BASIC PROBLEMS OF THERMOELASTICITY WITH MICROTEMPERATURES FOR THE CIRCLE

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Abstract

The present paper is devoted to the explicit solution of the Dirichlet type BVP for an elastic circle with microtemperatures. The regular solution of the system of equations for an isotropic materials with microtemperatures is constructed by means of the elementary (harmonic, bi-harmonic and meta-harmonic) functions. The Dirichlet type BVP for a circle is solved explicitly. The obtained solutions are presented as absolutely and uniformly convergent series.

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1 Introduction

The linear theory of thermoelasticity for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was established by Grot [1]. Iesan and Quintanilla have formulated the boundary value problems and presented an uniqueness result and a solution of the Boussinesq-Somigliana-Galerkin type [2]. The fundamental solutions of the equations of the three-dimensional (3D) theory of thermoelasticity with microtemperatures were constructed by Svanadze in [3]. The representations of the Galerkin type and general solutions of the system of static of the above theory were obtained by Scalia, Svanadze, and Tracina [4]. The linear theory for microstretch elastic materials with microtemperatures was constructed by Iesan [5], where the uniqueness and existence theorems in the dynamical case for isotropic materials are proved.

Some works of the 2D and 3D theories of elasticity for materials with microstructures can be seen in [6-25], in which give the main results and bibliographical data.

The present paper is devoted to the explicit solution of the Dirichlet type BVP for an elastic circle with microtemperatures. The regular solution of the system of equations for an isotropic materials with microtemperatures is constructed by means of the elementary (harmonic, bi-harmonic and meta-harmonic) functions. The Dirichlet type BVP for a circle is solved explicitly. The obtained solutions are presented as absolutely and uniformly convergent series.

2 Basic Equations. Boundary Value Problems.

We consider an isotropic elastic circle D bounded by the circumference S with center at the origin and radius R. Let us assume that the domain D is filled with an isotropic thermoelastic materials with microtemperatures. Let $\mathbf{x} = (x_1, x_2) \in D$.

The governing homogeneous system of the theory of thermoelasticity with microtemperatures has the form [1]-[3]

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\text{grad div}\mathbf{u} - \beta\text{grad}\theta = 0 \tag{1}$$

$$k_6\Delta\mathbf{w} + (k_4+k_5)\text{grad div}\mathbf{w} - k_3\text{grad}\theta - k_2\mathbf{w} = 0 \tag{2}$$

$$k\Delta\theta + k_1\text{div}\mathbf{w} = 0 \tag{3}$$

where $\mathbf{u} := (u_1, u_2)^\top$ denotes the displacement vector, $\mathbf{w} := (w_1, w_2)^\top$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 $(T_0 > 0)$ by the natural state (i.e. by the state of the absence of loads), λ, μ, β, k, k_j, $j = 1, ..., 6$, are constitutive coefficients, Δ is the 2D Laplace operator. The superscript $(.)^\top$ denotes transposition operation.

For the equations (1-3) we formulate the following BVP:

Problem 1: *Find a regular solution $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \theta)$ to the equations (1-3) in the domain D satisfying the following boundary conditions on S :*

$$\mathbf{u}^+ = \mathbf{f}^+(\mathbf{z}), \quad \mathbf{w}(\mathbf{x})^+ = \mathbf{F}^+(\mathbf{z}), \quad \theta^+ = f_3^+(\mathbf{z}),$$

where the vector-functions $\mathbf{F}(\mathbf{z})$, $\mathbf{f}(\mathbf{z},)$ and the function f_3, are prescribed functions on S, at $\mathbf{z}$. The symbol $\mathbf{U}^+$ denotes the limits of $\mathbf{U}(\mathbf{x})$ on $\mathbf{z} \in S$ from D

$$\mathbf{U}^+(\mathbf{z}) = \lim_{D\ni\mathbf{x}\to\mathbf{z}\in S} \mathbf{U}(\mathbf{x}).$$

3 Expansion of regular solutions of the system (1), (3).

In [9] it is proved that the regular solution of the system (2),(3) can be represented in the form

$$\mathbf{w}(\mathbf{x}) = -\frac{k}{k_1}\mathrm{grad}\,\vartheta_1(\mathbf{x}) - \frac{k_3}{k_2}\mathrm{grad}\,\vartheta(\mathbf{x}) + \overset{2}{\mathbf{w}} \tag{4}$$

$$\theta(\mathbf{x}) = \vartheta_1(\mathbf{x}) + \vartheta(\mathbf{x}) \tag{5}$$

where

$$\Delta\vartheta = 0 \quad (\Delta - s_1^2)\vartheta_1 = 0 \tag{6}$$

$$(\Delta - s_2^2)\overset{2}{\mathbf{w}} = 0, \quad \mathrm{div}\overset{2}{\mathbf{w}} = 0. \tag{7}$$

$$s_1^2 := \frac{k_2 k - k_1 k_3}{k_7 k} > 0, \quad s_2^2 := \frac{k_2}{k_6} > 0. \tag{8}$$

