ON THE NONLINEAR THEORY OF NON-SHALLOW SHELLS

T. Meunargia

I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2 University Str., Tbilisi 0186, Georgia
tengizmeunargia37@gmail.com

(Received 12.06.2019; accepted 16.10.2019)

Abstract

I. Vekua constructed several versions of the refined linear theory of thin and shallow shells, containing the regular processes by means of the method of reduction of 3-D problems of elasticity to 2-D ones. By means of I. Vekua's method the system of differential equations for the nonlinear theory of non-shallow shells is obtained. The general solutions of the approximation of Order $N = 0,\ 1,\ 2,\ 3,\ 4$ are obtained.

Keywords and phrases: Non-shallow shells, metric tensor and tensor of curvature midsurface of the shell.

AMS subject classification (2010): 72K25, 74B20.

1 Introduction

In the present paper, by means of Vekua's method, the system of differential equations for the nonlinear theory of non-shallow shells is obtained [1, 2].

By thin and shallow shells I.Vekua means 3-D shell type elastic bodies satisfying the following conditions

$$a_\alpha^\beta - x^3 b_\alpha^\beta \cong \alpha_\alpha^\beta \quad -h \le x^3 = x_3 \le b, \quad \alpha,\beta = 1,2, \tag{*}$$

where a_α^β and b_α^β are mixed components of the metric and curvature tensors of the midsurface of the shell, x^3 is the thickness coordinate and h is the semi-thickness.

The assumption of the type (*) means that the interior geometry of the shell does not vary in thickness and there such kind of shells are usually called non-varying geometry.

In the sequel, under non-shallow shells we wean elastic bodies free from the assumption of the type (*) or, more exactly, the bodies with the conditions

$$a_\alpha^\beta - x_3 b_\alpha^\beta \ne a_\alpha^\beta \Rightarrow |hb\beta_\alpha| \le q < 1.$$

Such kind of shells are called shells with varying-in-thickness geometry [5, 6], or non-shallow shells.

2 Non-Shallow and Shallow Shells

To construct the theory of non-shells is used the coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\vec{R}$ can be represented in the form

$$\vec{R}(x^1,x^2,x^3)=\vec{r}(x^1,x^2)+x^3\vec{n}(x^1,x^2)\ \ (x^3=x_3),$$

where $\vec{r}$ and $\vec{n}$ are respectively the radius-vector and the unit vector of the normal of the surface $S(x^3=0)$ and (x^1,x^2) are the Gaussian parameters of the midsurfaces S.

The covariant and contravariant basis vectors $\vec{R}_i$ and $\vec{R}^i$ of the surfaces $\hat{S}(x^3=\text{const})$, and the corresponding basis vectors $\vec{r}_i$ and $\vec{r}^{\,i}$ of the midsurface $S(x^3=0)$ are connected by the following relations:

$$\begin{gathered}\vec{R}_\alpha=\partial_\alpha\vec{R}=(a_\alpha^\beta-x_3b_\alpha^\beta)\vec{r}_\beta,\\ \vec{R}^\alpha=\frac{a_\beta^\alpha+x_3(b_\beta^\alpha-2Ha_\beta^\alpha)}{1-2Hx_3+Kx_3^2}\vec{r}^\beta,\\ \vec{R}_3=\vec{R}^3=\vec{n},\end{gathered}\tag{1}$$

where $(a_{\alpha\beta},a^{\alpha\beta},a_\alpha^\beta)$ and $(b_{\alpha,\beta,}b^{\alpha\beta},b_\alpha^\beta)$ are the components (covariant, contravariant and mixed) of the metric and curvature tensors of the midsurface S, H and K are, respectively, middle and Gaussian curvature of the surface S,

$$2H=b_\alpha^\alpha=b_1^1+b_2^2,\quad K=b_1^1b_2^2-b_2^1b_1^2.$$

The main quadratic forms of the midsurface S have the forms

$$I=ds^2=a_{\alpha\beta}dx^\alpha dx^\beta,\quad II=K_sds^2=b_{\alpha\beta}dx^\alpha dx^\beta,\tag{2}$$

where k_s is the normal courvative of the S and

$$a_{\alpha\beta}=\vec{r}_\alpha\vec{r}_\beta,\ \ b_{\alpha\beta}=-\vec{n}_\alpha\vec{r}_\beta,\ \ k_s=b_{\alpha\beta}s^\alpha s^\beta,\tag{3}$$

$$\vec{r}_\alpha=\partial_\alpha\vec{r},\ \ \vec{n}_\alpha=\partial_\alpha\vec{n}=-b_\alpha^\beta\vec{r}_\beta,\ \ s^\alpha=\frac{dx^\alpha}{ds}.$$

Note that, sometimes under non-shallow shells be meant the following approximate equalities

$$\vec{R}^\alpha\cong(a_\beta^\alpha+x_3b_\beta^\alpha)\vec{r}^\beta,\quad \text{(Koiter, Haghdi, Lurie)}$$

which are the first approximations of the general case (1).

In reality we have (general case) [4, 5, 6]

$$\vec{R}^\alpha = \frac{a^\alpha_\beta + x_3(b^\alpha_\beta - 2Ha^\alpha_\beta)}{1 - 2Hx_3 + Kx_3^2}\vec{r}^\beta = \frac{a^\alpha_\beta + x_3(b^\alpha_\beta - 2Ha^\alpha_\beta)}{[1-(H+\sqrt{E})x_3][1-(H-\sqrt{E})x_3]}\vec{r}^\beta$$

$$= \{a^\alpha_\beta + x_3 b^\alpha_\beta + x_3^2[(3H^2+E)a^\alpha_\beta + 2H(b^\alpha_\beta - 2Ha^\alpha_\beta)] + \cdots\}\vec{r}^\beta \quad \Rightarrow$$

$$\vec{R}^\alpha \cong (a^\alpha_\beta + x_3 b^\alpha_\beta)\vec{r}^\beta, \tag{4}$$

where $E = H^2 - K$ (Euler's difference).

For shallow shells it may be assumed that

$$\vec{R}^\alpha = \vec{r}^\alpha, \quad \vec{R}_\alpha = \vec{r}_\alpha.$$

The first and second quadratic forms of the surfaces $\hat{S}(x^3 = \text{const})$ are expressed by the formulas

$$I = d\hat{s}^2 = g_{\alpha\beta}dx^\alpha dx^\beta, \;\; II = \hat{K}_{\hat{S}}ds^2 = \hat{b}_{\alpha\beta}dx^\alpha dx^\beta, \tag{5}$$

where

$$\begin{aligned} g_{\alpha\beta} &= \vec{R}_\alpha\vec{R}_\beta = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2(2Hb_{\alpha\beta} - Ka_{\alpha\beta}), \\ \hat{b}_{\alpha\beta} &= (1 - 2Hx_3)b_{\alpha\beta} + x_3 K a_{\alpha\beta} \end{aligned} \tag{6}$$

and $\hat{k}_s = \hat{b}_{\alpha\beta}s^\alpha s^\beta$ is the normal courvature of the $\hat{S}$.

