ON ONE PROBLEM FOR THE PLATE

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Abstract

In this work we consider equations of equilibrium of the isotropic elastic plate. By means of Vekua's method, the system of differential equations for plates is obtained (approximation $N = 1$), when on upper and lower face surfaces displacements are assumed to be known. The general solution for approximations $N = 1$ is constructed. The concrete problem is solved.

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1 Introduction

One of the theories of shallow shells was constructed by Vekua by using the Cauchy-–Poisson method, which is based on the expansion of displacements and stresses into series in terms of a system of functions with respect to the thickness coordinate [1-2]. This method for non-shallow shells in case of geometrical and physical nonlinear theory was generalized by T. Meunargia [3, 4].

By means of Vekua's method, the system of differential equations for thin and shallow shells was obtained, when on upper and lower face surfaces displacements are assumed to be known [5].

The systems of equilibrium equations and stress-strain relations (Hooke's law) of the plates in the case of $N = 1$ approximation may be written in the following form [6]:

$$\begin{cases} \partial_\alpha \overset{(0)}{\sigma}_{\alpha\beta} + \frac{1}{h} \overset{(1)}{\sigma}_{\beta 3} + \overset{(0)}{\Phi}_\beta = 0, \\ \partial_\alpha \overset{(0)}{\sigma}_{\alpha 3} + \frac{1}{h} \overset{(1)}{\sigma}_{33} + \overset{(0)}{\Phi}_3 = 0, \end{cases} \tag{1}$$

$$\begin{cases} \partial_\alpha \overset{(1)}{\sigma}_{\alpha\beta} + \overset{(1)}{\Phi}_\beta = 0, \\ \partial_\alpha \overset{(1)}{\sigma}_{\alpha 3} + \overset{(1)}{\Phi}_3 = 0, \end{cases} \tag{2}$$

$$\begin{cases} \overset{(0)}{\sigma}_{\alpha\beta} = \lambda \left(\partial_\gamma \overset{(0)}{u}_\gamma \right) \delta_{\alpha\beta} + \mu \left(\partial_\beta \overset{(0)}{u}_\alpha + \partial_\alpha \overset{(0)}{u}_\beta \right) \\ + \frac{\lambda}{h} \left(\overset{(+)}{u}_3 - \overset{(-)}{u}_3 \right) \delta_{\alpha\beta}, \\ \overset{(0)}{\sigma}_{\alpha 3} = \mu \left(\partial_\alpha \overset{(0)}{u}_3 \right) + \frac{\mu}{h} \left(\overset{(+)}{u}_\alpha - \overset{(-)}{u}_\alpha \right), \\ \overset{(0)}{\sigma}_{33} = \lambda \left(\partial_\gamma \overset{(0)}{u}_\gamma \right) + \frac{\lambda + 2\mu}{h} \left(\overset{(+)}{u}_3 - \overset{(-)}{u}_3 \right), \end{cases} \tag{3}$$

$$\begin{cases} \overset{(1)}{\sigma}_{\alpha\beta} = \lambda \left(\partial_\gamma \overset{(1)}{u}_\gamma - \frac{3}{h} \overset{(0)}{u}_3 \right) \delta_{\alpha\beta} + \mu \left(\partial_\beta \overset{(1)}{u}_\alpha + \partial_\alpha \overset{(1)}{u}_\beta \right) \\ + \frac{3\lambda}{h} \left(\overset{(+)}{u}_3 + \overset{(-)}{u}_3 \right) \delta_{\alpha\beta}, \\ \overset{(1)}{\sigma}_{\alpha 3} = \mu \left(\partial_\alpha \overset{(1)}{u}_3 - \frac{3}{h} \overset{(0)}{u}_\alpha \right) + \frac{3\mu}{h} \left(\overset{(+)}{u}_\alpha + \overset{(-)}{u}_\alpha \right), \\ \overset{(1)}{\sigma}_{33} = \lambda \left(\partial_\gamma \overset{(1)}{u}_\gamma \right) - \frac{3(\lambda + 2\mu)}{h} \overset{(0)}{u}_3 + \frac{3(\lambda + 2\mu)}{h} \left(\overset{(+)}{u}_3 + \overset{(-)}{u}_3 \right), \end{cases} \tag{4}$$

where

$$\left(\overset{(m)}{\sigma}{}^{ij}, \overset{(m)}{u}{}^i, \overset{(m)}{\Phi}{}^i \right) = \frac{2m+1}{2h} \int_{-h}^{h} (\sigma^{ij}, u^i, \Phi^i) P_m \left(\frac{x_3}{h} \right) dx_3,$$

$$(m = 0, 1)$$

$$\overset{(\pm)}{u}{}^i = u^i(x^1, x^2, \pm h),$$

λ and μ are Lame's constants, σ^{ij} are contravariant components of the stress vectors, u^i are contravariant components of the displacement vector, Φ^i are contravariant components of the volume force, $P_m\left(\frac{x^3}{h}\right)$ are Legendre polynomials, h is the semi-thickness.

