ABOUT ONE PROBLEM OF POROUS COSSERAT MEDIA FOR SOLIDS WITH TRIPLE-POROSITY

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Abstract

The purpose of this paper is to consider the two-dimensional version of the linear theory of elasticity for solids with triple-porosity in the case of an elastic Cosserat medium. Using the analytic functions of a complex variable and solutions of the Helmholtz equation the second fundamental problem for the infinite plane with a circular hole are solved explicitly.

Key words and phrases: Triple-porosity, the elastic Cosserat medium.

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1 Introduction

In the past half century, the mathematical models of multi-porosity media, as originally developed for the mechanics of naturally fractured reservoirs, have found applications in many branches of civil engineering, geotechnical engineering, technology and, in recent years, biomechanics [1-5]. Significant progress has been made towards understanding and modeling of flow processes in fractured rock. However, fractured rock may be considered as a multiporous medium but the most studies have focused naturally fractured reservoirs with double and triple porosities [6-11].

The triple porosity model represents a new possibility for the study of important problems of engineering and mechanics. The intended applications of the theories of elasticity and thermoelasticity for materials with a triple porosity structure are to geological materials such as oil and gas reservoirs, rocks and soils, manufactured porous materials such as ceramics and pressed powders, and biomaterials such as bone [12, 4].

It should be noted that all the papers mentioned above dealt with a classical (symmetric) medium. We consider the problem of elasticity for solids with triple-porosity in the case of an elastic Cosserat medium [13].

2 Basic three-dimensional relations

Let an elastic body with triple-porosity occupy the domain $\overline{\Omega} \subset R^3$. Denote by (x_1, x_2, x_3) a point of the domain $\overline{\Omega}$ in the Cartesian coordinate system. Let the domain $\overline{\Omega}$ be filled with an elastic Cosserat medium having triple-porosity. The considered solid body is characterized by the displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)$, rotation vector $\boldsymbol{\omega}(\omega_1, \omega_2, \omega_3)$ and also by the fluid pressures $p_1(x_1, x_2, x_3)$, $p_2(x_1, x_2, x_3)$ and $p_3(x_1, x_2, x_3)$ occurring respectively in the pores and fissures of the porous medium.

Then a homogeneous system of static equilibrium equations is written in the form [14, 16]

$$\begin{cases} \partial_i \sigma_{ij} = 0, \\ \partial_i \mu_{ij} + \in_{jik} \sigma_{ik} = 0, \ j = 1, 2, 3 \end{cases} \quad \text{in} \quad \Omega, \tag{1}$$

where σ_{ij} are stress tensor components, μ_{ij} are moment stress tensor components, $\in_{jik}$ is the Levi-Civita symbol, $\partial_i \equiv \frac{\partial}{\partial x_i}$, the summation over the recurring index i is assumed to be done from 1 to 3.

Formulas that interrelate the stress and moment stress components, the displacement and rotation vector components and the pressures p_1,p_2, p_3 have the form

$$\begin{aligned} \sigma_{ij} &= (\lambda \operatorname{div} \boldsymbol{u} - \beta_1 p_1 - \beta_2 p_2 - \beta_3 p_3)\delta_{ij} + (\mu + \alpha)\partial_i u_j \\ &\quad + (\mu - \alpha)\partial_j u_i - 2\alpha \in_{ijk} \omega_k, \\ \mu_{ij} &= \sigma \operatorname{div} \boldsymbol{\omega} \delta_{ij} + (\nu + \beta)\partial_i \omega_j + (\nu - \beta)\partial_j \omega_i, \quad j = 1, 2, 3, \end{aligned} \tag{2}$$

where λ and μ are the Lam parameters, α, β, ν, σ are the constants characterizing the microstructure of the considered elastic medium, β_1, β_2 and β_3 are the effective stress parameters, δ_{ij} is the Kronecker delta.

In the stationary case, the values p_1, p_2 and p_3 satisfy the following system of equations

$$\begin{cases} a_1 \Delta p_1 + a_{12}(p_2 - p_1) + a_{13}(p_3 - p_1) = 0, \\ a_2 \Delta p_2 + a_{21}(p_1 - p_2) + a_{23}(p_3 - p_2) = 0, \\ a_3 \Delta p_3 + a_{31}(p_1 - p_3) + a_{32}(p_2 - p_3) = 0, \end{cases} \tag{3}$$

where a_{ij} is the fluid transfer rate between phase i and phase j, $a_1 = \frac{\kappa_1}{\mu'}$, j, $a_2 = \frac{\kappa_2}{\mu'}$, j, $a_3 = \frac{\kappa_3}{\mu'}$, (for the fluid phase, each phase i carries its respectively permeability κ_i), μ' is fluid viscosity, $\Delta \equiv \partial_{11} + \partial_{22} + \partial_{33}$ is the three-dimensional Laplace operator.

