ON ONE BOUNDARY VALUE PROBLEMS OF PLATES WITH DOUBLE POROSITY BY THE VEKUA METHOD FOR APPROXIMATIONS N=1

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We consider the three-dimensional system of the equations of elastic static equilibrium of bodies with double porosity. From this system of the equations, using a method of a reduction of I. Vekua, we receive the equilibrium equations for the plates having double porosity. The systems of the equations corresponding to approximations of $N = 1$ it is written down in a complex form and we express the general solution of these systems through analytic functions of complex variable and solutions of the Helmholtz equation. The Dirichlet boundary value problem are solved explicitly for the concentric circular ring.

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1 Introduction

The first theory of consolidation for elastic materials with double porosity was presented by Wilson and Aifantis [1]. This theory unifies the earlier proposed models of porous media with single [2] and double [3, 4] porosities. More general models of double porosity materials based on the Darcys law are introduced in [5-9] and studied by several authors [10-17]. Recently, on the basis of balance of equilibrated force the theory for elastic materials with a double porosity structure are presented by Iesan and Quintanilla [18] and the basic threedimensional BVPs of the equilibrium and steady vibrations of this theory are investigated in [19, 20].

The problem of elastic bodies with double porosity was the subject of study for some papers more than fifty years ago. Many authors have investigated the BVPs of the theory of elasticity for materials with double porosity, that are published in a large number of papers (some of these results can be seen in [21-28] and references therein).

2 Basic three-dimensional relations

Let an elastic body with double porosity occupy the domain $\overline{\Omega} \subset R^3$. Denote by (x^1, x^2, x^3) a point of the domain $\overline{\Omega}$ in the arbitrary curvilinear system of coordinates. Let the domain $\overline{\Omega}$ be filled with an elastic isotropic homogenous medium having double porosity. The considered solid body is characterized by the displacement vector $\mathbf{u} = (u^1, u^2, u^3)$, and also by the fluid pressures $p_1(x^1, x^2, x^3)$ and $p_2(x^1, x^2, x^3)$ occurring respectively in the pores and fissures of the porous medium.

Then a homogeneous system of static equilibrium equations is written in the form [6]

$$\partial_i \sigma_{ij} = 0, \tag{1}$$

where $\partial_i \equiv \frac{\partial}{\partial x_i}$, σ_{ij} are components of stress tensor, the summation over the recurring index i is assumed to be made from 1 to 3.

Formulas that interrelate the stress components, the displacement vector components and the pressures p_1, p_2 have the form [6]

$$\sigma_{ij} = (\lambda \partial_k u_k - \beta_1 p_1 - \beta_2 p_2)\delta_{ij} + 2\mu e_{ij}. \tag{2}$$

where λ and μ are the Lamé parameters; β_1 and β_2 are the effective stress parameters; δ_{ij} is the Kronecker delta, e_{ij} are components of the deformation tensor

$$e_{ij} = 0.5(\partial_i u_j + \partial_j u_i).$$

In the stationary case, the values p_1 and p_2 satisfy the following system of equations [6]

$$\begin{cases} (k_1\Delta_3 - \gamma)p_1 + (k_{12}\Delta_3 + \gamma)p_2 = 0, \\ (k_{21}\Delta_3 + \gamma)p_1 + (k_2\Delta_3 - \gamma)p_2 = 0 \end{cases} \quad \text{in} \quad \Omega, \tag{3}$$

where $k_1 = \frac{\kappa_1}{\mu'}, k_2 = \frac{\kappa_2}{\mu'}, k_{12} = \frac{\kappa_{12}}{\mu'}, k_{21} = \frac{\kappa_{21}}{\mu'}$; μ' is fluid viscosity; κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity; k_{12} and k_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases; $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures; Δ_3 is the three-dimensional Laplace operator.