Substituting these expressions (3) and (4) into equation (1) and (2), we obtain the system of second-order partial differential equations:

$$\begin{cases} \mu\Delta \overset{(0)}{u}_1 + (\lambda+\mu)\partial_1 \overset{(0)}{\theta} + \dfrac{1}{h}\left(\mu\partial_1 \overset{(1)}{u}_3 - \dfrac{3\mu}{h}\overset{(0)}{u}_1\right) = \overset{(0)}{\Psi}_1, \\ \mu\Delta \overset{(0)}{u}_2 + (\lambda+\mu)\partial_2 \overset{(0)}{\theta} + \dfrac{1}{h}\left(\mu\partial_2 \overset{(1)}{u}_3 - \dfrac{3\mu}{h}\overset{(0)}{u}_2\right) = \overset{(0)}{\Psi}_2, \\ \mu\Delta \overset{(0)}{u}_3 + \dfrac{1}{h}\left(\lambda \overset{(1)}{\theta} - \dfrac{3(\lambda+2\mu)}{h}\overset{(0)}{u}_3\right) = \overset{(0)}{\Psi}_3, \end{cases} \tag{5}$$

$$\begin{cases} \mu\Delta \overset{(1)}{u}_1 + (\lambda+\mu)\partial_1 \overset{(1)}{\theta} - \dfrac{3\lambda}{h}\partial_1 \overset{(0)}{u}_3 = \overset{(1)}{\Psi}_1, \\ \mu\Delta \overset{(1)}{u}_2 + (\lambda+\mu)\partial_2 \overset{(1)}{\theta} - \dfrac{3\lambda}{h}\partial_2 \overset{(0)}{u}_3 = \overset{(1)}{\Psi}_2, \\ \mu\Delta \overset{(1)}{u}_3 - \dfrac{3\mu}{h}\overset{(0)}{\theta} = \overset{(1)}{\Psi}_3, \end{cases} \tag{6}$$

where $\overset{(m)}{\Psi}_i$ are the known values and

$$\overset{(m)}{\theta} = \partial_1 \overset{(m)}{u}_1 + \partial_2 \overset{(m)}{u}_2, \quad m = 0, 1.$$

Introducing the well-known differential operators

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2),$$

where $z = x_1 + ix_2$.

System (5) and (6) can be written in complex form:

a) for the tension-pressure of plates

$$\begin{cases} \mu\Delta \overset{(0)}{u}_+ + 2(\lambda+\mu)\partial_{\bar z} \overset{(0)}{\theta} + \dfrac{1}{h}\left(2\mu\partial_{\bar z} \overset{(1)}{u}_3 - \dfrac{3\mu}{h}\overset{(0)}{u}_+\right) = \overset{(0)}{\Psi}_+, \\ \mu\Delta \overset{(1)}{u}_3 - \dfrac{3\mu}{h}\overset{(0)}{\theta} = \overset{(1)}{\Psi}_3, \end{cases} \tag{7}$$

b) for the bending of plates

$$\begin{cases} \mu\Delta \overset{(1)}{u}_+ + 2(\lambda+\mu)\partial_{\bar z} \overset{(1)}{\theta} - \dfrac{6\lambda}{h}\partial_{\bar z} \overset{(0)}{u}_3 = \overset{(1)}{\Psi}_+, \\ \mu\Delta \overset{(0)}{u}_3 + \dfrac{1}{h}\left(\lambda \overset{(1)}{\theta} - \dfrac{3(\lambda+2\mu)}{h}\overset{(0)}{u}_3\right) = \overset{(0)}{\Psi}_3, \end{cases} \tag{8}$$

where $\Delta = 4\dfrac{\partial^2}{\partial z \partial \bar z}$ and

$$\overset{(m)}{u}_+ = \overset{(m)}{u}_1 + i\overset{(m)}{u}_2, \quad \overset{(m)}{\theta} = \partial_z \overset{(m)}{u}_z + \partial_{\bar z} \overset{(m)}{\bar u}_+, \quad \overset{(m)}{\Psi}_+ = \overset{(m)}{\Psi}_1 + i\overset{(m)}{\Psi}_2.$$