3 System of Geometrically Nonlinear Equations for Non-Shallow Shells

The equations of equilibrium of an elastic shells-type bodies in a vector form may be written as

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\vec{T}^i}{dx^i} + \vec{\Psi} = 0, \quad (i = 1, 2, 3) \tag{7}$$

where g is the discriminant of the metric quadratic form of the 3-D domain, $\vec{T}^i$ are the contravariant constituents of the stress vector, $\vec{\Psi}$ is an external force

$$g = \vartheta^2 a, \quad \vartheta = 1 - 2Hx_3 + Kx_3^2, \quad a = a_{11}a_{22} - a_{12}^2. \tag{8}$$

The stress-strain relations the geometrically nonlinear theory of elasticity has the form

$$\vec{T}^i = T^{ij}(\vec{R}_j + \partial_j\vec{U}) = E^{ijpq}e_{pq}(\vec{R}_j + \partial_j\vec{U}), \tag{9}$$

where T^{ij} are contravariant components of the stress tensor, e_{ij} are covariant components of the strain tensor, $\vec{U}$ is the displacement vector. E^{ijpq} and e_{ij} are defined by the formulas:

$$\begin{aligned} &E^{ijpq} = \lambda g^{ij} g^{\mu q} + \mu(g^{ip} g^{jq} + g^{iq} g^{jp}), \quad (g^{ij} = \vec{R}^i \vec{R}^j), \\ &e_{ij} = \frac{1}{2}(\vec{R}_i \partial_j \vec{U} + \vec{R}_j \partial_i \vec{U} + \partial_i \vec{U} \partial_j \vec{U}) \quad (i,j,p,q = 1,2,3). \end{aligned} \tag{10}$$

To reduce the 3-D problems of the theory of elasticity to 2-D ones, it is necessary to rewrite the relation (7-10) in terms of the bases of the midsurface S of the shell Ω.

Relation (7) can be written as follows:

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a} \vartheta \vec{T}^{\alpha}}{\partial x^{\alpha}} + \frac{\partial \vartheta \vec{T}^3}{\partial x^3} + \vartheta \vec{\Psi} = 0, \quad (\vartheta = 1 - 2Hx_3 + kx_3^2). \tag{11}$$

Let $\vec{\sigma}^i = \vartheta \vec{T}^i$ and $\vec{\Phi} = \vartheta \vec{\Psi}$, then making use of tensor notation the equilibrium equation (11) can be written as follows:

$$\nabla_\alpha \vec{\sigma}^\alpha + \partial_3 \vec{\sigma}^3 + \vec{\Phi} = 0 \ \Rightarrow$$

$$\begin{cases} \nabla_\alpha \sigma^{\alpha\beta} - b_\alpha^\beta \sigma^{\alpha 3} + \partial_3 \sigma^{3\beta} + \Phi^\beta = 0, \\ \nabla_\alpha \sigma^{\alpha 3} - b_{\alpha\beta} \sigma^{\alpha\beta} + \partial_3 \sigma^{33} + \Phi^3 = 0, \end{cases} \tag{12}$$

where v are covariant derivatives to the coordinates (x^1, x^2):

$$\sigma^{\alpha\beta} = \vec{\sigma}^\alpha \vec{r}^\beta, \ \ \sigma^{\alpha 3} = \vec{\sigma}^\alpha \vec{n}, \ \ \sigma^{3\alpha} = \vec{\sigma}^3 \vec{r}^\alpha, \ \ \sigma^{33} = \vec{\sigma}^3 \vec{n}.$$

4 Isometric System of Coordinates

The isometrical system of coordinates in the surface S is of the special interest, since in this system we can obtain bases equations of the theory of shells in a complex form, which in turn, allows one for a rather wide class of problems to construct complex representation of general solutions by means of analytic functions of one variable $z = x^1 + ix^2$. This circumstance makes is possible to apply the methods developed by N. Muskhelishvili and his disciples by means of the theory of functions of a complex variable and integral equations.

The main quadratic forms in this of coordinates are of the type

$$I = ds^2 = \Lambda(x^1, x^2)[(dx^1)^2 + (dx^2)^2] = \Lambda(z, \bar{z}) dz d\bar{z}, \quad (\Lambda > 0)$$

$$II = b_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{2}[\bar{Q} dz^2 + 2H dz d\bar{z} + Q d\bar{z}^2],$$

where

$$Q = \frac{1}{2}(b_1^1 - b_2^2 + 2ib_2^1), \quad 2H = b_1^1 + b_2^2.$$

Introducing the well-known differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)$$

and the notations

$$\vec{\sigma}^i = \vartheta \vec{T}^i, \quad \vec{\Phi} = \vartheta \vec{\Psi}.$$

For the nonlinear theory of non-shallow shells from (12) we obtain the following complex writing both for the system of equations of equilibrium and for stress-strain relation

$$\begin{aligned} &\frac{1}{\Lambda}\left[\partial_z\left(\Lambda\vec{\sigma}^+\vec{r}_+\right) + \partial_{\bar{z}}\left(\Lambda\bar{\vec{\sigma}}^+\vec{r}_+\right)\right] - \Lambda\left[H\vec{\sigma}^+ + Q\bar{\vec{\sigma}}^+\right]\vec{n} \\ &+\partial_3\left(\vec{\sigma}^3\vec{r}_+\right) + \vec{\Phi}\vec{r}_+ = 0, \\ &\frac{1}{\Lambda}\left[\partial_z\left(\Lambda\vec{\sigma}^+\vec{n}\right) + \partial_{\bar{z}}\left(\Lambda\bar{\vec{\sigma}}^+\vec{n}\right)\right] - \mathrm{Re}\left[H\vec{\sigma}^+\bar{\vec{r}}_+ + Q\bar{\bar{\vec{\sigma}}}^+\vec{r}_+\right]\vec{n} \\ &+\partial_3\left(\vec{\sigma}^3\vec{n}\right) + \vec{\Phi}\vec{n} = 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &\vec{\sigma}^+ = \vec{\sigma}^1 + i\vec{\sigma}^2 = \vartheta\left\{\left[\lambda\theta + \mu\left(\vec{R}^+\partial_z\vec{U} + \bar{\vec{R}}^+\partial^z\vec{U} + 2\partial_z\vec{U}\partial^z\vec{U}\right)\right]\right. \\ &\times\left(\vec{R}^+ + 2\partial_{\bar{z}}\vec{U}\right) + \mu\left[\vec{R}^+\partial_3\vec{U} + 2\left(\vec{n} + \partial_3\vec{U}\right)\partial^z\vec{U}\right]\left.\left(\vec{n} + \partial_3\vec{U}\right)\right\}, \\ &\vec{\sigma}^3 = \vartheta\left\{\left[\lambda\theta + 2\mu\left(\vec{n}\partial^3\vec{U} + \frac{1}{2}\left(\partial_3\vec{U}\right)^2\right)\right]\left(\vec{n} + \partial_3\vec{U}\right)\right. \\ &+\mu\left[\frac{1}{2}\vec{R}_+\partial_3\vec{U} + \vec{n}\partial_{\bar{z}}\vec{U} + \partial_z\vec{U}\partial_3\vec{U}\right]\left.\left(\vec{R}^+ + 2\partial_z\vec{U}\right)\right\}, \end{aligned} \tag{14}$$

