THE ITERATION STAGE OF A NUMERICAL ALGORITHM FOR A TIMOSHENKO TYPE BEAM EQUATION

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Abstract

An initial boundary value problem for a differential equation describing the beam oscillation is considered. As a result of application of a projection method and a difference scheme, a nonlinear system of equations is obtained, which is solved by the Newton iteration. The convergence conditions and the iteration error estimate are studied.

Key words and phrases: Timoshenko beam equation, Newton iteration method, convergence, error estimate.

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1 Statement of the Problem

Let us consider the nonlinear differential equation

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^4 u}{\partial x^4}(x,t)-h\frac{\partial^4 u}{\partial x^2\partial t^2}(x,t)\\ -\left(\lambda+\int_0^L\left(\frac{\partial u}{\partial \xi}(\xi,t)\right)^2 d\xi\right)\frac{\partial^2 u}{\partial x^2}(x,t)=f(x,t),\\ 0<x<L,\quad 0<t\leq T.\end{gathered}\tag{1}$$

with the initial boundary conditions

$$\begin{gathered}u(x,0)=u^0(x),\quad \frac{\partial u}{\partial t}(x,0)=u^1(x),\\ u(0,t)=u(L,t)=0,\quad \frac{\partial^2 u}{\partial x^2}(0,t)=\frac{\partial^2 u}{\partial x^2}(L,t)=0,\\ 0\leq x\leq L,\quad 0\leq t\leq T,\end{gathered}\tag{2}$$

where $h > 0$ and $\lambda \geq 0$ are some given constants, $u^0(x)$, $u^1(x)$ and $f(x,t)$ are the given sufficiently smooth functions and $u(x,t)$ is the unknown function. Equation (1), which describes the oscillation of a beam is given in E. Henriques de Brito [3] and belongs to the class of equations based on the Timoshenko theory [9]. For $\lambda = 0$ (1) is derived in [6] by passing to the limit in the one-dimensional version of the von Karman system describing approximately the plane motion of a uniform prismatic beam. The existence and uniqueness of a generalized solution of an operator equation, the particular case of which is equation (1) is proved in [3] and [1].

Here we consider one of the questions of approximate algorithms for equation (1). Numerical methods for nonlinear integro-differential dynamic beam equations are investigated in [2] and [7].

2 Algorithm

a. projection method. An approximate solution of problem (1), (2) is written in the form

$$u_n(x,t) = \sum_{i=1}^{n} u_{ni}(t) \sin \frac{i\pi x}{L},$$

where the coefficients $u_{ni}(t)$ are defined by Galerkin method from the system of nonlinear differential equations

$$\begin{aligned} &\left(1 + h\left(\frac{\pi i}{L}\right)^2\right)u''_{ni}(t) + \left(\frac{\pi i}{L}\right)^2 u_{ni}(t) + \left(\lambda + \frac{L}{2}\sum_{j=1}^{n}\left(\frac{\pi i}{L}\right)^2 u_{nj}^2(t)\right) \\ &\quad \times \left(\frac{\pi i}{L}\right)^2 u_{ni}(t) = f_i(t), \\ &\quad i = 1, 2, \ldots, n, \quad 0 < t \leq T, \end{aligned} \tag{3}$$

and the initial conditions

$$u_{ni}(0) = u_i^0, \quad u'_{ni}(0) = u_i^1, \quad i = 1, 2, \ldots, n. \tag{4}$$

Here

$$f_i(t) = \frac{2}{L}\int_0^L f(x,t) \sin \frac{i\pi x}{L} dx, \quad u_i^l = \frac{2}{l}\int_0^L u^l(x) \sin \frac{i\pi x}{L} dx, \quad l = 0, 1.$$

The convergence of the Galerkin method for equation (1) is considered in [8] and [4].

b. difference scheme. Let us solve problem (3), (4) by the difference method. On the time interval $[0,T]$ we introduce the net with step $\tau = \frac{T}{M}$,