The three-dimensional system of equations (1), (2) and (3) describes the static equilibrium of a porous elastic Cosserat medium with triple-porosity. Substituting relations (2) into (1), we obtain equilibrium equations with respect to the components of the displacement and rotation vectors

$$\left\{\begin{array}{l} (\mu+\alpha)\Delta u_j + (\lambda+\mu-\alpha)\partial_j(\partial_k u_k) - 2\alpha \in_{ijk} \partial_i \omega_k \\ -\partial_j(\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3) = 0, \\ (\nu+\beta)\Delta\omega_j + (\sigma+\nu-\beta)\partial_j(\partial_k\omega_k) + 2\alpha \in_{jik} \partial_i u_k \\ -4\alpha\omega_j = 0, \quad j = 1,2,3 \end{array}\right. \quad \text{in } \Omega,$$

If to the system of equilibrium equations we add boundary conditions on the boundary $\partial\Omega$ of the domain Ω, then we can consider various classical boundary value problems.

3 The plane deformation case

From the basic three-dimensional equations we obtain the basic equations for the case of plane deformation. Let Ω be a sufficiently long cylindrical body with generatrix parallel to the Ox_3-axis. Denote by V the crosssection of this cylindrical body, thus $V \subset R^2$. In the case of plane deformation $u_3 = 0$, $\omega_1 = 0$, $\omega_2 = 0$, while the functions u_1, u_2, ω_3, p_1, p_2 and p_3 do not depend on the coordinate x_3 [17].

As follows from formulas (2), in the case of plane deformation

$$\sigma_{\alpha 3} = 0,\ \sigma_{3\alpha} = 0,\ \mu_{\alpha\beta} = 0,\ \mu_{33} = 0, \quad \alpha,\beta = 1,2.$$

Therefore the system of equilibrium equations (1) takes the form

$$\left\{\begin{array}{l} \partial_1\sigma_{11} + \partial_2\sigma_{21} = 0, \\ \partial_1\sigma_{12} + \partial_2\sigma_{22} = 0, \\ \partial_1\mu_{13} + \partial_2\mu_{23} + (\sigma_{12} - \sigma_{21}) = 0, \end{array}\right. \quad \text{in } V, \tag{4}$$

Relations (2) are rewritten as

$$\begin{array}{l} \sigma_{11} = \lambda\theta + 2\mu\partial_1 u_1 - \beta_1 p_1 - \beta_2 p_2 - \beta_3 p_3, \\ \sigma_{22} = \lambda\theta + 2\mu\partial_2 u_2 - \beta_1 p_1 - \beta_2 p_2 - \beta_3 p_3, \\ \sigma_{12} = (\mu+\alpha)\partial_1 u_2 + (\mu-\alpha)\partial_2 u_1 - 2\alpha\omega_3, \\ \sigma_{21} = (\mu+\alpha)\partial_2 u_1 + (\mu-\alpha)\partial_1 u_2 + 2\alpha\omega_3, \\ \sigma_{33} = \lambda\theta - \beta_1 p_1 - \beta_2 p_2 - \beta_3 p_3, \\ \mu_{13} = (\nu+\beta)\partial_1\omega_3, \quad \mu_{23} = (\nu+\beta)\partial_2\omega_3, \\ \mu_{31} = (\nu-\beta)\partial_1\omega_3, \quad \mu_{32} = (\nu-\beta)\partial_2\omega_3, \end{array} \tag{5}$$

where $\theta = \partial_1 u_1 + \partial_2 u_2$.

Equations (3) take the form

$$\begin{cases} a_1\Delta_2 p_1 + a_{12}(p_2 - p_1) + a_{13}(p_3 - p_1) = 0, \\ a_2\Delta_2 p_2 + a_{21}(p_1 - p_2) + a_{23}(p_3 - p_2) = 0, \\ a_3\Delta_2 p_3 + a_{31}(p_1 - p_3) + a_{32}(p_2 - p_3) = 0, \end{cases} \tag{6}$$

where $\Delta_2 = \partial_{11} + \partial_{22}$ is the Laplace operator in two dimensions.

