APPROXIMATE SOLUTION SOME MIXED BOUNDARY VALUE PROBLEMS OF THE PLANE MOMENT THEORY OF ELASTICITY

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Abstract

In the paper some plane mixed boundary value problems of the Cosserat theory are considered. The domain is the square with circular hole. Some sides of the square are rigidly clamped. In the other sides stresses and moment stresses are given. The hole contours are free from the action of external stresses. The formulated problems are solved approximately by using the method of fundamental solutions.

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1 Introduction

In the work [1] an algorithm was proposed for the approximate solution of plane boundary value problems of the moment theory of elasticity[2-4]. In the same work with the help of this algorithm, the problems of stress concentration in square domain with a circular hole were solved, when stresses and moment stresses are given on the boundary of the domain. In the present paper, using the method of fundamental solutions [5-6] and the results of [1], approximate solutions of mixed boundary value problems are constructed, when the components of displacement and rotation are given on some sides of the square.

2 Two Dimensional System of Equilibrium Equations of Cosserat Theory and its General solution

Let $Oxyz$ be a Cartesian coordinate system with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Let the cylindrical domain Ω occupied by a Cosserat elastic medium be in the plane-deformed state which is parallel to the Oxy plane. Then the displacement vector $\mathbf{u}$ rotation vector $\mathbf{w}$ the form

$$\mathbf{u} = u(x,y)\mathbf{i} + v(x,y)\mathbf{j}, \quad \mathbf{w} = \omega(x,y)\mathbf{k}.$$

In this case, a homogeneous static system of equilibrium equations is written in terms of displacement and rotation components as [1]

$$\begin{cases} (\mu+\alpha)\Delta u + (\lambda+\mu-\alpha)\partial_x(\partial_x u + \partial_y v) + 2\alpha\partial_y\omega = 0, \\ (\mu+\alpha)\Delta v + (\lambda+\mu-\alpha)\partial_y(\partial_x u + \partial_y v) - 2\alpha\partial_x\omega = 0, \\ (\nu+\beta)\Delta\omega + 2\alpha(\partial_x v - \partial_y u) - 4\alpha\omega = 0, \end{cases} \tag{1}$$

where λ, μ are Lame constants; α, β, ν are the constants characterizing the micro structure of the considered medium; $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}; \triangle \equiv \partial_{xx} + \partial_{yy}$ is the two-dimensional Laplace operator.

In [1], a general solution of system (1) is represented as follows

$$\begin{aligned} 2\mu u &= \tfrac{\lambda+3\mu}{\lambda+\mu}\varphi - x\partial_x\varphi - y\partial_x\psi - \partial_y\chi, \\ 2\mu v &= \tfrac{\lambda+3\mu}{\lambda+\mu}\psi - y\partial_y\psi - x\partial_y\varphi + \partial_x\chi, \\ 2\mu\omega &= \tfrac{2\mu}{\nu+\beta}\chi + \tfrac{\lambda+2\mu}{\lambda+\mu}(\partial_x\psi - \partial_y\varphi), \end{aligned} \tag{2}$$

where $\varphi(x,y)$ and $\psi(x,y)$ are arbitrary harmonic functions, and $\chi(x,y)$ is an arbitrary solution of the following Helmholtz equation

$$\Delta\chi - \tfrac{4\mu\alpha}{(\nu+\beta)(\mu+\alpha)}\chi = 0. \tag{3}$$

Expressions of stresses and moment stresses through functions φ, ψ and χ can see in [1].

3 Formulation and Solution of Boundary Value Problem

Let domain Ω be a square with a circular hole $\Omega = \Omega_1 \setminus \overline{\Omega}_0$, where $\Omega_1 = \{(x,y)| -4 < x < 4, -4 < y < 4\}$, $\Omega_0 = \{(x,y)|x^2+y^2 < 4\}$. Let $\mathbf{l}$ the unit outward normal vector of the contour of hole, $\mathbf{k} \perp \mathbf{l}$ and $\mathbf{l} \times \mathbf{s} = \mathbf{k}$ (Figure 1).

It is assumed that the considered domain is filled with a non-symmetric elastic medium having the following elastic constants: $\lambda = 82.098765432$ GPa, $\mu = 35.185185185$ GPa, $\alpha = 7.037037036$ GPa, $\nu+\beta = 0.7037037036$ GN. These data correspond to the elastic characteristics of brass.