Here

$$\vec{\sigma}^+\vec{r}_+ = \left(\vec{\sigma}^1 + i\vec{\sigma}^2\right)\left(\vec{r}_1 + i\vec{r}_2\right), \quad \vec{\sigma}^+\vec{n} = \sigma_3^1 + i\sigma_3^2, \quad \vec{\sigma}^3\vec{r}_+ = \sigma_1^3 + i\sigma_2^3$$

$$\vec{R}_+ = \vec{R}_1 + i\vec{R}_2 = (1 - Hx_3)\vec{r}_+ + x_3 Q\bar{\vec{r}}_+,$$

$$\vec{R}^+ = \vec{R}^1 + i\vec{R}^2 = \vartheta^{-1}[(1 - Hx_3)\vec{r}^+ + x_3 Q\bar{\vec{r}}_+]$$

$$\theta = 2\mathrm{Re}\left[\left(\vec{R}^+\partial^{\bar{z}}\vec{U}\right)\partial_z\vec{U}\right] + \partial_3 U_3 + \frac{1}{2}(\partial_3\vec{U})^2,$$

$$2\partial^z\vec{U} = (\vec{R}^+\bar{\vec{R}}^+)\partial_z\vec{U}_+ + (\bar{\vec{R}}^+\vec{R}^+)_{\partial_{\bar{z}}}\vec{U},$$

$$U_+ = \vec{U}\vec{r}_+ = U_1 + iU_2, \quad U^+ = \vec{U}\vec{r}^+ = U^1 + iU^2.$$

$$
\begin{aligned}
&\vec{r}^{+} = \vec{r}^{1} + i\vec{r}^{2}, \quad \vec{r}_{+} = \vec{r}_{1} + i\vec{r}_{2}, \quad \vec{r}^{+}\bar{\vec{r}}^{+} = \frac{2}{\Lambda}, \quad \vec{r}_{+}\bar{\vec{r}}_{+} = 2,\\
&\vec{R}^{+}\vec{R}^{+} = \frac{4x_3}{\Lambda}\frac{\lambda - Hx_3}{\vartheta^2}Q, \quad \vec{R}^{+}\bar{\vec{R}}^{+} = \frac{2}{\Lambda}\frac{\vartheta + 2x_3^2 Q\bar{Q}}{\vartheta^2},\\
&\vec{R}^{+}\vec{r}_{+} = \frac{2Q}{\vartheta}x_3, \quad \bar{\vec{R}}^{+}\vec{r}_{+} = \frac{2}{\vartheta}(1 - Hx_3),\\
&\vec{r}^{+}\partial_z\vec{U} = \frac{1}{\lambda}\partial_z U_{+} - HU_3, \quad \vec{r}^{+}\partial_{\bar{z}}\vec{U} = \partial_{\bar{z}}U^{+} - QU_3,\\
&\vec{n}\partial_{\bar{z}}\vec{U} = \partial_{\bar{z}}U_3 + \frac{1}{2}(\bar{Q}U_{+} + H\bar{U}_{+}),\\
&U_{(l)} = \vec{U}\vec{l}, \quad U_{(s)} = \vec{U}\vec{s}.
\end{aligned}
\tag{15}
$$

5 Vekua's Method of Reduction

There are many different methods of reducing 3-D problems of the theory of elasticity to 2-D one of the theory shells [2, 4].

In the present paper, we realize the reduction by the method suggested by I. Vekua. Since the system of Legender polynomials $\{P_m(\frac{x_3}{h})\}$ is complete in the interval [-h,h] for equation (11) we obtain the equivalent infinite system of 2-D equations

$$
\int_{-h}^{h}\left[\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\vartheta\vec{T}^{\alpha}}{\partial x^{\alpha}} + \frac{\partial\vartheta\vec{T}^{3}}{\partial x^{3}} + \vartheta\vec{\Psi}\right]P_m\left(\frac{x_3}{h}\right)dx_3 = 0,
$$

or in the form

$$
\nabla_\alpha \overset{(m)}{\vec{\sigma}}{}^{\alpha} - \frac{2m+1}{2h}\left(\overset{(m-1)}{\vec{\sigma}_3} + \overset{(m-3)}{\vec{\sigma}_3} + \cdots\right) + \overset{(m)}{\vec{F}} = 0, \tag{16}
$$

where

$$
\left(\overset{(m)}{\vec{\sigma}}{}^{i}, \overset{(m)}{\vec{\Phi}}\right) = \frac{2m+1}{2h}\int_{-h}^{h}(\vartheta\vec{T}^{i}, \vartheta\vec{\Psi})P_m\left(\frac{x_3}{h}\right)dx_3,
$$

$$
\overset{(m)}{\vec{F}} = \overset{(m)}{\vec{\Phi}} + \frac{2m+1}{2h}\left[\overset{(+)}{\vartheta}\overset{(+)}{\vec{T}^{3}} - (-1)^m\overset{(-)}{\vartheta}\overset{(-)}{\vec{T}^{3}}\right],
$$

$$
\overset{(\pm)}{\vartheta} = 1 \pm 2Hh + Kh^2, \quad \overset{(\pm)}{\vec{T}^{3}} = T^3(x^1, x^2, \pm h), \quad -h \le x_3 \le h
$$

and ∇_α are covariant derivatives on the surface $S(x^3 = 0)$.

6 Vekua's Method of Reduction

3-D shell-type bodies are characterized by inequalities of the type [4, 7]

$$|hb^{\alpha}_{\beta}| \leq q < 1, \quad (\alpha, \beta = 1, 2). \tag{17}$$

Therefore, they can be represented in the form

$$|\varepsilon b^{\alpha}_{\beta} R| \leq q < 1, \tag{18}$$

where ε is a small parameter which is expressed in the form

$$\varepsilon = \frac{h}{R} \leq q < 1. \tag{19}$$

Here h is the semi-thickness of the shell and R is a certain characteristic radius of curvature of the midsurface S.