If relations (5) are substituted into system (4), then we obtain the following system of equilibrium equations with respect to the functions u_1, u_2 and ω_3

$$\begin{cases} (\mu+\alpha)\Delta_2 u_1 + (\lambda+\mu-\alpha)\partial_1\theta + 2\alpha\partial_2\omega_3 \\ -\partial_1(\beta_1 p_1 + \beta_1 p_3 + \beta_1 p_3) = 0, \\ (\mu+\alpha)\Delta_2 u_2 + (\lambda+\mu-\alpha)\partial_2\theta + 2\alpha\partial_1\omega_3 \qquad \text{in } V, \\ -\partial_2(\beta_1 p_1 + \beta_1 p_3 + \beta_1 p_3) = 0, \\ (\nu+\beta)\Delta_2\omega + 2\alpha(\partial_1 u_2 - \partial_2 u_1) - 4\alpha\omega_3 = 0, \end{cases} \tag{7}$$

On the plane Ox_1x_2, we introduce the complex variable $z = x_1 + ix_2 = re^{i\vartheta}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2)$, $\partial_{\bar z} = 0.5(\partial_1 + i\partial_2)$, $\bar z = x_1 - ix_2$, and $\Delta_2 = 4\partial_z\partial_{\bar z}$.

To write system (4) in the complex form, the second equation of this system is multiplied by i and summed up with the first equation

$$\begin{cases} \partial_z(\sigma_{11} - \sigma_{22} + i(\sigma_{12} + \sigma_{21})) + \partial_{\bar z}(\sigma_{11} + \sigma_{22} + i(\sigma_{12} - \sigma_{21})) = 0, \\ \partial_z(\mu_{13} + i\mu_{23}) + \partial_{\bar z}(\mu_{13} - i\mu_{23}) + \sigma_{12} - \sigma_{21} = 0, \end{cases} \tag{8}$$

where by formulas (5)

$$\begin{aligned} &\sigma_{11} - \sigma_{22} + i(\sigma_{12} + \sigma_{21}) = 4\mu\partial_{\bar z} u_+, \\ &\sigma_{11} + \sigma_{22} + i(\sigma_{12} - \sigma_{21}) = 2(\lambda+\mu-\alpha)\theta + 4\alpha\partial_z u_+ - 4\alpha i\omega_3 \\ &-2(\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3), \\ &\mu_{13} + i\mu_{23} = 2(\nu+\beta)\partial_{\bar z}\omega_3, \quad \mu_{31} + i\mu_{32} = 2(\nu-\beta)\partial_{\bar z}\omega_3, \end{aligned} \tag{9}$$

$$u_+ = u_1 + iu_2, \quad \theta = \partial_z u_+ + \partial_{\bar z}\bar u_+.$$

We write equations (6)

$$\Delta_2 p - Ap = 0, \quad A = \begin{pmatrix} b_1/a_1 & -a_{12}/a_1 & -a_{13}/a_1 \\ -a_{21}/a_2 & b_2/a_2 & -a_{23}/a_2 \\ -a_{31}/a_3 & -a_{32}/a_3 & b_3/a_3 \end{pmatrix} \tag{10}$$

$$p = (p_1, p_2, p_3)^T,$$

$$b_1 = a_{12} + a_{13}, \quad b_2 = a_{21} + a_{23}, \quad b_3 = a_{31} + a_{32}.$$

If relations (9) are substituted into system (8), then system (7) is written in the complex form

$$\begin{cases} 2(\mu+\alpha)\partial_{\bar z}\partial_z u_+ + (\lambda+\mu-\alpha)\partial_{\bar z}\theta - 2\alpha i\partial_{\bar z}\omega_3 \\ -\partial_{\bar z}(\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3) = 0, \\ 2(\nu+\beta)\partial_{\bar z}\partial_z\omega_3 + \alpha i(\theta - 2\partial_z u_+) - 2\alpha\omega_3 = 0, \end{cases} \quad \text{in } V. \tag{11}$$

4 The general solution of system (10-11)

In this section, we construct the analogues of the Kolosov-Muskhelishvili formulas [17] for system (10-11).

Equations (10) imply that

$$\begin{aligned} p_1 &= f'(z) + \overline{f'(z)} + l_{11}\chi_1(z,\bar z) + l_{12}\chi_2(z,\bar z), \\ p_2 &= f'(z) + \overline{f'(z)} + l_{21}\chi_1(z,\bar z) + l_{22}\chi_2(z,\bar z), \\ p_3 &= f'(z) + \overline{f'(z)} + l_{31}\chi_1(z,\bar z) + l_{32}\chi_2(z,\bar z), \end{aligned} \tag{12}$$

where $f(z)$ is an arbitrary analytic functions of a complex variable z in the domain V and $\chi_\alpha(z,\bar z)$ is an arbitrary solution of the Helmholtz equation

$$\Delta_2\chi_\alpha(z,\bar z) - \kappa_\alpha\chi_\alpha(z,\bar z) = 0,$$

κ_α are eigenvalues and (l_{11}, l_{21}, l_{31}), (l_{12}, l_{22}, l_{32}) are eigenvectors of the matrix A.