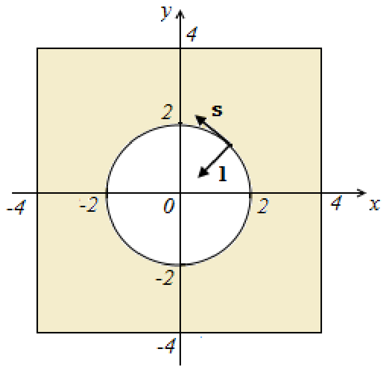

Fig. 1. The considered domain Ω

Problem 1. We consider a boundary value problem for system (1) with the following boundary conditions (Fig. 2):

$$\begin{aligned} &x=-4,\ -4<y<4: \quad u=0, v=0, \omega=0,\\ &x=4,\ -4<y<4: \quad \sigma_{xx}=1.0, \sigma_{xy}=0, \mu_{xz}=0,\\ &y=\pm 4,\ -4<x<4: \quad \sigma_{yy}=0, \sigma_{yx}=0, \mu_{yz}=0,\\ &x^2+y^2=4: \quad \sigma_{ll}=0,\ \sigma_{ls}=0,\ \mu_{lz}=0, \end{aligned} \tag{4}$$

where $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}, \sigma_{yx}$ are stress tensor components; μ_{xz}, μ_{yz} are moment stress tensor components;

$$\sigma_{ll}=\sigma_{xx}\cos^2\vartheta+(\sigma_{xy}+\sigma_{yx})\sin\vartheta\cos\vartheta+\sigma_{yy}\sin^2\vartheta,$$

$$\sigma_{ls}=(\sigma_{yy}-\sigma_{xx})\sin\vartheta\cos\vartheta+\sigma_{xy}\cos^2\vartheta-\sigma_{yx}\sin^2\vartheta,$$

$$\mu_{lz}=\mu_{xz}\cos\vartheta+\mu_{yz}\sin\vartheta,$$

where ϑ be the angle between the vector **l** and the positive direction of the $Ox-$axis.

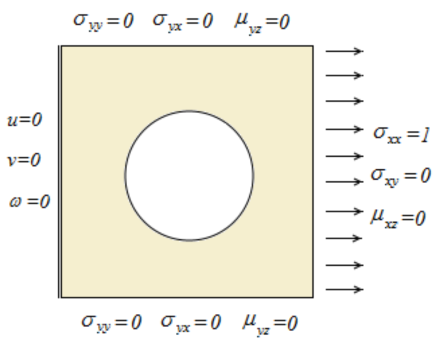

Fig. 2. The loaded domain Ω

To construct an approximate solution of the formulated problem, we use the method of fundamental solutions and the algorithm proposed in the work [1]. In the latter work, the approximate solution is constructed by using a general representation of the solutions (2) of system (1).

An approximate solution is sought in the form

$$
\begin{aligned}
\bar{u}(x,y) = \sum_{j=1}^{88} & \left(\tfrac{\lambda+3\mu}{\lambda+\mu} \ln \sqrt{(x-\xi_j)^2+(y-\eta_j)^2} - \tfrac{(x-\xi_j)^2}{(x-\xi_j)^2+(y-\eta_j)^2} \right) a_j \\
& - \tfrac{(x-\xi_j)(y-\eta_j)}{(x-\xi_j)^2+(y-\eta_j)^2} b_j + \tfrac{y-\eta_j}{\sqrt{(x-\xi_j)^2+(y-\eta_j)^2}} K_1(\gamma\sqrt{(x-\xi_j)^2+(y-\eta_j)^2}) c_j, \\
\bar{v}(x,y) = \sum_{j=1}^{88} & \left(\tfrac{\lambda+3\mu}{\lambda+\mu} \ln \sqrt{(x-\xi_j)^2+(y-\eta_j)^2} - \tfrac{(y-\eta_j)^2}{(x-\xi_j)^2+(y-\eta_j)^2} \right) b_j \\
& - \tfrac{(x-\xi_j)(y-\eta_j)}{(x-\xi_j)^2+(y-\eta_j)^2} a_j - \tfrac{x-\xi_j}{\sqrt{(x-\xi_j)^2+(y-\eta_j)^2}} K_1(\gamma\sqrt{(x-\xi_j)^2+(y-\eta_j)^2}) c_j, \\
\bar{\omega}(x,y) = \sum_{j=1}^{88} & \left(-\tfrac{\lambda+2\mu}{\lambda+\mu} \tfrac{y-\eta_j}{(x-\xi_j)^2+(y-\eta_j)^2} \right) a_j + \left(\tfrac{\lambda+2\mu}{\lambda+\mu} \tfrac{x-\xi_j}{(x-\xi_j)^2+(y-\eta_j)^2} \right) b_j \\
& + \tfrac{2\mu}{\nu+\beta} K_0(\gamma\sqrt{(x-\xi_j)^2+(y-\eta_j)^2}) c_j,
\end{aligned}
\tag{5}
$$