Having introduced a small parameter, we represent the system of equations (13)-(15) of approximation of order in the complex form

$$\begin{aligned}
&\frac{h}{\Lambda}\frac{\partial}{\partial z}\left(\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + i\left(\overset{(m)}{\sigma}_{12} + \overset{(m)}{\sigma}_{21}\right)\right) \\
&+h\frac{\partial}{\partial \bar{z}}\left(\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + i\left(\overset{(m)}{\sigma}_{12} + \overset{(m)}{\sigma}_{21}\right)\right) - \varepsilon\left(H\overset{(m)}{\sigma}{}^{\cdot 3}_{+\cdot} + Q\overset{(m)}{\bar{\sigma}}{}^{3\cdot}_{\cdot +}\right)R \\
&-(2m+1)\left(\overset{(m-1)}{\sigma}{}^{3\cdot}_{\cdot +} + \overset{(m-3)}{\sigma}{}^{3\cdot}_{\cdot +} + \cdots\right) + h\overset{(m)}{F}_{+} = 0, \\
&\frac{h}{\Lambda}\left(\frac{\partial \overset{(m)}{\sigma}{}^{\cdot 3}_{+\cdot}}{\partial z} + \frac{\partial \overset{(m)}{\bar{\sigma}}{}^{\cdot 3}_{+\cdot}}{\partial \bar{z}}\right) \\
&+\varepsilon\left\{H\overset{(m)}{\sigma}{}^{\alpha}_{\alpha} + \mathrm{Re}\left[\bar{Q}\left(\overset{(m)}{\sigma}{}^{1}_{1} - \overset{(m)}{\sigma}{}^{2}_{2} + i\left(\overset{(m)}{\sigma}{}^{1}_{2} + \overset{(m)}{\sigma}{}^{2}_{1}\right)\right)\right]\right\}R \\
&-(2m+1)\left(\overset{(m-1)}{\sigma}{}^{3}_{3} + \overset{(m-3)}{\sigma}{}^{3}_{3} + \cdots\right) + h\overset{(m)}{F}_{3} = 0 \\
&\qquad\qquad (m = 0,\ 1,\ \cdots, N).
\end{aligned} \tag{20}$$

where (now we write only linear part in explicit form)

$$\begin{aligned}
&h\left(\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + i\left(\overset{(m)}{\sigma}_{12} + \overset{(m)}{\sigma}_{21}\right)\right) = h\vec{\sigma}^{\cdot}_{+}\vec{r}_{+} \\
&= 4\mu\Lambda\left(h\partial_{\bar{z}}\overset{(m)}{U}{}^{+} - \varepsilon Q\overset{(m)}{U}_{3}R\right)
\end{aligned}$$

$$+2\Lambda\sum_{s=0}^{N}\left\{\left(\overset{(m,s)}{I_1}-H\overset{(m,s)}{I_2}\right)Q\left[(\lambda+\mu)\left(h\overset{(s)}{\theta}-2H\varepsilon\overset{(s)}{U_3}R\right)\right]\right.$$
$$+2\mu\left(\frac{h}{\Lambda}\partial_{\bar z}\overset{(s)}{U_\alpha}-\varepsilon H\overset{(s)}{U_3}R\right)+\overset{(m,s)}{I_2}Q\left[(\lambda+\mu)\left(h\partial_z\overset{(s)}{\bar U^+}-\varepsilon\bar Q\overset{(s)}{U_3}R\right)\right. \tag{21}$$
$$\left.\left.+(\lambda+3\mu)\bar Q\left(h\partial_z\overset{(s)}{\bar U^+}-\varepsilon Q\overset{(s)}{U_3}R\right)\right]+2\lambda\overset{(m,s)}{I_3}Q\overset{(s)}{U_3'}+\overset{(m)}{L_1}\left(\overset{(s)}{U}\right)\right\},$$

$$h\left(\overset{(m)}{\sigma_1^1}-\overset{(m)}{\sigma_2^2}+i\left(\overset{(m)}{\sigma_2^1}+\overset{(m)}{\sigma_1^2}\right)\right)=h\vec\sigma^{+}_{\cdot}\vec r_+=2(\lambda+\mu)\left(\overset{(m)}{\theta}-2H\varepsilon Q\overset{(m)}{U_3}R\right)$$
$$+2\sum_{s=0}^{N}\left\{\left(\overset{(m,s)}{I_1}-H\overset{(m,s)}{I_2}\right)\left[(\lambda+3\mu)\bar Q(\lambda+\mu)\left(h\partial_{\bar z}\overset{(s)}{U^+}-\varepsilon Q\overset{(s)}{U_3}R\right)\right.\right.$$
$$\left.+(\lambda+\mu)Q\left(h\partial_z\overset{(s)}{\bar U^+}-\varepsilon\bar Q\overset{(s)}{U_3}R\right)\right]+\overset{(m,s)}{I_2}Q\bar Q\left[(\lambda+\mu)\left(h\overset{(s)}{\theta}-2H\varepsilon\overset{(s)}{U_3}R\right)\right.$$
$$\left.\left.+2\mu\left(\frac{h}{\Lambda}\partial_z\overset{(s)}{\bar U_+}-\varepsilon H\overset{(s)}{U_3}R\right)\right]+2\lambda\left(\delta_m^s-H\overset{(m,s)}{I_3}\right)\overset{(s)}{U_3'}+\overset{(m)}{L_2}\left(\overset{(s)}{U}\right)\right\},$$

$$h\vec\sigma^{3}_{\cdot +}\vec r_+=\mu\left[2h\partial_{\bar z}\overset{(m)}{U_3}+\varepsilon\left(H\overset{(m)}{U_+}+Q\overset{(m)}{\bar U_+}\right)R\right]$$
$$+2\mu\sum_{s=0}^{N}\left\{\overset{(m,s)}{I_2}Q\bar Q\left[2h\partial_{\bar z}\overset{(s)}{U_3}+\varepsilon\left(H\overset{(s)}{U_+}+Q\overset{(s)}{\bar U_+}\right)R\right]\right.$$
$$+\left(\overset{(m,s)}{I_1}-H\overset{(m,s)}{I_2}\right)Q\left[2h\partial_z\overset{(s)}{U_3}+\varepsilon\left(H\overset{(s)}{\bar U_+}+\bar Q\overset{(s)}{U_+}\right)R\right]$$
$$\left.+\overset{(m)}{U_3'}\delta_m^s-\left(H\overset{(s)}{U_+}+Q\overset{(s)}{\bar U_+}\right)\overset{(m,s)}{I_3}R+\overset{(m)}{L_3}\left(\overset{(s)}{U}\right)\right\}, \tag{22}$$
$$h\vec\sigma^{3}_{+\cdot}\vec n=\mu\left[2h\partial_{\bar z}\overset{(m)}{U_3}+\varepsilon\left(H\overset{(m)}{\bar U_+}+\bar Q\overset{(m)}{U_+}\right)R\right]$$
$$+2\mu\sum_{s=0}^{N}\left\{\overset{(m,s)}{I_3}Q\left(h\partial_{\bar z}\overset{(s)}{U_3}+\varepsilon\frac{H\overset{(s)}{\bar U_+}+\bar Q\overset{(s)}{U_+}}{2}R\right)\right.$$
$$\left.-H\left(h\partial_z\overset{(s)}{U_3}+\varepsilon\frac{H\overset{(s)}{U_+}+Q\overset{(s)}{\bar U_+}}{2}R\right)+\overset{(m,s)}{I_4}\overset{(s)}{U_+'}+\overset{(m)}{L_4}\left(\overset{(s)}{U}\right)\right\},$$