Theorem. *The general solution of the system of equations (12) is represented as follows:*

$$\begin{aligned} 2\mu u_+ &= \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}(f(z) + z\overline{f'(z)}) \\ &\quad + 2i\partial_{\bar z}\tau(z,\bar z) + \frac{4\mu}{\lambda+2\mu}\partial_{\bar z}[\delta_1\chi_1(z,\bar z) + \delta_2\chi_2(z,\bar z)], \\ 2\mu\omega_3 &= \frac{2\mu}{\nu+\beta}\tau(z,\bar z) - \frac{\varkappa+1}{2}i(\varphi'(z) - \overline{\varphi'(z)}), \end{aligned} \tag{13}$$

where $\varkappa = \dfrac{\lambda+3\mu}{\lambda+\mu}$, $\delta_\alpha := \dfrac{l_{1\alpha}}{\kappa_\alpha}\beta_1 + \dfrac{l_{2\alpha}}{\kappa_\alpha}\beta_2 + \dfrac{l_{3\alpha}}{\kappa_\alpha}\beta_3$, $\varphi(z)$ *and* $\psi(z)$ are *arbitrary analytic functions of a complex variable* z *in the domain* V, $\chi(z,\bar z)$ *is an arbitrary solution of the Helmholtz equation*

$$\Delta_2\tau(z,\bar z) - \xi^2\tau(z,\bar z) = 0,$$

where

$$\xi^2 := \frac{4\mu\alpha}{(\nu+\beta)(\mu+\alpha)} > 0.$$

Proof. We take the operator $\partial_{\bar{z}}$ out of the brackets in the left-hand part of the first equation of system (11)

$$\begin{aligned}&\partial_{\bar{z}}(2(\mu+\alpha)\partial_z u_+ + (\lambda+\mu-\alpha)\theta - 2\alpha i\omega_3\\&-(\beta_1 p_1+\beta_2 p_2+\beta_3 p_3)) = 0.\end{aligned} \tag{14}$$

Since (14) is a system of Cauchy-Riemann equations, we have

$$\begin{aligned}&2(\mu+\alpha)\partial_z u_+ + (\lambda+\mu-\alpha)\theta - 2\alpha i\omega_3\\&=(\varkappa+1)\varphi'(z)+\beta_1 p_1+\beta_2 p_2+\beta_3 p_3,\end{aligned} \tag{15}$$

where $\varphi(z)$ is an arbitrary analytic function of z.

A conjugate equation to (15) has the form

$$\begin{aligned}&2(\mu+\alpha)\partial_{\bar{z}}\bar{u}_+ + (\lambda+\mu-\alpha)\theta + 2\alpha i\omega_3\\&=(\varkappa+1)\overline{\varphi'(z)}+\beta_1 p_1+\beta_2 p_2+\beta_3 p_3,\end{aligned} \tag{16}$$

Summing up equations (15) and (16) and taking into account that

$$\theta = \partial_z u_+ + \partial_{\bar{z}}\bar{u}_+,$$

we obtain

$$\theta = \frac{1}{\lambda+\mu}(\varphi'(z)+\overline{\varphi'(z)}) + \frac{1}{\lambda+2\mu}(\beta_1 p_1+\beta_2 p_2+\beta_3 p_3). \tag{17}$$

If from equation (15) we subtract equation (16) and write the expression $i(\partial_z u_+ - \partial_{\bar{z}}\bar{u}_+)$, then we have

$$i(\partial_z u_+ - \partial_{\bar{z}}\bar{u}_+) = \frac{\varkappa+1}{2(\mu+\alpha)}i(\varphi'(z)-\overline{\varphi'(z)}) - \frac{2\alpha}{\mu+\alpha}\omega_3. \tag{18}$$

The second equation of system (11) is written as

$$4\partial_z\partial_{\bar{z}}\omega_3 - \frac{2\alpha}{\nu+\beta}i(\partial_z u_+ - \partial_{\bar{z}}\bar{u}_+)\omega_3 - \frac{4\alpha}{\nu+\beta}\omega_3 = 0. \tag{19}$$

Substituting formula (18) into formula (19) we obtain the equation

$$\Delta_2\omega_3 - \xi^2\omega_3 = \frac{\alpha(\varkappa+1)}{(\nu+\beta)(\mu+\alpha)}i(\varphi'(z)-\overline{\varphi'(z)}). \tag{20}$$

The general solution of equation (20) is written in the form

$$2\mu\omega_3 = \frac{2\mu}{\nu+\beta}\tau(z,\bar{z}) - \frac{\varkappa+1}{2}i(\varphi'(z)-\overline{\varphi'(z)}), \tag{21}$$

where $\tau(z,\bar z)$ is a general solution of the Helmholtz equation

$$\Delta_\tau - \xi^2\tau = 0. \tag{22}$$

The multiplier $\dfrac{2\mu}{\nu+\beta}$ has been introduced for convenience in writing our subsequent formulas.