The complex representation of the general solutions of the homogenous systems (7) end (8) are written in the following form [2, 5]:

$$\begin{cases} \overset{(0)}{u}_+ = f(z) + z\overline{f'(z)} + \dfrac{4(\lambda+2\mu)h^2}{3\mu}\overline{f''(z)} + \overline{g'(z)} - \dfrac{ih}{3}\dfrac{\partial\omega(z,\bar z)}{\partial \bar z}, \\ \overset{(1)}{u}_3 = \dfrac{3}{2h}\left(\bar z f(z) + z\overline{f(z)}\right) + \dfrac{3}{2h}\left(g(z) + \overline{g(z)}\right), \end{cases} \tag{9}$$

$$\begin{cases} \overset{(1)}{u}_+ = \dfrac{5\lambda+6\mu}{3\lambda+2\mu}\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \dfrac{\lambda h}{2(\lambda+\mu)}\dfrac{\partial\chi(z,\bar z)}{\partial\bar z}, \\ \overset{(0)}{u}_3 = \chi(z,\bar z) + \dfrac{2\lambda h}{3(3\lambda+2\mu)}\left(\varphi'(z) + \overline{\varphi'(z)}\right), \end{cases} \tag{10}$$

where $f(z)$, $g(z)$, $\varphi(z)$ and $\psi(z)$ are any analytic functions of z, $\omega(z,\bar z)$ and $\chi(z,\bar z)$ are the general solutions of the following Helmholtz's equations, respectively:

$$\Delta\omega - \gamma^2\omega = 0, \quad \left(\gamma^2 = \frac{3}{h^2}\right),$$

$$\Delta\chi - \nu^2\chi = 0, \quad \left(\chi^2 = \frac{12(\lambda+\mu)h^2}{\lambda+2\mu}\right).$$

From eqs. (3), (4) the following relations follow

$$\begin{cases} \overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{22} = 2(\lambda+\mu)\overset{(0)}{\theta}, \\ \overset{(0)}{\sigma}_{11} - \overset{(0)}{\sigma}_{22} + 2i\overset{(0)}{\sigma}_{12} = 4\mu\partial_{\bar z}\overset{(0)}{u}_+, \\ \overset{(1)}{\sigma}_{11} + \overset{(1)}{\sigma}_{22} = 2(\lambda+\mu)\overset{(1)}{\theta} - \dfrac{6\lambda}{h}\overset{(0)}{u}_3, \\ \overset{(1)}{\sigma}_{11} - \overset{(1)}{\sigma}_{22} + 2i\overset{(1)}{\sigma}_{12} = 4\mu\partial_{\bar z}\overset{(1)}{u}_+, \\ \overset{(0)}{\sigma}_{13} + i\overset{(0)}{\sigma}_{23} = 2\mu\partial_{\bar z}\overset{(0)}{u}_3, \\ \overset{(1)}{\sigma}_{13} + i\overset{(1)}{\sigma}_{23} = 2\mu\partial_{\bar z}\overset{(1)}{u}_3 - \dfrac{3\mu}{h}\overset{(0)}{u}_+. \end{cases} \tag{11}$$

2 The Problem for the Infinite Plane with a Circular Hole

Now let us have an infinite plane with a circular hole (Fig. 2). Assume that the origin of coordinates is at the center of the hole of radius R.

The boundary problem (in stresses) takes the form [3]:

$$\begin{cases} \overset{(m)}{\sigma}_{rr} + i\overset{(m)}{\sigma}_{r\alpha} \\ = \dfrac{1}{2}\left[\overset{(m)}{\sigma}_{11} + \overset{(m)}{\sigma}_{22} - \left(\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\overset{(m)}{\sigma}_{12}\right)\left(\dfrac{d\bar z}{ds}\right)^2\right] = \overset{(m)}{F}_+, \\ \overset{(m)}{\sigma}_{rn} = -\mathrm{Im}\left(\overset{(m)}{\sigma}_{+3}\dfrac{d\bar z}{ds}\right) = \overset{(m)}{F}_3, \quad \left(\overset{(m)}{\sigma}_{+3} = \overset{(m)}{\sigma}_{13} + i\overset{(m)}{\sigma}_{23}\right). \end{cases} \tag{12}$$

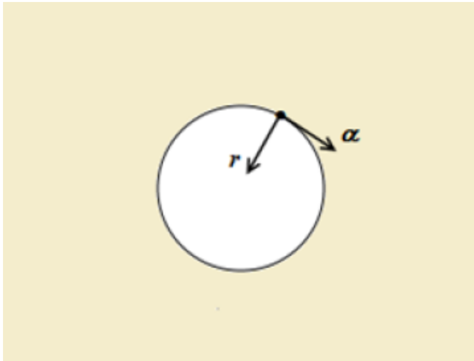

Figure 1:

Conditions at infinity are

$$\overset{(0)}{\sigma}_{11} = \Gamma_1, \quad \overset{(0)}{\sigma}_{22} = \Gamma_2, \quad \overset{(0)}{\sigma}_{12} = \Gamma_3,$$

$$\overset{(1)}{\sigma}_{11} = \Gamma_4, \quad \overset{(1)}{\sigma}_{22} = \Gamma_5, \quad \overset{(1)}{\sigma}_{12} = \Gamma_6.$$

Use eqs. (9) and (10) the boundary conditions are written as

$$\begin{cases} (\lambda+\mu)(f'(z)+\overline{f'(z)}) + \Big(2\mu z\overline{f''(z)} + \dfrac{8(\lambda+2\mu)}{3}\overline{f'''(z)} \\ +2\mu\overline{g''(z)} - \dfrac{2\mu i h}{3}\dfrac{\partial^2\omega(z,\bar z)}{\partial \bar z^2}\Big)e^{-2i\alpha} = \displaystyle\sum_{-\infty}^{+\infty} A_{n1}e^{in\alpha}, \quad r=R, \\ \dfrac{\mu}{2h}\Big(ih\dfrac{\partial\omega(z,\bar z)}{\partial \bar z} - \dfrac{4(\lambda+2\mu)h^2}{3\mu}\overline{f''(z)}\Big)e^{-i\alpha} \\ -\dfrac{\mu}{2h}\Big(ih\dfrac{\partial\omega(z,\bar z)}{\partial z} + \dfrac{4(\lambda+2\mu)h^2}{3\mu}f''(z)\Big)e^{i\alpha} = \displaystyle\sum_{-\infty}^{+\infty} B_{n1}e^{in\alpha}, \quad r=R, \end{cases} \tag{13}$$

$$\begin{cases} 2\mu(\varphi'(z)+\overline{\varphi'(z)}) - \dfrac{3\lambda\mu}{(\lambda+2\mu)h}\chi(z,\bar z) \\ +2\mu\Big(\dfrac{\lambda h}{2(\lambda+\mu)}\dfrac{\partial^2\chi(z,\bar z)}{\partial\bar z^2} - z\overline{\varphi''(z)} - \overline{\psi'(z)}\Big)e^{-2i\alpha} = \displaystyle\sum_{-\infty}^{+\infty} A_{n2}e^{in\alpha}, \quad r=R, \\ \Big(\mu\dfrac{\partial\chi(z,\bar z)}{\partial\bar z} + \dfrac{2\lambda\mu h}{3(3\lambda+2\mu)}\overline{\varphi''(z)}\Big)e^{-i\alpha} \\ +\Big(\mu\dfrac{\partial\chi(z,\bar z)}{\partial z} + \dfrac{2\lambda\mu h}{3(3\lambda+2\mu)}\varphi''(z)\Big)e^{i\alpha} = \displaystyle\sum_{-\infty}^{+\infty} B_{n2}e^{in\alpha}, \quad r=R. \end{cases} \tag{14}$$

In this case the analytic functions

Inside of the domain the analytic functions $f'(z)$, $g'(z)$, $\varphi'(z)$, $\psi'(z)$ and the metaharmonic functions $\omega(z,\bar{z})$, $\chi(z,\bar{z})$ are represented as a series:

$$f'(z)=\sum_{n=0}^{+\infty}\frac{a_n}{z_n},\quad g'(z)=\sum_{n=0}^{+\infty}\frac{b_n}{z_n}, \tag{15}$$

$$\varphi'(z)=\sum_{n=0}^{+\infty}\frac{c_n}{z_n},\quad \psi'(z)=\sum_{n=0}^{+\infty}\frac{d_n}{z_n}, \tag{16}$$

$$\omega(z,\bar{z})=\sum_{-\infty}^{+\infty}\alpha_n K_n(\gamma r)e^{in\alpha}, \tag{17}$$

$$\chi(z,\bar{z})=\sum_{-\infty}^{+\infty}\beta_n K_n(\nu r)e^{in\alpha}, \tag{18}$$

where $K_n(\cdot)$ is the modified Bessel function of the second kind of n-th order.