$$h\sigma_3^3=\lambda\left(h\overset{(m)}{\theta}-2H\varepsilon Q\overset{(m)}{U_3}R\right)+2\sum_{s=0}^{N}\left\{\lambda\left[Q\left(h\partial_z\overset{(s)}{\bar U^+}-\varepsilon Q\overset{(s)}{U_3}R\right)\right.\right.$$

$$
\begin{gathered}
+\bar{Q}\left(h\partial_{\bar{z}}\overset{(s)}{U^{+}}-\varepsilon Q\overset{(s)}{U_3}R\right)-H\left(h\overset{(s)}{\theta}-2H\varepsilon\overset{(s)}{U_3}R\right)\overset{(m,s)}{I_3}\Bigg]\\
+(\lambda+2\mu)\overset{(m,s)}{I_4}\overset{(s)}{U'_{+}}+\overset{(m)}{L_5}\left(\overset{(s)}{U}\right)\Bigg\},\\
(m=0,\ 1,\ \cdots,N).
\end{gathered}
\tag{23}
$$

where $\overset{(m)}{L_i}(\overset{s}{U})$ $(i=1,\cdots,s)$ are the nonlinear parts of relations (21-23).

Then we have (general case of non-shallow shells)

$$
\overset{(m)}{\theta}=\frac{1}{\Lambda}(\partial_z\overset{(m)}{U_{+}}+\partial_{\bar{z}}\overset{(m)}{\bar{U}_{+}})+\overset{(m)}{U'_3},\qquad \overset{(m)}{U'_i}=(2m+1)(\overset{(m+1)}{U_i}+\overset{(m+3)}{U_i}+\cdots),
$$

$$
\begin{aligned}
\overset{(m,s)}{I_1}&=\frac{2m+1}{2h}\int_{-h}^{h}\frac{x_3P_mP_sdx_3}{1-2Hx_3+Kx_3^2},\\
\overset{(m,s)}{I_2}&=\frac{2m+1}{2h}\int_{-h}^{h}\frac{x_3^2P_mP_sdx_3}{1-2Hx_3+Kx_3^2},\\
\overset{(m,s)}{I_3}&=\frac{2m+1}{2h}\int_{-h}^{h}x_3P_mP_sdx_3,\\
\overset{(m,s)}{I_4}&=\frac{2m+1}{2h}\int_{-h}^{h}(1-2Hx_3+Kx_3^2)P_mP_sdx_3.
\end{aligned}
\tag{24}
$$

The above integrals can be calculated explicitly and their expressions with regard to ξ have the form, for example

$$
\begin{gathered}
\overset{(m,s)}{I_1}=\frac{2m+1}{2\sqrt{E}}\sum_{r=0}^{[\frac{m}{2}]}\sum_{p=0}^{\infty}\overset{(m,s)}{M_{rp}}\varepsilon^{s-m+2(r+p)}\left\{\left[(H+\sqrt{E})R\right]^{s-m+2(r+p)}\right.\\
\left.-[(H-\sqrt{E})R]^{s-m+2(r+p)}\right]\Bigg\},
\end{gathered}
$$

where

$$
\overset{(m,s)}{M_{rp}}=2^{s-m}\frac{(-1)^r(2m-2r)!(s+p)!(s+2p)!}{r!(m-r)!(m-2r)!p!(2s+2p+1)!}.
$$

Now, following Signiorini we assume the validity of the expansions for approximation of order N:

$$
\left(\overset{(m)}{\vec{\sigma}^i},\overset{(m)}{\vec{U}},\overset{(n)}{\vec{F}}\right)=\sum_{n=1}^{\infty}\left(\overset{(m,n)}{\vec{\sigma}^i},\overset{(m,n)}{\vec{U}},\overset{(m,n)}{\vec{F}}\right)\varepsilon^n,\quad (m=0,1,\cdots,N).
$$

Substituting the above expansions into the relations (20-23), then equalizing the coefficients of expansion for ε^n we obtain the following 2-D infinite system of equilibrium equations with respect to components of displacement vector in the isometric coordinates:

$$\begin{aligned}
&4\mu\partial_z\left(\Lambda^{-1}\partial_z \overset{(m,n)}{U_+}\right)+2(\lambda+\mu)\partial_{\bar z}\overset{(m,n)}{\theta}+\frac{2\lambda}{h}\partial_{\bar z}\overset{(m,n)}{U'_3}\\
&-\frac{2m+1}{h}\mu\left[2\partial_{\bar z}\left(\overset{(m-1,n)}{U_3}+\overset{(m-3,n)}{U_3}+\cdots\right)\right.\\
&\left.+\left(\overset{(m-1,n)}{U_+}+\overset{(m-3,n)}{U_+}+\cdots\right)\right]+\overset{(m,n)}{F_+}=0,\\
&\mu\left(\nabla^2\overset{(m,n)}{U_3}+\overset{(m,n)}{\theta'}\right)-\frac{2m+1}{h}\left[\lambda\left(\overset{(m-1,n)}{\theta}+\overset{(m-3,n)}{\theta}+\cdots\right)\right.\\
&\left.(\lambda+2\mu)\left(\overset{(m-1,n)}{U'_3}+\overset{(m-3,n)}{U'_3}+\cdots,\right)\right]+\overset{(m,n)}{F_3}=0,
\end{aligned}\tag{25}$$

where

$$\overset{(m)}{U_+}=\overset{(m)}{U_1}+i\overset{(m)}{U_2},\quad \overset{(m)}{\theta}=\Lambda^{-1}\left(\partial_z\overset{(m)}{U_+}+\partial_{\bar z}\overset{(m)}{\bar U_+}\right),$$

$$\overset{(m)}{U'_i}=\frac{2m+1}{h}\left(\overset{(m+1)}{U_i}+\overset{(m+3)}{U_i}+\cdots\right),\quad \nabla^2=\frac{4}{\Lambda}\frac{\partial^2}{\partial z\partial\bar z}.$$

Obviously, in passing from the n-th step of approximations to the $n+1$-th step only the right-hand sides of equations are changed and the boundary conditions after the initial step may be considered homogenous.

Below it will be omit upper index n.