Substituting formulas (17) and (21) into equation (16) and taking into account that $\tau(z,\bar z)$ is a solution of equation (22), we obtain

$$\begin{aligned} 2\mu\partial_z u_+ &= \varkappa\varphi'(z) - \overline{\varphi'(z)} + 2i\partial_z\partial_{\bar z}\tau(z,\bar z) \\ &+ \frac{\mu}{\lambda+2\mu}(\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3). \end{aligned} \tag{23}$$

From formulas (12) we find the following expression for the combination $\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3$

$$\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 = (\beta_1+\beta_2+\beta_3)(f'(z) + \overline{f'(z)}) + 4\partial_z\partial_{\bar z}[\delta_1\chi_1(z,\bar z) + \delta_2\chi_2(z,\bar z)].$$

Substituting the latter formula into (23), integrating over z we obtain formula (13) which we are proving.

Substituting expressions (13) and (14) into formulas (9), for combinations of stress tensor components we obtain the following formulas

$$\begin{aligned} &\sigma_{11} + \sigma_{22} + i(\sigma_{12} - \sigma_{21}) \\ &= 2\left[\varphi'(z) + \overline{\varphi'(z)} - 2i\partial_z\partial_{\bar z}\tau - \frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}(f'(z) + \overline{f'(z)})\right] \\ &- \frac{8\mu}{\lambda+2\mu}\partial_z\partial_{\bar z}[\delta_1\chi_1(z,\bar z) + \delta_2\chi_2(z,\bar z)], \end{aligned}$$

$$\begin{aligned} &\sigma_{11} - \sigma_{22} + i(\sigma_{12} + \sigma_{21}) = 2\left[-z\overline{\varphi''(z)} - \overline{\psi'(z)} + 2i\partial_{\bar z}\partial_{\bar z}\tau\right] \\ &+ \frac{2\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu} z\overline{f''(z)} + \frac{8\mu}{\lambda+2\mu}\partial_{\bar z}\partial_{\bar z}[\delta_1\chi_1(z,\bar z) + \delta_2\chi_2(z,\bar z)], \end{aligned}$$

$$\begin{aligned} &\sigma_{33} = \frac{\lambda}{\lambda+\mu}\left[\varphi'(z) + \overline{\varphi'(z)}\right] - \frac{8\mu}{\lambda+2\mu}\partial_z\partial_{\bar z}[\delta_1\chi_1(z,\bar z) + \delta_2\chi_2(z,\bar z)] \\ &- \frac{2\mu}{\lambda+2\mu}(\beta_1+\beta_2+\beta_3)(f'(z) + \overline{f'(z)}), \end{aligned}$$

$$\mu_{13} + i\mu_{23} = 2\partial_{\bar z}\tau + \frac{(\varkappa+1)(\nu+\beta)}{2\mu} i\overline{\varphi''(z)},$$

$$\mu_{31} + i\mu_{32} = \frac{2(\nu-\beta)}{\nu+\beta}\partial_{\bar z}\tau + \frac{(\varkappa+1)(\nu-\beta)}{2\mu} i\overline{\varphi''(z)}.$$

Thus, the general solution of a two-dimensional system of differential equations that describes the static equilibrium of a porous elastic medium with triple-porosity is represented by means of three analytic functions of a complex variable and three solutions of the Helmholtz equation. By an appropriate choice of these functions we can satisfy five independent classical boundary conditions.

Let mutually perpendicular unit vectors $\mathbf{l}$ and $\mathbf{s}$ be such that

$$\mathbf{l} \times \mathbf{s} = \mathbf{e}_3,$$

where e_3 is the unit vector directed along the x_3-axis. The vector $\mathbf{l}$ forms the angle ϑ with the positive direction of the x_1-axis. Then the displacement components $u_l = \mathbf{u} \cdot \mathbf{l}$, $u_s = \mathbf{u} \cdot \mathbf{s}$ as well as the stress and moment stress components acting on an area of arbitrary orientation are expressed by the formulas

$$\begin{aligned}
&u_l + iu_s = e^{-i\vartheta} u_+, \\
&\sigma_{ll} + i\sigma_{ls} = \frac{1}{2}\Big[\sigma_{11} + \sigma_{22} + i(\sigma_{12} - \sigma_{21}) \\
&+(\sigma_{11} - \sigma_{22} + i(\sigma_{12} + \sigma_{21}))e^{-2i\vartheta}\Big], \\
&\mu_{l3} = \frac{1}{2}\left[(\mu_{13} + i\mu_{23})e^{-i\vartheta} + (\mu_{13} - i\mu_{23})e^{i\vartheta}\right].
\end{aligned}$$

5 Solution of the second fundamental problem for the infinite plane with a circular hole

Let the origin of coordinates be at the centre of the hole of radius R.