From (1) we find

$$\mu_0\Delta\mathrm{div}\mathbf{u} = \beta\Delta\theta = \beta s_1^2\vartheta_1. \tag{9}$$

$$\mathrm{div}\mathbf{u} = \psi + \frac{\beta}{\mu_0}\vartheta_1, \tag{10}$$

where $\Delta\psi = 0$

From (1) for $\mathbf{u}$, after obvious transformations, we get the following nonhomogeneous equation

$$\Delta\mathbf{u} = -\mathrm{grad}\left[\frac{\lambda+\mu}{\mu}\psi - \frac{\beta}{\mu}\vartheta - \frac{\beta}{\mu_0}\vartheta_1\right]. \tag{11}$$

The general solution of the equation (11) has the form

$$\mathbf{u} = \mathbf{\Psi} + \mathbf{u}_0,$$

where $\mathbf{\Psi}$ is an arbitrary harmonic function, $\mathbf{u}_0$ is a particular solution of equation (11)

$$\mathbf{u}_0 = -\mathrm{grad}\left[\frac{\lambda+\mu}{\mu}\psi_0 - \frac{\beta}{\mu}\vartheta_0 - \frac{\beta}{s_1^2\mu_0}\vartheta_1\right], \tag{12}$$

the functions Ψ, ψ_0 and ϑ_0 are chosen such that

$$\Delta\mathbf{\Psi} = 0, \ \ \Delta\vartheta_0 = \vartheta, \quad \Delta\psi_0 = \psi, \quad \mathrm{div}\mathbf{\Psi} = \frac{\mu_0}{\mu}\psi - \frac{\beta}{\mu}\vartheta.$$

From the above reasoning we have proved the following theorem:

Theorem 1. *The regular solution $\boldsymbol{U} := (\boldsymbol{u}, \boldsymbol{w}, \theta)$ of the equations(1) -(3) admits in the domain of regularity a representation*

$$\mathbf{u} = \mathbf{\Psi} - \mathrm{grad}\left[\frac{\lambda+\mu}{\mu}\psi_0 - \frac{\beta}{\mu}\vartheta_0 - \frac{\beta}{s_1^2\mu_0}\vartheta_1\right],$$

$$\mathbf{w}(\mathbf{x}) = -\mathrm{grad}\left[\frac{k_3}{k_2}\vartheta(\mathbf{x}) + \frac{k}{k_1}\vartheta_1(\mathbf{x})\right] + \overset{2}{\mathbf{w}}(\mathbf{x}) \tag{13}$$

$$\theta(\mathbf{x}) = \vartheta_1(\mathbf{x}) + \vartheta(\mathbf{x}),$$

where

$$\Delta\vartheta = 0, \quad (\Delta - s_1^2)\vartheta_1 = 0, \qquad \Delta\mathbf{\Psi} = 0, \quad (\Delta - s_2^2)\overset{\mathbf{2}}{\mathbf{w}} = 0,$$

$$\Delta\vartheta_0 = \vartheta, \quad \Delta\psi_0 = \psi, \quad \Delta\psi = 0, \quad \mathrm{div}\mathbf{\Psi} = \frac{\mu_0}{\mu}\psi - \frac{\beta}{\mu}\vartheta,$$

$$\mathrm{div}\overset{\mathbf{2}}{\mathbf{w}} = 0, \quad \mathrm{div}\mathbf{u} = \psi + \frac{\beta}{\mu_0}\vartheta_1, \quad \mathrm{div}\mathbf{w} = -\frac{k}{k_1}s_1^2\theta_1(\mathbf{x}).$$

4 Solution of Problem 1 for the Circle

Let us introduce the polar coordinates

$$x_1 = \rho\cos\eta, \quad x_2 = \rho\sin\eta, \quad \rho = \sqrt{x_1^2 + x_2^2}, \quad 0 \le \eta \le 2\pi,$$

First of all we find the functions $\vartheta, \vartheta_1, \psi$.