It is easy to show that if $\gamma > 0$, $k_1k_2 - k_{12}k_{21} > 0$, then the system of equations (3) is equivalent to two independent equations: to the Laplace equation [32]

$$\Delta_3\widetilde{p}_1 = 0 \quad in \quad \Omega \tag{4}$$

and to the Helmholtz equation

$$\Delta_3\widetilde{p}_2 - \zeta^2\widetilde{p}_2 = 0 \quad in \quad \Omega, \tag{5}$$

where

$$\widetilde{p}_1 := (k_1 + k_{21})p_1 + (k_2 + k_{12})p_2, \quad \widetilde{p}_2 := p_1 - p_2,$$

$$\zeta^2 := \frac{\gamma(k_1 + k_2 + k_{12} + k_{21})}{k_1k_2 - k_{12}k_{21}} > 0.$$

Adding the first equation of system (3) to the second equation of this system, we immediately obtain equation (4).

3 Approximation $N = 1$

I. Vekua [29, 30] constructed hierarchical models for elastic prismatic shells, in particular, plates of variable thickness, when on the face surfaces either stresses are known. In [31] we apply I. Vekua's method for a reduction of the equations (1-5).

Consider approximation of the order $N = 1$.

We introduce the complex variable $z = x_1 + ix_2$ $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2), \partial_{\bar{z}} = 0.5(\partial_1 + i\partial_2)$ $(\bar{z} = x_1 - ix_2)$. The two-dimensional Laplace operator is expressed as $\Delta = 4\partial_z\partial_{\bar{z}}$.

The homogenous system of equation of the elastic plate may be written in the following complex form [31]:

$$\begin{cases} \mu\Delta \overset{(0)}{u}_+ + 2(\lambda+\mu)\partial_{\bar{z}} \overset{(0)}{\vartheta} - \dfrac{2\lambda}{h}\partial_{\bar{z}} \overset{(1)}{u}_3 - 2\partial_{\bar{z}}\Big(\beta_1^* \overset{(0)}{\widetilde{p}}_1 + \beta_2^* \overset{(0)}{\widetilde{p}}_2\Big) = 0, \\ \mu\Delta \overset{(1)}{u}_3 - \dfrac{3\lambda}{h} \overset{(0)}{\vartheta} - \dfrac{3(\lambda+2\mu)}{h^2} \overset{(1)}{u}_3 + \dfrac{3}{h}\Big(\beta_1^* \overset{(0)}{\widetilde{p}}_1 + \beta_2^* \overset{(0)}{\widetilde{p}}_2\Big) = 0, \end{cases} \tag{6}$$

$$\begin{cases} \mu\Delta \overset{(1)}{u}_+ + 2(\lambda+\mu)\partial_{\bar{z}} \overset{(1)}{\vartheta} - \dfrac{6\mu}{h}\partial_{\bar{z}} \overset{(0)}{u}_3 - \dfrac{3\mu}{h^2} \overset{(1)}{u}_+ - 2\partial\bar{z}\Big(\beta_1^* \overset{(1)}{\widetilde{p}}_1 + \beta_2^* \overset{(1)}{\widetilde{p}}_2\Big) = 0, \\ \mu\Delta \overset{(0)}{u}_3 + \dfrac{\mu}{h} \overset{(1)}{\vartheta} = 0. \end{cases} \tag{7}$$

$$\begin{aligned} \Delta \overset{(0)}{\widetilde{p}}_1 - \frac{3}{h^2} \overset{(0)}{\widetilde{p}}_1 &= 0, \\ \Delta \overset{(0)}{\widetilde{p}}_2 - \left(\frac{3}{h^2} + \zeta^2\right) \overset{(0)}{\widetilde{p}}_2 &= 0, \end{aligned} \tag{8}$$

$$\Delta \overset{(1)}{\widetilde{p}_1} - \frac{15}{h^2} \overset{(1)}{\widetilde{p}_1} = 0,$$
$$\Delta \overset{(1)}{\widetilde{p}_2} - \Big(\frac{15}{h^2} + \zeta^2\Big) \overset{(1)}{\widetilde{p}_2} = 0, \tag{9}$$