$0 < \tau < 1$, and nodes $t_m = m\tau$, $m = 0, 1, \ldots, M$. Denote by u_{ni}^m an approximate value of the function $u_{ni}(t)$ at the m-th node, i.e. for $t = t_m$, by u_{ni}^m and assume that $f_i^m = f_i(t_m)$, we use a Crank-Nicolson type scheme

$$\begin{aligned}
&\Big(1 + h\Big(\frac{i\pi}{L}\Big)^2\Big)\frac{u_{ni}^m - 2u_{ni}^{m-1} + u_{ni}^{m-2}}{\tau^2} + \Big\{\Big(\frac{i\pi}{L}\Big)^4 + \Big(\frac{i\pi}{L}\Big)^2 \\
&\times\Big[\lambda + \frac{L}{4}\sum_{j=1}^{n}\Big(\frac{j\pi}{L}\Big)^2\Big(\Big(\frac{u_{nj}^m + u_{nj}^{m-1}}{2}\Big)^2 \\
&+\Big(\frac{u_{nj}^{m-1} + u_{nj}^{m-2}}{2}\Big)^2\Big)\Big]\Big\}\frac{u_{ni}^m + 2u_{ni}^{m-1} + u_{ni}^{m-2}}{4} = f_i^{m-1}, \\
&m = 2, 3, \ldots, M, \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{5}$$

with the initial conditions

$$u_{ni}^0 = u_i^0, \quad u_{ni}^1 = u_i^0 + \tau u_i^1, \quad i = 1, 2, \ldots, n. \tag{6}$$

c. iteration method. Let us introduce the notations

$$\begin{gathered}
v_{ni}^m = \frac{u_{ni}^m - u_{ni}^{m-1}}{\tau}, \quad w_{ni}^m = \frac{i\pi}{L}\frac{u_{ni}^m + u_{ni}^{m-1}}{2}, \\
m = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, n,
\end{gathered} \tag{7}$$

and rewrite system (5), (6) as follows

$$\begin{aligned}
&\Big(1 + h\Big(\frac{i\pi}{L}\Big)^2\Big)\frac{v_{ni}^m - v_{ni}^{m-1}}{\tau} + \Big\{\Big(\frac{i\pi}{L}\Big)^3 \\
&+\tfrac{i\pi}{L}\Big[\lambda + \tfrac{L}{4}\textstyle\sum_{j=1}^{n}\Big((w_{nj}^m)^2 + (w_{nj}^{m-1})^2\Big)\Big]\Big\} \\
&\times\frac{w_{ni}^m + w_{ni}^{m-1}}{2} = f_i^{m-1}, \\
&\frac{w_{ni}^m - w_{ni}^{m-1}}{\tau} = \frac{i\pi}{L}\frac{v_{ni}^m + v_{ni}^{m-1}}{2}, \quad m = 2, 3, \ldots, M, \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{8}$$

and

$$v_{ni}^1 = u_i^1, \quad w_{ni}^1 = \frac{i\pi}{L}\Big(w_i^0 + \frac{1}{2}\tau w_i^1\Big).$$

From (9) we get

$$v_{ni}^m = -v_{ni}^{m-1} + \frac{2L}{i\pi}\frac{w_{ni}^m - w_{ni}^{m-1}}{\tau}, \quad m = 2, 3, \ldots, M. \tag{9}$$

Let us put (10) into (8). As a result for each fixed m, $m = 2, 3, \ldots, M$, we obtain the following system of discrete equations

$$\varphi_i(w_{n1}^m, w_{n2}^m, \ldots, w_{nm}^m) = 0, s \quad i = 1, 2, \ldots, n, \tag{10}$$

where

$$\varphi_i = 2\left(1+h\left(\frac{i\pi}{L}\right)^2\right)\frac{L}{i\pi}\frac{w_{ni}^m - w_{ni}^{m-1}}{\tau^2}$$
$$+\left\{\left(\frac{i\pi}{L}\right)^3 + \frac{i\pi}{L}\left[\lambda + \frac{L}{4}\sum_{j=1}^n\left(\left(w_{nj}^m\right)^2 + \left(w_{nj}^{m-1}\right)^2\right)\right]\right\}\frac{w_{ni}^m}{2}$$
$$+\left(\sum_{j=1}^n\left(w_{nj}^m\right)^2\right)\frac{i\pi}{8}w_{ni}^{m-1} + \psi_i,$$
$$\psi_i = -2\left(1+h\left(\frac{i\pi}{L}\right)^2\right)\frac{v_{ni}^{m-1}}{\tau}$$
$$+\left\{\left(\frac{i\pi}{i}\right)^3 + \frac{i\pi}{L}\left[\lambda + \frac{L}{4}\sum_{j=1}^n\left(w_{nj}^{m-1}\right)^2\right]\right\}\frac{w_{ni}^{m-1}}{2} - f_i^{m-1},$$
$$i = 1,2,\ldots,n.$$