We are looking for a solution of the system (1)-(3) in the form (13),where

$$\vartheta(\mathbf{x}) = \frac{E_0}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^n (E_n\cos n\eta + M_n\sin n\eta), \quad \rho < R.$$

$$\vartheta_1(\mathbf{x}) = \frac{C_{10}}{2}J_0(is_1\rho) + \sum_{n=1}^{\infty} J_n(is_1\rho)(C_{1n}\cos n\eta + D_{1n}\sin n\eta), \quad \rho < R.$$

$$\psi(\mathbf{x}) = \frac{A_0'}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^n (A_n'\cos n\eta + B_n'\sin n\eta), \quad \rho < R,$$

$$\mathbf{\Psi}(\mathbf{x}) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^n (A_n\cos n\eta + B_n\sin n\eta), \quad \rho < R.$$

$$\overset{\mathbf{2}}{\mathbf{w}}(\mathbf{x}) = \frac{C_{10}}{2}J_0(is_2\rho) + \sum_{n=1}^{\infty} J_n(is_2\rho)(C_{1n}'\cos n\eta + D_{1n}'\sin n\eta), \tag{14}$$

We introduce the following notations:

$$\mathrm{div}\boldsymbol{f} = h_1, \quad \mathrm{div}\boldsymbol{F} = h_2, \quad when \;\; \rho = R.$$

In what follows we assume that the functions $h_k, \quad k = 1, 2$ can be expanded in the Fourier series.

On the basis of equations $\Delta\vartheta_0 = \vartheta$, and $\Delta\psi_0 = \psi$, the functions ϑ_0, and ψ_0 are represented in the following form

$$\vartheta_0(\mathbf{x}) = \frac{\rho^2 E_0}{8} + \frac{\rho^2}{4}\sum_{n=1}^{\infty}\frac{1}{n+1}\left(\frac{\rho}{R}\right)^n (E_n\cos n\eta + M_n\sin n\eta), \quad \rho < R.$$

$$\psi_0(\mathbf{x}) = \frac{\rho^2 A'_0}{8} + \frac{\rho^2}{4}\sum_{n=1}^{\infty}\frac{1}{n+1}\left(\frac{\rho}{R}\right)^n (A'_n\cos n\eta + B'_n\sin n\eta), \quad \rho < R, \tag{15}$$

Making use the latest notations, from (13), passing to the limit as $\rho \longrightarrow R$, we obtain the following system of equations:

$$\vartheta^+ + \vartheta_1^+ = f_3^+, \quad \psi^+ + \frac{\beta}{\mu_0}\vartheta_1^+ = h_1, \quad -\frac{k}{k_1}s_1^2\vartheta_1^+(\mathbf{x}) = h_2 \tag{16}$$

Solving system (16) we get

$$\begin{aligned} &\vartheta_1^+ = -\frac{k_1}{ks_1^2}h_2 = q_1, \quad \vartheta^+ = f_3^+ + \frac{k_1}{ks_1^2}h_2 = q, \\ &\psi^+ = h_1 + \frac{\beta}{\mu_0}\frac{k_1}{ks_1^2}h_2 = q_2, \end{aligned} \tag{17}$$

On the other hand taking into account (14),we obtain

$$\begin{aligned} &\frac{E_0}{2} + \sum_{n=1}^{\infty}(E_n\cos n\eta + M_n\sin n\eta) = q, \\ &\frac{C_{10}}{2}J_0(is_1R) + \sum_{n=1}^{\infty}J_n(is_1R)(C_{1n}\cos n\eta + D_{1n}\sin n\eta) = q_1, \\ &\frac{A'_0}{2} + \sum_{n=1}^{\infty}(A'_n\cos n\eta + B'_n\sin n\eta) = q_2, \end{aligned} \tag{18}$$

where E_k, M_k, ,... are the Fourier coefficients of the functions q and q_j

respectively.

$$E_k = \frac{1}{\pi}\int_0^{2\pi} q(\eta)\cos k\eta d\eta, \quad M_k = \frac{1}{\pi}\int_0^{2\pi} q(\eta)\sin k\eta d\eta,$$

$$C_{1n} = \frac{1}{\pi J_n(is_1 R)}\int_0^{2\pi} q_1(\eta)\cos k\eta d\eta, \quad D_{1n} = \frac{1}{\pi J_n(is_1 R)}\int_0^{2\pi} q_1(\eta)\sin k\eta d\eta,$$

$$A'_k = \frac{1}{\pi}\int_0^{2\pi} q_2(\eta)\cos k\eta d\eta, \quad B'_k = \frac{1}{\pi}\int_0^{2\pi} q_2\eta)\sin k\eta d\eta, \tag{19}$$