On the boundary of the considered domain the values of pressures p_1, p_2, p_3, the displacement and rotation vectors are given.

We consider the following problem

$$p_j|_{r=R} = P_j = \sum_{-\infty}^{+\infty} A_{nj} e^{in\vartheta}, \; A_{nj} = \overline{A}_{-nj}, \; j = \overline{1,3}, \tag{24}$$

$$\begin{aligned}
2\mu u_+|_{r=R} = 2\mu(G_1 + iG_2) = \sum_{-\infty}^{+\infty} B_n e^{in\vartheta}, \\
2\mu\omega_3|_{r=R} = 2\mu G_3 = \sum_{-\infty}^{+\infty} C_n e^{in\vartheta}, \; C_n = \overline{C}_{-n}.
\end{aligned} \tag{25}$$

The analytic function $f'(z)$ and the metaharmonic functions $\chi_\alpha(z, \bar{z})$ is represented as a series

$$f'(z) = \sum_{n=0}^{\infty} a_n z^{-n}, \quad \chi_\alpha(z, \bar{z}) = \sum_{-\infty}^{+\infty} \alpha_{n\alpha} K_n(\kappa_\alpha r) e^{in\vartheta},$$

where $K_n(\cdot)$ is modified Bessel function of n-th order, and are substituted in the boundary conditions (24) we have

$$\sum_{n=0}^{\infty}\left(\frac{a_n}{R^n}e^{-in\vartheta}+\frac{\bar{a}_n}{R^n}e^{in\vartheta}\right)$$
$$+\sum_{-\infty}^{+\infty}[l_{j1}\alpha_{n1}K_n(\kappa_1 R)+l_{j2}\alpha_{n2}K_n(\kappa_2 R)]e^{in\vartheta}=\sum_{-\infty}^{+\infty}A_{nj}e^{in\vartheta},$$

Compare the coefficients at identical degrees. We obtain the following systems of equations

$$\begin{aligned}&a_0+\bar{a}_0+l_{j1}K_0(\kappa_1 R)\alpha_{01}+l_{j2}K_0(\kappa_2 R)\alpha_{02}=A_{0j},\quad j=1,2,3,\\&\frac{1}{R^n}\bar{a}_n+l_{j1}K_n(\kappa_1 R)\alpha_{n\,1}+l_{j2}K_n(\kappa_2 R)\alpha_{n\,2}=A_{nj},\quad j=1,2,3.\end{aligned}\tag{26}$$

It is also assumed that a_0 is a real value; that is, $a_0=\bar{a_0}$. The coefficients $a_n\ \alpha_{n1},\ \alpha_{n2}$, are found by solving (26).

The analytic functions $\varphi(z)$, $\psi(z)$ and the metaharmonic functions $\tau(z,\bar{z})$ are represented as series

$$\varphi'(z)=\sum_{n=0}^{\infty}b_n z^{-n},\quad \psi'(z)=\sum_{n=0}^{\infty}c_n z^{-n},\quad \tau(z,\bar{z})=\sum_{-\infty}^{+\infty}\alpha_n K_n(\xi r)e^{in\vartheta}$$

and are substituted in the boundary conditions (25) we have

$$\begin{aligned}&\varkappa\left(R\,b_0e^{i\vartheta}+\ln R\,b_1+b_1 i\vartheta-\sum_{n=2}^{\infty}\frac{b_n e^{i(1-n)\vartheta}}{(n-1)R^{n-1}}\right)\\&-\sum_{n=0}^{\infty}\frac{\bar{b}_n}{R^{n-1}}e^{i(n+1)\vartheta}-R\,\bar{c}_0e^{-i\vartheta}-\ln R\,\bar{c}_1+\bar{c}_1 i\vartheta\\&+\sum_{n=2}^{\infty}\frac{\bar{c}_n e^{i(n-1)\vartheta}}{(n-1)R^{n-1}}-i\xi\sum_{-\infty}^{\infty}\alpha_n K_{n+1}(\xi R)e^{i(n+1)\vartheta}\\&+\frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}\left[\ln R\,a_1+i\vartheta a_1-\sum_{n=2}^{\infty}\frac{a_n}{(n-1)R^{n-1}}e^{i(1-n)\vartheta}\right.\\&\left.+(a_0+\bar{a}_0)Re^{i\vartheta}+\sum_{n=1}^{\infty}\frac{\bar{a}_n}{R^{n-1}}e^{i(n+1)\vartheta}\right]-\frac{4\mu}{\lambda+2\mu}\\&\times\sum_{-\infty}^{+\infty}\left(\frac{\delta_1\kappa_1}{2}K_{n+1}(\kappa_1 R)\alpha_{n1}+\frac{\delta_2\kappa_2}{2}K_{n+1}(\kappa_2 R)\alpha_{n2}\right)e^{i(n+1)\vartheta}\\&=\sum_{-\infty}^{+\infty}B_n e^{in\vartheta},\end{aligned}\tag{27}$$