We will solve systems (11) node-by-node. Assuming that at the $(m-1)$th node the solution, i.e. w_{ni}^{m-1}, $i=1,2,\ldots,n$ has been obtained, to find w_{ni}^m, $i=1,2,\ldots,n$, we use an iteration method, which will be chosen below. For the sake of simplicity, we suppose, that the iteration method at each node will be applied until it becomes possible to neglect the error of the final iteration approximation. After finding w_{ni}^m, v_{ni}^m is calculated by means of (10). Applying v_{ni}^m, w_{ni}^m we obtain the sought values of u_{ni}^m by using the formula

$$u_{ni}^m = \frac{L}{i\pi}w_{ni}^m + \frac{1}{2}\tau v_{ni}^m, \quad i=1,2,\ldots,m,$$

which follows from (7).

To solve system (11) we rewrite it in the vector form

$$\varphi(\mathbf{w_n^m}) = \mathbf{0}, \tag{11}$$

where $\mathbf{w_n^m} = (\mathbf{w_{ni}^m})_{\mathbf{n=1}}^{\mathbf{n}}$, $\mathbf{0}$ is a n-dimensional zero-vector,

$$\varphi(\mathbf{w_n^m}) = \begin{pmatrix} \varphi_1(w_{n1}^m, w_{n2}^m, \ldots, w_{nn}^m) \\ \varphi_2(w_{n1}^m, w_{n2}^m, \ldots, w_{nn}^m) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_n(w_{n1}^m, w_{n2}^m, \ldots, w_{nn}^m) \end{pmatrix},$$

and apply the Newton iteration method. Denoting the k-th iteration approximation, $k = 0, 1, , \dots,$ of w_{ni}^m by w_{nik}^m we write

$$\mathbf{w_{nk+1}^m} = \mathbf{w_{nk}^m} - \mathbf{J^{-1}}(\mathbf{w_{nk}^m})\varphi(\mathbf{w_{nk}^m}), \tag{12}$$

where the vector

$$\mathbf{w_{nk}^m} = (\mathbf{w_{nik}^m})_{\mathbf{i=1}}^{\mathbf{n}}$$

and the Jacobi matrix

$$J(\mathbf{w_n^m}) = (\mathbf{J_{ij}})_{\mathbf{i,j=1}}^{\mathbf{n}}, \quad \mathbf{J_{ij}} = \frac{\partial \varphi_{\mathbf{i}}(\mathbf{w_n^m})}{\partial \mathbf{w_{nj}^m}},$$

are used.

Suppose that the initial approximation in (13) is chosen by the rule

$$\mathbf{w_{n0}^m} = \mathbf{w_n^{m-1}}. \tag{13}$$

3 Iteration Method Error

Let us introduce the vector and function norms $\|\mathbf{v}\| = \max\limits_{\mathbf{1\le i\le n}} |\mathbf{v_i}|$, $\|f(x)\| = \Big(\int\limits_0^L f^2(x)dx\Big)^{\frac{1}{2}}$, where $\mathbf{v} = (\mathbf{v_i})_{\mathbf{i=1}}^{\mathbf{n}}$, $f(x) \in L_2[0, L]$.

Applying a Kantorowich's results [5] we obtain following

Theorem. Assume the step τ in scheme (5), (6) is so small that

$$0 < \tau < \min\left(1, \frac{1}{\rho^{\frac{1}{2}}}\left(\frac{2}{\alpha_0\alpha_1}\right)^{\frac{1}{4}}\right),$$

$$s_0 = 2c_0\tau^4 n \prod_{l=0}^{1}\Bigg(lc_5(n+1) + \tau^{2l}\Bigg(c_{5l+1}\frac{(n+1)^{2l+\frac{3}{2}}}{\tau} + \sum_{p=0}^{2} c_{5l+p+2}(n+1)^{2l+\frac{1}{2}(p^2+1)}\Bigg)\Bigg) \le 1.$$