The substitution $E_k,\ M_k,\ E'_k, ...$ from (19) into (14) gives

$$\vartheta(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 2\sum_{k=1}^{\infty}\left(\frac{\rho}{R}\right)^k \cos k(\eta-\eta_0)\right] q(\eta)d\eta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-\rho^2)q(\eta)d\eta}{R^2 - 2R\rho\cos(\eta-\eta_0)+\rho^2},$$

$$\vartheta_1(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{J_0(is_1\rho)}{J_0(is_1 R)} + 2\sum_{k=1}^{\infty}\frac{J_k(is_1\rho)}{J_k(is_1 R)}\cos k(\eta-\eta_0)\right] q_1(\eta)d\eta, \tag{20}$$

$$\psi(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 2\sum_{k=1}^{\infty}\left(\frac{\rho}{R}\right)^k \cos k(\eta-\eta_0)\right] q_2(\eta)d\eta =$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-\rho^2)q_2(\eta)d\eta}{R^2 - 2R\rho\cos(\eta-\eta_0)+\rho^2},$$

$$\vartheta_0(\mathbf{x}) = \frac{\rho^2}{8\pi}\int_0^{2\pi}\left[1 + 2\sum_{k=1}^{\infty}\frac{1}{k+1}\left(\frac{\rho}{R}\right)^k \cos k(\eta-\eta_0)\right] q(\eta)d\eta.$$

$$\psi_0(\mathbf{x}) = \frac{\rho^2}{8\pi}\int_0^{2\pi}\left[1 + 2\sum_{k=1}^{\infty}\frac{1}{k+1}\left(\frac{\rho}{R}\right)^k \cos k(\eta-\eta_0)\right] q_2(\eta)d\eta.$$

Now let us find the functions $\mathbf{\Psi}$ and $\overset{2}{\mathbf{w}}$ from (13) when $\rho = R$

$$\begin{aligned}
&\mathbf{\Psi} = \mathbf{f} + \operatorname{grad}\left[\frac{\lambda+\mu}{\mu}\psi_0 - \frac{\beta}{\mu}\vartheta_0 - \frac{\beta}{s_1^2\mu_0}\vartheta_1\right] = g, \quad \rho = R \\
&\overset{2}{\mathbf{w}}(\mathbf{x}) = \mathbf{F} + \operatorname{grad}\left[\frac{k_3}{k_2}\theta(\mathbf{x}) + \frac{k}{k_1}\theta_1(\mathbf{x})\right] = g_1, \quad \rho = R.
\end{aligned} \tag{21}$$

On the other fend

$$\mathbf{\Psi}(\mathbf{x}) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos n\eta + B_n \sin n\eta) = g, \quad \rho = R.$$

$$\overset{2}{\mathbf{w}}(\mathbf{x}) = \frac{C'_{10}}{2} J_0(is_2R) + \sum_{n=1}^{\infty} J_n(is_2R)(C'_{1n}\cos n\eta + D'_{1n}\sin n\eta) = g_1, \tag{22}$$

where

$$C'_{1n} = \frac{1}{\pi J_n(is_2R)}\int_0^{2\pi} g_1(\eta)\cos k\eta d\eta, \quad D'_{1n} = \frac{1}{\pi J_n(is_2R)}\int_0^{2\pi} g_1(\eta)\sin k\eta d\eta,$$

$$A_k = \frac{1}{\pi}\int_0^{2\pi} g(\eta)\cos k\eta d\eta, \quad B_k = \frac{1}{\pi}\int_0^{2\pi} g(\eta)\sin k\eta d\eta,$$

Thus, for $\mathbf{\Psi}$ and $\overset{2}{\mathbf{w}}$ we get

$$\mathbf{\Psi}(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 2\sum_{k=1}^{\infty}\left(\frac{\rho}{R}\right)^k \cos k(\eta - \eta_0)\right] g(\eta) d\eta,$$

$$\overset{2}{\mathbf{w}}(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{J_0(is_2\rho)}{J_0(is_2R)} + 2\sum_{k=1}^{\infty}\frac{J_k(is_2\rho)}{J_k(is_2R)}\cos k(\eta - \eta_0)\right] g_1(\eta) d\eta,$$

For the obtained series to converge absolutely and uniformly it is suffices to require the following:

$$\mathbf{f}, \mathbf{F}, f_3 \in C^3(S).$$

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