where $K_n\left(\gamma\sqrt{(x-\xi_j)^2+(y-\eta_j)^2}\right)$ are modified Bessel functions (Macdonald functions) of order n; $\gamma = \sqrt{\frac{4\mu\alpha}{(\nu+\beta)(\mu+\alpha)}}$; the points $(\xi_j,\eta_j), j = 1,2,...,88$, lie around the domain Ω on the circumference of radius 6.5 and inside the hole on the circumference of radius 0.5 (the asterisks in fig. 3).Thus the boundary conditions (4) are satisfied at the points $(x_i,y_i), i = 1,2,...,88$ (the points in Fig. 3). As a result we obtain a system of 264 linear algebraic equations with 264 unknowns $(a_1,...,a_{88},b_1,...,b_{88},c_1,...,c_{88})$. After solving this system, we substitute the found coefficient values into formulas (5) and define all the components of displacement and rotation.This means that all the components of stresses and moment stresses are determined by us.

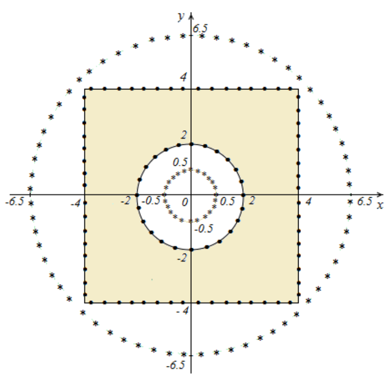

Fig. 3. The domain Ω with the marked points on the boundary and singular points around

In the case of of our problem it is of special interest to investigate the distribution of stress σ_{ss} along the hole contour. Fig. 4 show the graph of the stress σ_{ss} along the hole contour. As seen from the graph, the maximal concentration of stresses observed at the points $\vartheta = 1.4$ and $\vartheta = 4.8$ and $(\sigma_{ss})_{max} = 5$. For comparison, see [1], fig. 9.

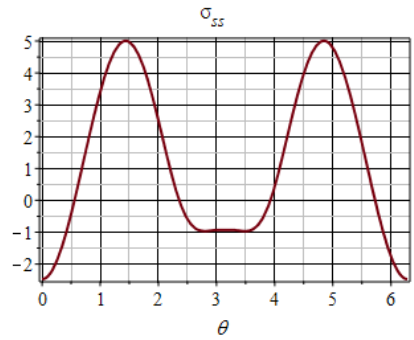

Fig. 4. Graphs of the distribution of stress σ_{ss} along the hole contour

Problem 2. We consider a boundary value problem for system (1) with the following boundary conditions (Fig. 5):

$$\begin{aligned} &x = \pm 4,\ -4 < y < 4: \quad \sigma_{xx} = 1.0, \sigma_{xy} = 0, \mu_{xz} = 0, \\ &y = \pm 4,\ -4 < x < 4: \quad v = 0, u = 0, \omega = 0, \\ &x^2 + y^2 = 4: \quad \sigma_{ll} = 0,\ \sigma_{ls} = 0,\ \mu_{lz} = 0. \end{aligned} \tag{6}$$

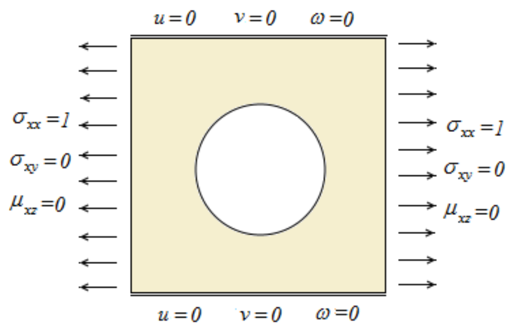

Fig. 5. Problem 2. The loaded domain Ω

Fig. 6 show the graph of the stress σ_{ss} along the hole contour in the case of problem 2. As seen from the graph, the maximal concentration of stresses observed at the points $\vartheta = 1.1$ and $\vartheta = 2.05$ and $(\sigma_{ss})_{max} = 2.65$. As was to be expected, in comparison with Problem 1, the value of stress concentration on the contour of the hole was significantly reduced.

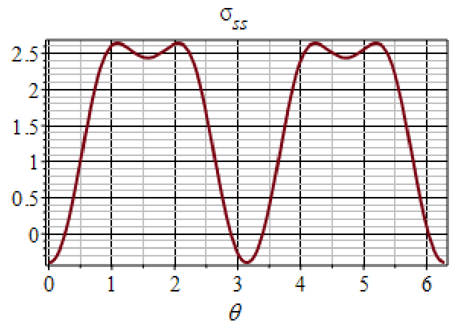

Fig. 6. Problem 2. Graphs of the distribution of stress σ_{ss}

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