The beharmonic solution of the homogeneous system (25) we can find the form [1]

$$\begin{aligned}
\overset{(m)}{U_+}&=\partial_{\bar z}\left(\overset{(m)}{V_1}+i\overset{(m)}{V_2}\right)+\left(\frac{1}{\pi}\iint\limits_S\frac{\overline{\varphi'_0(\xi)}-k_1\varphi'_0(\xi)}{\bar\xi-\bar z}ds_s-\overline{\psi'_0\xi}\right)\overset{m}{\delta_0}\\
&+k_2\overline{\varphi''_0(z)}\,\overset{m}{\delta_2}-\frac{1}{\pi}\left(\iint\limits_S\frac{\varphi'_1(\xi)+\overline{\varphi'_1(\xi)}}{\bar\xi-\bar z}dS_\xi+\eta_1\overline{\varphi''_1(z)}-2\overline{\psi'_1(z)}\right)\delta_1^m\\
&+\eta_2\overline{\varphi''_1(z)}\delta_2^m,
\end{aligned}\tag{26}$$

$$\overset{(m)}{U_3}=\overset{(m)}{V_3}-\left[\frac{1}{\pi}\iint\limits_S\left(\varphi'_1(\xi)+\overline{\varphi'_1(\xi)}\right)\ln|\xi-z|dS_\xi-\left(\psi_1(z)+\overline{\psi_1(z)}\right)\right]\delta_0^m$$

$$-\frac{3}{2}k_2\left[\left(\varphi_0'(z)+\overline{\varphi_0'(z)}\right)\delta_1^m+\left(\varphi_1'(z)+\overline{\varphi_1'(z)}\right)\delta_2^m\right],$$

$$(m=0,1,\cdots,N)$$

$$\overset{0}{V}_1+\overset{0}{V}_2=0,\quad \overset{0}{U}_3=\psi_1(z)+\overline{\psi_1(z)},\quad if\ \ N=0,$$

where V_i $(i=1.2.3)$ are unknown metaharmonic functions, $\varphi_0,\varphi_0,\psi_0,\psi_1$ are analytic functions of z, δ_i^j - Kroneeker delta, $dS_\xi=\Lambda(\xi,\bar{\xi})d\xi d\eta, \xi=\xi+i\eta$, then

$$k_1=\begin{cases}\dfrac{\lambda+3\mu}{\lambda+\mu},\ N=0\\ \dfrac{5\lambda+6\mu}{3\lambda+2\mu},\ N\neq 0,\end{cases}\qquad \eta_1=\begin{cases}\dfrac{\lambda+\mu}{\mu},\ N=1\\ 4\dfrac{\lambda+\mu}{\lambda+2\mu},\ N=2,\\ \dfrac{23\lambda+24\mu}{5(\lambda+2\mu)},\ N=3,\end{cases}$$

$$k_2=\frac{4}{3}\frac{\lambda}{3\lambda+2\mu},\quad \eta_2=\frac{4}{15}\frac{3\lambda+4\mu}{\lambda+2\mu}.$$

7 Approximation of Order $N=0,1,2,3,4$

For them the general solutions of the homogeneous system (26) can be represented by formulas:

Case $N=0$

$$\overset{(0)}{U}_+=\frac{1}{\pi}\iint\limits_S\frac{\overline{\varphi'(\zeta)}-\varkappa\varphi'(\zeta)}{\bar{\zeta}-\bar{z}}dS_\zeta-\overline{\psi(z)},$$

$$\overset{(0)}{U}_3=f(z)+\overline{f(z)},$$

$$\left(\zeta=\xi+i\eta,\quad dS_\zeta=\Lambda(\zeta,\bar{\zeta})d\xi d\eta\right),$$

where $\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}$, $f(z)$, $\varphi(z)$, $\psi(z)$ are the holomorphic functions of $z=x^1+ix^2$.

Note that for approximation of order $N=0$, when $\lambda(z,\bar{z})=1$, the expression for $\overset{(0)}{U}_+$, coincides with well-known representation of Kolosov-Muskhelishvili for plane deformation

$$\overset{(0)}{U}_+=\varkappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\varphi(z)}.$$

Case $N = 1$

$$\overset{(0)}{U}_{+} = -\frac{\lambda h}{6(\lambda+\mu)}\partial_{\bar z}\omega + \frac{1}{\pi}\iint_S \frac{\overline{\varphi'(\zeta)} - \overset{*}{\varkappa}\varphi'(\zeta)}{\bar\zeta - \bar z} dS_\zeta - \overline{\psi(z)},$$

$$\overset{(1)}{U}_{+} = i\partial_{\bar z}\chi + \frac{1}{\pi}\iint_S \frac{\overline{\Phi'(\zeta)} - \Phi'(\zeta)}{\bar\zeta - \bar z} dS_\zeta + \frac{2(\lambda+2\mu)}{3\mu}\overline{\Phi'(\xi)} - 2h\overline{\Psi'(z)}$$

$$\overset{(0)}{U}_3 = \Psi(z) + \overline{\Psi(z)} - \frac{1}{\pi}\iint_S \left[\Phi'(\zeta) + \overline{\Phi'(\zeta)}\right]\ln|\zeta - z| dS_\zeta,$$

$$\overset{(1)}{U}_3 = \omega(z,\bar z) + \frac{2\lambda h}{3\lambda+2\mu}\left[\varphi'(z) + \overline{\varphi'(z)}\right],$$

where

$$\nabla^2\chi - \frac{3}{h^2}\chi = 0, \quad \nabla^2\omega - \frac{3(\lambda+\mu}{(\lambda+2\mu)h}\omega = 0, \quad \overset{*}{\varkappa} = \frac{5\lambda+6\mu}{3\lambda+2\mu}.$$

Case $N = 2$

$$\overset{(0)}{U}_{+} = \frac{1}{\pi}\iint_S \frac{\overline{\varphi'(\zeta)} - \kappa\varphi'(\zeta)}{\bar\zeta - \bar z} dS_\zeta - \overline{\psi(z)} - \frac{2\lambda}{\lambda+2\mu}\sum_{k=1}^{2}\frac{1}{\alpha_k}\partial_{\bar z}V_k,$$

$$\overset{(1)}{U}_{+} = i\partial_{\bar z}\chi + \frac{4}{3}\frac{\lambda+\mu}{\mu}\overline{\Phi''(z)} - \frac{1}{\pi}\iint_S \frac{\Phi'(\zeta) + \overline{\Phi'(\zeta)}}{\bar\zeta - \bar z} dS_\zeta - 2h\overline{\Psi'(z)} - \frac{\lambda}{10(\lambda+\mu)}\partial_{\bar z}w,$$

$$\overset{(2)}{U}_{+} = \frac{2}{3}\left(i\partial_{\bar z}\omega + \sum_{k=1}^{2}\frac{\alpha_{3-k}}{\alpha_k}\partial_{\bar z}V_k + \frac{2\lambda}{3\lambda+2\mu}\overline{\varphi''(z)}\right),$$

$$\overset{(0)}{U}_3 = \Psi(z) + \overline{\Psi(z)} - \frac{1}{\pi}\iint_S \left(\overline{\Phi'(\zeta)} + \Phi(\zeta)\right)\ln|(\zeta) - z| dS_\zeta + \frac{\lambda}{2(\lambda+\mu)}w,$$

$$\overset{(1)}{U}_3 = V_1 + V_2 - \frac{2\lambda}{3\lambda+2\mu}(\varphi'(z) + \overline{\varphi'(z)}),$$

$$\overset{(2)}{U}_3 = w - \frac{2\lambda}{3\lambda+2\mu}(\Phi'(z) + \overline{\Phi'(z)}),$$

where

$$\nabla^2 V_k = \alpha_k V_k, \quad \alpha_k^2 - \frac{12(\lambda+\mu)}{\lambda+2\mu}\alpha_k + \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2} = 0, \quad (k=1,2)$$

$$\nabla^2 w = \frac{60(\lambda+\mu)}{\lambda+2\mu} w, \quad \nabla^2 \chi = 3\chi, \quad \nabla^2 \omega = 15\omega.$$