In the boundary conditions (13) we substitute the corresponding expressions (15), (17) and compare the coefficients at identical degrees. We obtain the following system of equations

$$\begin{cases}
(\lambda+\mu)\dfrac{a_n}{R^n}-\dfrac{\mu i h\gamma^2}{6}K_{-n+2}(\gamma R)\alpha_{-n}=A_{-n1},\;\; n\geq 3\\
\dfrac{i\mu\gamma}{4}\Big(K_{n-1}(\gamma R)-K_{n+1}(\gamma R)\Big)\alpha_n-\dfrac{2(\lambda+2\mu)hn}{3}\dfrac{\bar{a}_n}{R^{n+1}}=B_{n1},\\
\left[(\lambda+\mu)\dfrac{1}{R^n}+2\mu n\dfrac{1}{R^n}+\dfrac{8(\lambda+2\mu)}{3}\dfrac{n(n+1)}{R^{n+2}}\right]\bar{a}_n\\
+2\mu\dfrac{\bar{b}_{n+2}}{R^{n+2}}-\dfrac{\mu i h\gamma^2}{6}K_{n+2}(\gamma R)\alpha_n=\bar{A}_{n1},\;\; n\geq 0\\
(\lambda+\mu)\dfrac{a_1}{R}+2\mu\dfrac{b_1}{R}-\dfrac{\mu i h\gamma^2}{6}K_1(\gamma R)\alpha_{-1}=A_{-11},\\
(\lambda+\mu)\dfrac{a_2}{R^2}+2\mu\bar{b}_0-\dfrac{\mu i h\gamma^2}{6}K_0(\gamma R)\alpha_{-2}=A_{-21}.
\end{cases} \tag{19}$$

From Conditions at infinity we have

$$a_0=\Gamma,\quad b_0=\Gamma', \tag{20}$$

where Γ, Γ' are known quantities, specifying the stress distribution at infinity (It is also assumed that a_0 is a real value).

We use the condition of single-valuedness of the displacements which in the present case is expressed as

$$a_1+\bar{b}_1=0. \tag{21}$$

Now by substituting (16), (18) into (14) obtain the system of algebraic equations:

$$\begin{cases} 2\mu\dfrac{c_n}{R^n} + \left(\dfrac{3\lambda\mu}{4(\lambda+2\mu)}K_{-n+2}(\nu R) - \dfrac{3\lambda\mu}{(\lambda+2\mu)h}K_{-n}(\nu R)\right)\beta_{-n} \\ = A_{-n2}, \quad n \geq 3 \\ 2\mu(n+1)\dfrac{\bar{c}_n}{R^n} + \left(\dfrac{3\lambda\mu}{4(\lambda+2\mu)}K_{n+2}(\nu R) - \dfrac{3\lambda\mu}{(\lambda+2\mu)h}K_n(\nu R)\right)\beta_n \\ -2\mu\frac{\bar{d}_{n+2}}{R^{n+2}} = A_{n2}, \quad n \geq 0 \\ -\dfrac{\mu\nu}{2}\Big(K_{n+1}(\nu R) + K_{n-1}(\nu R)\Big)\beta_n - \dfrac{2\lambda\mu h n}{3(3\lambda+2\mu)}\dfrac{\bar{c}_n}{R^{n+1}} = B_{n2}, \\ 2\mu\dfrac{c_1}{R} + \left(\dfrac{3\lambda\mu}{4(\lambda+2\mu)}K_1(\nu R) - \dfrac{3\lambda\mu}{(\lambda+2\mu)h}K_{-1}(\nu R)\right)\beta_{-1} \\ -2\mu\frac{\bar{d}_1}{R} = A_{-21}, \\ 2\mu\dfrac{c_2}{R^2} + \left(\dfrac{3\lambda\mu}{4(\lambda+2\mu)}K_0(\nu R) - \dfrac{3\lambda\mu}{(\lambda+2\mu)h}K_{-2}(\nu R)\right)\beta_{-2} \\ -2\mu\bar{d}_0 = A_{-22}. \end{cases} \tag{22}$$

From Conditions at infinity we have

$$c_0 = \Gamma^1, \quad d_0 = \Gamma^2, \tag{23}$$

where Γ^1, Γ^2 are known quantities, specifying the stress distribution at infinity (It is also assumed that c_0 is a real value).

We use the condition of single-valuedness of the displacements which in the present case is expressed as

$$\frac{5\lambda+6\mu}{3\lambda+2\mu}c_1 + \bar{d}_1 = 0. \tag{24}$$

The coefficients a_n, b_n, c_n, d_n, α_n and β_n are found by solving (19)-(24).

It is easy to prove that the absolute and uniform convergence of the series obtained in the circle (including the contours) when the functions set on the boundaries have sufficient smoothness.

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