$$\frac{2\mu}{\nu+\beta}\sum_{-\infty}^{+\infty}\alpha_n K_n(\xi R)e^{in\vartheta}+\frac{\varkappa+1}{2}i\sum_{n=0}^{\infty}\left(\frac{a_n}{R^n}e^{-in\vartheta}-\frac{\bar{a}_n}{R^n}e^{in\vartheta}\right)$$
$$=\sum_{-\infty}^{+\infty}C_n e^{in\vartheta}. \tag{28}$$

Compare the coefficients at identical degrees. Then one obtains from the constant term and from those involving $e^{i\vartheta}$, $e^{-i\vartheta}$ and $e^{2i\vartheta}$ respectively

$$\begin{cases}\varkappa \ln R\,\bar{b}_1-\ln R\,c_1+i\xi K_0(\xi R)\alpha_1=\bar{B}'_0,\\ \dfrac{(\varkappa+1)i}{2R}\bar{b}_1+\dfrac{2\mu}{\nu+\beta}K_1(\xi R)\alpha_1=C_1,\end{cases} \tag{29}$$

$$\begin{cases}\varkappa R\,b_0-R\,\bar{b}_0+\dfrac{1}{R}\bar{c}_2-i\xi K_1(\xi R)\alpha_0=\bar{B}'_1,\\ \dfrac{(\varkappa+1)i}{2}(\bar{b}_0-b_0)+\dfrac{2\mu}{\nu+\beta}K_0(\xi R)\alpha_0=C_0,\end{cases} \tag{30}$$

$$\begin{cases}-\dfrac{\varkappa}{R}b_2-R\,\bar{c}_0-i\xi K_{-1}(\xi R)\alpha_{-2}=B'_{-1},\\ \dfrac{(\varkappa+1)i}{2R^2}\bar{b}_2+\dfrac{2\mu}{\nu+\beta}K_2(\xi R)\alpha_2=C_2,\end{cases} \tag{31}$$

where

$$B'_0=B_0-\frac{\mu(\beta_1+\beta_2+\beta_3)\ln R}{\lambda+2\mu}a_1$$
$$+\frac{4\mu}{\lambda+2\mu}\left(\frac{\delta_1\kappa_1}{2}K_0(\kappa_1 R)\alpha_{-11}+\frac{\delta_2\kappa_2}{2}K_0(\kappa_2 R)\alpha_{-12}\right),$$
$$B'_1=B_1-\frac{\mu(\beta_1+\beta_2+\beta_3)R}{\lambda+2\mu}(a_0+\bar{a}_0)$$
$$+\frac{4\mu}{\lambda+2\mu}\left(\frac{\delta_1\kappa_1}{2}K_1(\kappa_1 R)\alpha_{01}+\frac{\delta_2\kappa_2}{2}K_1(\kappa_2 R)\alpha_{02}\right),$$
$$B'_{-1}=B_{-1}+\frac{\mu(\beta_1+\beta_2+\beta_3)}{(\lambda+2\mu)R}a_2$$
$$+\frac{4\mu}{\lambda+2\mu}\left(\frac{\delta_1\kappa_1}{2}K_{-1}(\kappa_1 R)\alpha_{-21}+\frac{\delta_2\kappa_2}{2}K_{-1}(\kappa_2 R)\alpha_{-22}\right).$$