Case $N = 3$

$$\overset{(0)}{U}_+ = \frac{1}{\pi} \iint\limits_S \frac{\overline{\varphi'(\varsigma)} - æ\varphi'(\varsigma)}{\bar{\varsigma} - \bar{z}} dS_\varsigma - \overline{\psi(z)} - \frac{2\lambda}{\lambda+2\mu} \sum_{k=1}^{3} \frac{1}{\alpha_k} \partial_{\bar{z}} V_k,$$

$$\overset{(1)}{U}_+ = \sum_{k=1}^{2} \left(i \frac{\chi_{3-k}}{3} \partial_{\bar{z}} \chi_k - \frac{6\lambda}{\lambda+2\mu} \frac{1}{\gamma_k} \partial_{\bar{z}} w_k \right) - \frac{1}{\pi} \iint\limits_S \frac{\Phi'(\varsigma) + \overline{\Phi'(\xi)}}{\bar{\varsigma} - \bar{z}} dS_\varsigma$$

$$-\frac{4}{15} \frac{23\lambda+2\mu}{\lambda+2\mu} \overline{\Psi'(z)},$$

$$\overset{(2)}{U}_+ = \frac{2}{3} \left(i\partial_{\bar{z}}\omega + \sum_{k=1}^{3} + \overset{(2)}{A}_k \partial_{\bar{z}} V_k + \frac{2\lambda}{3\lambda+2\mu} \overline{\varphi''(z)} \right),$$

$$\overset{(3)}{U}_+ = \sum_{k=1}^{2} \left(i\partial_{\bar{z}}\chi_k + \frac{2}{3} \frac{\gamma_{3-k}}{\gamma_k} \partial_{\bar{z}} w_k \right) - \frac{4}{15} + \frac{3\lambda+2\mu}{\lambda+2\mu} \overline{\varphi''(z)},$$

$$\overset{(0)}{U}_3 = \sum_{k=1}^{2} \left(\frac{3\lambda}{\lambda+2\mu} - \frac{\gamma_{3-k}}{5} \right) \frac{1}{\gamma_k} w_k$$

$$-\frac{1}{\pi} \iint\limits_S (\Phi'(\varsigma) + \overline{\Phi'(\varsigma)}) \ln|(\varsigma) - z| dS_\varsigma + \Psi(z) + \overline{\Psi(z)},$$

$$\overset{(1)}{U}_3 = \sum_{k=1}^{2} \overset{(1)}{A}_k V_k - \frac{2\lambda}{3\lambda+2\mu} (\varphi'(z) + \overline{\varphi'(z)})),$$

$$\overset{(2)}{U}_3 = \sum_{k=1}^{2} w_k - -\frac{2\lambda}{3(\lambda+2\mu)} (\Phi'(z) + \overline{\Phi'(z)})),$$

$$\overset{(3)}{U}_3 = \sum_{k=1}^{3} V_k,$$

where

$$\left[\begin{array}{l} \nabla^2\omega + 15\omega = 0 \\ \nabla^2 V_k + a_k V_k = 0, \\ (k = 1, 2, 3) \end{array} \right.$$

$$\left[\begin{array}{l}\alpha_k^3-\dfrac{180(\lambda+\mu)}{\lambda+2\mu}\alpha_k^2+\dfrac{120(\lambda+\mu)(7\lambda+15\mu)}{(\lambda+2\mu)^2}\alpha_k+\dfrac{2700\mu^2(\lambda+\mu)}{(\lambda+2\mu)^2}=0,\\ \overset{(1)}{A}_k=\dfrac{3}{\lambda+2\mu}\left[(5\lambda+4\mu)\alpha_k-\dfrac{60\mu(\lambda+\mu)}{\lambda+2\mu}\right]\\ \times\left[\alpha_k^2-\dfrac{12(\lambda+\mu)}{\lambda+2\mu}\alpha_k+\dfrac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\right]^{-1},\\ \overset{(2)}{A}_k=-\dfrac{10}{\lambda+2\mu}\left[\lambda\alpha_k-12(\lambda+\mu)\right]\left[\alpha_k^2-\dfrac{12(\lambda+\mu)}{\lambda+2\mu}\alpha_k+\dfrac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\right]^{-1},\end{array}\right.$$

$$\left[\begin{array}{l}\nabla^2\omega_k=\gamma_k w_k,\\ \nabla^2\chi_k=æ_k\chi_k,\\ (k=1,2),\end{array}\right.$$

$$\left[\begin{array}{l}\gamma_k^2-60\dfrac{\lambda+\mu}{3\lambda+2\mu}\gamma_k+120\dfrac{\mu(\lambda+\mu)}{(\lambda+2\mu)^2}=0,\\ æ_k^2-45æ_k+105=0,\\ (k=1,2).\end{array}\right.$$

Case $N=4$ (for the plate)

$$\left[\begin{array}{l}\mu\Delta\overset{(1)}{u}_+ + 2(\lambda+\mu)\partial_{\bar z}\overset{(0)}{\theta}+\dfrac{2\lambda}{h}\partial_{\bar z}\Big(\overset{(1)}{u}_3+\overset{(3)}{u}_3\Big)=0,\\ \mu\Delta\overset{(2)}{u}_+ + 2(\lambda+\mu)\partial_{\bar z}\overset{(2)}{\theta}+\dfrac{10\lambda}{h}\partial_{\bar z}\overset{(3)}{u}_3-\dfrac{5\mu}{h}\Big[\partial_{\bar z}\overset{(1)}{u}_3+\dfrac{3}{h}\Big(\overset{(3)}{u}_+ +\overset{(4)}{u}_+\Big)\Big]=0,\\ \mu\Delta\overset{(4)}{u}_+ + 2(\lambda+\mu)\partial_{\bar z}\overset{(4)}{\theta}-\dfrac{9\lambda}{h}\left[2\partial_{\bar z}\Big(\overset{(1)}{u}_3+\overset{(3)}{u}_3\Big)+\dfrac{3}{h}\overset{(2)}{u}_+ +\dfrac{10}{h}\overset{(4)}{u}_+\right]=0,\\ \mu\left[\Delta\overset{(1)}{u}_3+\dfrac{3}{h}\Big(\overset{(2)}{\theta}+\overset{(4)}{\theta}\Big)\right]-\dfrac{3}{h}\left[\lambda\overset{(0)}{\theta}+\dfrac{\lambda+2\mu}{h}\Big(\overset{(1)}{u}_3+\overset{(3)}{u}_3\Big)\right]=0,\\ \mu\Big(\Delta\overset{(3)}{u}_3+\dfrac{7}{h}\overset{(4)}{\theta}\Big)-\dfrac{7}{h}\Big[\lambda\Big(\overset{(1)}{\theta}+\overset{(2)}{\theta}\Big)+\dfrac{\lambda+2\mu}{h}\Big(\overset{(1)}{u}_3+6\overset{(3)}{u}_3\Big)\Big]=0,\end{array}\right.$$

