) we will receive

$$\bar{u}(t,\omega) = \frac{\bar{f}(\omega)}{\rho} t^2 E_{2-\beta,3}(-\omega^2 t^{2-\beta}). \tag{18}$$

Formula (19) implies that a solution of our problem (15) has the form

$$\begin{cases} u(t,x) = \dfrac{1}{\rho} \displaystyle\int_{-\infty}^{\infty} G(t, x-\xi) f(\xi) d\xi, \\ G(t,x) = \dfrac{1}{\pi} \displaystyle\int_{0}^{\infty} t^2 E_{2-\beta,3}(-\omega^2 t^{2-\beta}) \cos \omega x d\omega. \end{cases} \tag{19}$$

Note that in order to obtain the later formula, we have used the evenness of the function

$$t^2 E_{2-\beta,3}(-\omega^2 t^{2-\beta})$$

with respect to the variable ω.

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