For $e^{in\vartheta}$ $(n=\pm2,\ \pm3,...)$ gives

$$\begin{gathered}-\frac{\varkappa}{(n-1)R^{n-1}}\bar{b}_n+i\xi K_{n-1}(\xi R)\alpha_n=\bar{B'}_{-n+1},\quad n\geq 3\\ \frac{(\varkappa+1)i}{2R^n}\bar{b}_n+\frac{2\mu}{\nu+\beta}K_n(\xi R)\alpha_n=C_n,\quad n\geq 3\end{gathered} \tag{32}$$

$$\frac{1}{(n-1)R^{n-1}}\bar{c}_n-\frac{1}{R^{n-3}}\bar{b}_{n-2}-i\xi K_{n-1}(\xi R)\alpha_{n-2}=B'_{n-1},\quad n\geq 3 \tag{33}$$

where

$$B'_{-n} = B_{-n} - \frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}\frac{a_{n+1}}{nR^n}$$
$$+\frac{4\mu}{\lambda+2\mu}\left(\frac{\delta_1\kappa_1}{2}K_{-n-1}(\kappa_1 R)\alpha_{-n-11}+\frac{\delta_2\kappa_2}{2}K_{-n-1}(\kappa_2 R)\alpha_{-n-12}\right),$$
$$B'_n = B_n + \frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}\frac{(n-2)\bar{a}_{n-1}}{R^{n-2}}$$
$$+\frac{4\mu}{\lambda+2\mu}\left(\frac{\delta_1\kappa_1}{2}K_n(\kappa_1 R)\alpha_{n-11}+\frac{\delta_2\kappa_2}{2}K_n(\kappa_2 R)\alpha_{n-12}\right).$$

It is known that

$$b_0 = \Gamma - \frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}(a_0+\bar{a}_0), \quad c_0 = \Gamma',$$

where Γ, Γ' are known quantities, specifying the stress distribution at infinity (It is also assumed that b_0 is a real value (see [17])). Hence, by the formulae (30-31)

$$\alpha_0 = \frac{\nu+\beta}{2\mu K_0(\xi R)}C_0, \quad c_2 = -R^2(\varkappa-1)b_0 - i\xi R K_1(\xi R)\alpha_0 + R\bar{A}_1,$$

$$\bar{b}_2 = 2R^2\frac{(\nu+\beta)i\xi K_1(\xi R)C_2 - 2\mu K_2(\xi R)(\bar{A}_{-1}+Rb_0)}{4\mu R\varkappa K_2(\xi R)-(\nu+\beta)(\varkappa+1)\xi K_1(\xi R)},$$

$$\alpha_2 = \frac{(\varkappa+1)i(\bar{A}_{-1}+Rb_0)+2R\varkappa C_2}{4\mu R\varkappa K_2(\xi R)-(\nu+\beta)(\varkappa+1)\xi K_1(\xi R)}(\nu+\beta).$$

In order to find expressions for b_1 and c_1, it is necessary to refer to the condition for single-valuedness of the displacements

$$\varkappa b_1 + \bar{c}_1 = -\frac{\lambda(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}a_1,$$

which in combination with the second relation of (29) gives

$$b_1 = \frac{B'_0 - \frac{\lambda(\beta_1+\beta_2+\beta_3)\ln R}{\lambda+2\mu}a_1 + i\xi K_0(\xi R)\bar{C}_1}{2\varkappa\ln R + \frac{(\varkappa+1)\xi K_0(\xi R)}{2R}},$$

$$c_1 = -\varkappa\bar{b}_1 - \frac{\lambda(\beta_1+\beta_2+\beta_3)\ln R}{\lambda+2\mu}\bar{a}_1,$$

$$\alpha_1 = C_1 - \frac{(\varkappa+1)i}{2R}\bar{b}_1,$$

and, finally, (32-33) determines all coefficients $b_n,\ c_n,\ \alpha_n$:

$$b_n = \frac{\Delta_{1n}}{\Delta_n}, \quad \alpha_n = \frac{\Delta_{2n}}{\Delta_n}, \quad n>2,$$

$$c_n = (n-1)R^{n-1}\left(\frac{b_{n-2}}{R^{n-3}} - i\xi K_{n-1}(\xi R)\bar{\alpha}_{n-2} + \bar{B}'_{n-1}\right), \quad n > 2,$$

where

$$\Delta_{1n} = -\frac{2\mu}{\nu+\beta}K_n(\xi R)B'_{1-n} - i\xi K_{n-1}(\xi R)C_{-n},$$

$$\Delta_{2n} = \frac{(\varkappa+1)i}{2R^n}\bar{B}'_{1-n} + \frac{\varkappa}{(n-1)R^{n-1}}C_n,$$

$$\Delta_n = \frac{2\mu\varkappa K_n(\xi R)}{(\nu+\beta)(n-1)R^{n-1}} - \frac{(\varkappa+1)\xi K_{n-1}(\xi R)}{2R^n}.$$

It is easy to prove the absolute and uniform convergence of the series obtained in the circular ring (including the contours) when the functions set on the boundaries have sufficient smoothness.

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