$$\overset{(k)}{U}_+=\overset{(k)}{U}_1+i\overset{(k)}{U}_2,\quad \overset{(k)}{\theta}=\partial_z\overset{(k)}{U}_+ +\partial_{\bar z}\overset{(k)}{\bar U}_+,$$

$$\partial_1=\partial_z+\partial_{\bar z},\ \ \partial_z=i(\partial_z-\partial_{\bar z}),$$

$$\left[\begin{array}{l} \overset{(0)}{u}_+ = \dfrac{5\lambda+6\mu}{3\lambda+2\mu}\varphi(z) - \overline{z\varphi'(z)} - \bar{\Psi'}(z) - \dfrac{2\lambda}{\lambda+2\mu}\displaystyle\sum_{k=1}^{4}\dfrac{1+\beta_k}{\gamma_k h}\partial_{\bar z}\chi_k, \\ \overset{(2)}{u}_+ = i\partial_{\bar z}(V_1+V_2) + 2\displaystyle\sum_{k=1}^{4}\dfrac{(c_1+c_2\beta_k)\gamma_k + c_3\beta_k}{\gamma_4}\partial_{\bar z}\chi_k + \dfrac{4h^2}{3}\dfrac{\lambda}{3\lambda+2\mu}\overline{\varphi''(z)}, \\ \overset{(4)}{u}_+ = \dfrac{ih^2}{15}\partial_{\bar z}\left[\left(æ_1 - \dfrac{15}{h^2}\right)V_1 + \left(æ_2 - \dfrac{15}{h^2}\right)V_2\right] \\ +2\displaystyle\sum_{k=1}^{4}\dfrac{(d_1+d_2\beta_k)\gamma_k + d_1 + d_4\beta_k}{\gamma_k}\partial_{\bar z}\chi_k, \\ \overset{(1)}{u}_3 = \displaystyle\sum_{k=1}^{4}\chi_k - \dfrac{2\lambda h}{3\lambda+2\mu}(\varphi' + \overline{z\varphi'}), \\ \overset{(3)}{u}_3 = \displaystyle\sum_{k=1}^{4}\beta_k\chi_k, \end{array}\right.$$

$$\beta_k = \frac{\gamma_k^2 + a_1\gamma_k + a_3}{a_2\gamma_k + a_k},$$

$$\left\{\begin{array}{l} \Delta\chi_k - \gamma_4\chi_k = 0, \quad \Delta = 4\frac{\partial^2}{\partial z\partial\bar z}, \quad (k=1,2,3,4) \\ \gamma_k^4 + (a_1+b_2)\gamma_k^3 + (a_1b_2 - a_2b_1 + a_3 + b_4)\gamma_k^2 \\ +(a_1b_4 - a_4b_1 + a_3b_2 - a_2b_3)\gamma_k + a_3b_4 - a_4b_3 = 0, \end{array}\right.$$

$$\left\{\begin{array}{l} a_1 = \dfrac{12\lambda^2+59\lambda\mu+12\mu^2}{(\lambda+\mu)(\lambda+2\mu)h^2}, \quad a_2 = \dfrac{3(9\lambda^2+8\lambda\mu+4\mu^2)}{(\lambda+\mu)(\lambda+2\mu)h^2}, \\ a_3 = \dfrac{3\cdot 260(\lambda+\mu)}{(\lambda+2\mu)^2h^2} \quad a_4 = \dfrac{3}{h^4}\left[\dfrac{260\mu(\lambda+\mu)}{(\lambda+2\mu)^2} + \dfrac{175\mu}{\lambda+\mu}\right], \end{array}\right.$$

$$\left\{\begin{array}{l} b_1 = \dfrac{7}{3}c_1, \quad b_2 = 7\dfrac{10[\lambda\mu - (\lambda+2\mu)^2] - 4(\lambda+\mu)^2}{(\lambda+\mu)(\lambda+2\mu)h^2}, \quad h_3 = \dfrac{39}{49}a_3, \\ b_4 = \dfrac{7^2\cdot 5}{h^4}\left[\dfrac{4\mu(\lambda+\mu)}{(\lambda+2\mu)^2} + \dfrac{5\mu}{\lambda+\mu}\right], \end{array}\right.$$

$$\left\{\begin{array}{l} c_1 = -\dfrac{\mu h}{\lambda+\mu}, \quad c_2 = -\dfrac{3}{7}c_1, \quad c_3 = -\dfrac{5}{h}\dfrac{\lambda+2\mu}{\lambda+\mu}, \\ d_1 = \dfrac{\lambda}{\mu}c_1, \quad d_2 = -c_2 = \dfrac{3}{7}c_1, \quad d_3 = \dfrac{4}{h}\dfrac{\lambda+\mu}{\lambda+2\mu}, \quad d_4 = d_3 - c_3, \end{array}\right.$$

$$\begin{cases} \Delta V_1 = æ_1 V_1, \\ \Delta V_2 = æ_2 V_2, \end{cases}$$

$$\begin{cases} æ_1 + æ_2 = \dfrac{3\cdot 5\cdot 7}{h^2}, \\ æ_1 æ_2 = \dfrac{3\cdot 5\cdot 7\cdot 9}{h^4}, \end{cases} \qquad æ^2 - \frac{3\cdot 5\cdot 7}{h^2}æ + \frac{3\cdot 5\cdot 7\cdot 9}{h^4} = 0.$$

Acknowledgement

The designated project has been fulfilled by a financial support of Shota Rustaveli National Science Foundation (Grant SRNSF/FR/358/5-109/14). Any idea in this publication is possessed by the author and may not represent the opinion of Shota Rustaveli National Science Foundation itself.

References

1. Muskhelishvili N.I. *Some Basic Problems of the Mathematical Theory of Elasticity*. Nauka, Moscow, 1966.

2. Vekua I.N. *Shell theory general methods of construction*. Pitman Advanced Publishing Program, London, Boston, Melbourn, 1985.

3. Bitsadze A.V. *The Boundary Value Problem for the Elliptical System Equations of the Second Order*. Moscow, Nauka, 1966.

4. Ciarlet P.G. *Mathematical Elasticity, I. Three-Dimensional Elasticity*. Studies in Mathematics and its Applications, Nort-Holland Publishing Co, Amsterdam, **20** (1988). III. Theory of shells, Amsterdam, 2000.

5. Goldenveizer A.L. *Theory of Elastic Thin Shells*. Moscow, Nauka, 1976.

6. Oden J. *Finite Elements of Nonlinear Continua*. Mir, Moscow, 1976.

7. Meunargia T.V. A Small Parameter Method for I, Vekua's Nonlinear and Non-Shallow Shell. *Proceedings of the IUTAM Symposium, Springer Science, Tbilisi*, 2007.