Then for the initial approximation (14) Newton iteration method (13) converges and the limit vector $\mathbf{w_n^m} = \lim\limits_{\mathbf{k\to\infty}} \mathbf{w_{nk}^m}$ is the unique solution of system (12) in the domain $\{\mathbf{w_n} \in \mathbb{R}^{\ltimes} | \, \|\mathbf{w_n} - \mathbf{w_{n0}^m}\| \le \mathbf{2q_0}\}$, $q_0 = \Big(c_1\frac{(n+1)^{\frac{3}{2}}}{\tau} + c_2(n+1)^{\frac{1}{2}} + c_3(n+1) + c_4(n+1)^{\frac{5}{2}}\Big)\tau^2$. The error estimate

$$\|\mathbf{w_{nk}^m} - \mathbf{w_n^m}\| \le \mathbf{q_0 s_0^{2^k-1}}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{k-1}}, \quad \mathbf{k = 1, 2, \dots},$$

is fulfilled for the iteration method. Here

$$\alpha_0 = (e_1 + e_2)\exp\Big(\frac{2\mathrm{T}}{1-\tau}\Big), \quad \alpha_1 = \frac{2}{L}\Big((\lambda^2 + \alpha_0 L)^{\frac{1}{2}} - \lambda\Big),$$

$$e_1 = \frac{4}{L}\Big[\lambda\|u^{0\prime}(x)\|^2 + \|u^{0\prime\prime}(x)\|^2 + \frac{1}{2}\|u^1(x)\|^2 + h\|u^{1\prime}(x)\|^2\Big]$$

$$+\frac{1}{L}\Big[\tau^2\Big(\lambda\|u^{1\prime}(x)\|^2 + \|u^{1\prime\prime}(x)\|^2\Big) + \Big(\|u^{0\prime}(x)\|^2 + \frac{1}{4}\tau^2\|u^{1\prime}(x)\|^2\Big)^2\Big],$$

$$e_2 = \Big((1-\tau)L\Big(1 + h\Big(\frac{\pi}{L}\Big)^2\Big)\Big)^{-1}$$

$$\times\Big(\int_0^T\int_0^L f^2(x,t)dxdt + \tau^2\frac{\mathrm{T}}{12}\max_{0\le t\le T}\Big|\frac{d^2}{dt^2}\int_0^L f^2(x,t)dx\Big|\Big),$$

$$c_0 = \frac{1}{2\rho}\Big(1 + \frac{\tau^2}{2\rho}(1+\alpha_0)L\Big(\frac{\alpha_0}{6}\Big)^{\frac{1}{2}}\Big(1 - \tau^2\frac{1}{\rho}\Big(\frac{\alpha_0\alpha_1}{2}\Big)^{\frac{1}{2}}\Big)^{-1}\Big),$$

$$\rho = \begin{cases} L\pi^{-1} + h\pi L^{-1}, & \text{if} \quad 0 < L(\pi\sqrt{h})^{-1} \le 1, \\ 2h, & \text{if} \quad 1 \le L(\pi\sqrt{h})^{-1} \le n, \\ L(n\pi)^{-1} + hn\pi L^{-1}, & \text{if} \quad n \le L(\pi\sqrt{h})^{-1}, \end{cases}$$

$$c_1 = 2\Big(\frac{1}{3}\alpha_0\max\Big(1, h\Big(\frac{\pi}{L}\Big)^2\Big)\Big)^{\frac{1}{2}}c_0,$$

$$c_2 = \Big(\alpha_0^{\frac{1}{2}}\Big(\lambda + \alpha_1\frac{L}{2}\Big) + \frac{2}{L}\max_{0\le t\le T}\Big(\int_0^L f^2(x,t)dx\Big)^{\frac{1}{2}}\Big)c_0,$$

$$c_3 = \Big(\frac{2}{3}\alpha_0\Big(\lambda + \alpha_1\frac{L}{2}\Big)\Big)^{\frac{1}{2}}\frac{\pi}{L}c_0, \quad c_4 = \Big(\frac{1}{5}\alpha_0\Big)^{\frac{1}{2}}\Big(\frac{\pi}{L}\Big)^2 c_0,$$

$$c_5 = \frac{15}{16}\alpha_0^{\frac{1}{2}}L, \quad c_l = \frac{\pi}{2}c_{l-5}, \quad l = 6,7,8,9.$$

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