where

$$\overset{(k)}{\widetilde{p}_\alpha}(x^1, x^2) = \left(k + \frac{1}{2}\right) \frac{1}{h} \int\limits_{-h}^{h} \widetilde{p}_\alpha(x^1, x^2, x^3) P_k\left(\frac{x^3}{h}\right) dx^3,$$

$$\overset{(k)}{u_i} = \left(k + \frac{1}{2}\right) \frac{1}{h} \int\limits_{-h}^{h} u_i(x^1, x^2, x^3) P_k\left(\frac{x^3}{h}\right) dx^3,$$

$$\overset{(k)}{u_+} = \overset{(k)}{u_1} + i \overset{(k)}{u_2}, \quad \overset{(k)}{\vartheta} = \partial_z \overset{(k)}{u_+} + \partial_{\bar{z}} \overline{\overset{(k)}{u_+}}, \quad k = 0, 1,$$

$$\beta_1^* = \frac{\beta_1 + \beta_2}{k_0}, \quad \beta_2^* = \frac{\beta_1(k_2 + k_{12}) - \beta_2(k_1 + k_{21})}{k_0},$$

$$k_0 = k_1 + k_2 + k_{12} + k_{21},$$

$P_k\left(\frac{x^3}{h}\right)$ is the Legendre polynomials of order k.

We take the operator $2\partial_{\bar{z}}$ out of the brackets in the left-hand part of the first equation of system (6)

$$2\partial_{\bar{z}}\Big(2\mu\partial_z \overset{(0)}{u_+} + (\lambda + \mu) \overset{(0)}{\vartheta} + \frac{\lambda}{h} \overset{(1)}{u_3} - \beta_1^* \overset{(0)}{\widetilde{p}_1} - \beta_2^* \overset{(0)}{\widetilde{p}_2}\Big) = 0. \tag{10}$$

Since (10) is a system of Cauchy-Riemann equations, we have

$$2\mu\partial_z \overset{(0)}{u_+} + (\lambda + \mu) \overset{(0)}{\vartheta} + \frac{\lambda}{h} \overset{(1)}{u_3} - \beta_1^* \overset{(0)}{\widetilde{p}_1} - \beta_2^* \overset{(0)}{\widetilde{p}_2} = a\varphi'(z), \tag{11}$$

where $\varphi'(z)$ are an arbitrary analytic function of z, a, are arbitrary nonzero constant. Summing equation (11) with the conjugate equation, we will obtain

$$(\lambda + 2\mu) \overset{(0)}{\vartheta} + \frac{\lambda}{h} \overset{(1)}{u_3} - \beta_1^* \overset{(0)}{\widetilde{p}_1} - \beta_2^* \overset{(0)}{\widetilde{p}_2} = \frac{a}{2}(\varphi'(z) + \overline{\varphi'(z)}). \tag{12}$$

From the second equation of system (6) we will define $\overset{(0)}{\vartheta}$

$$\overset{(0)}{\vartheta} = \frac{\mu h}{3\lambda} \Delta \overset{(1)}{u_3} - \frac{\lambda + 2\mu}{\lambda h} \overset{(1)}{u_3} + \frac{1}{\lambda}\Big(\beta_1^* \overset{(0)}{\widetilde{p}_1} + \beta_2^* \overset{(0)}{\widetilde{p}_2}\Big) \tag{13}$$

(13) we will substitute in (12), we obtain the following equation

$$\Delta\left(2\mu\overset{(1)}{u_3}\right)-\frac{12(\lambda+\mu)}{(\lambda+2\mu)h^2}\left(2\mu\overset{(1)}{u_3}\right)=a\frac{3\lambda}{(\lambda+2\mu)h}(\varphi'(z)+\overline{\varphi'(z)})$$
$$-\frac{12\mu}{(\lambda+2\mu)h}\left(\beta_1^*\overset{(0)}{\widetilde{p}_1}+\beta_2^*\overset{(0)}{\widetilde{p}_2}\right) \tag{14}$$

The general solution of the equation (14) taking into account that $\overset{(0)}{\widetilde{p}_1}$ and $\overset{(0)}{\widetilde{p}_2}$ satisfy to the equations (8), will have a form

$$2\mu\overset{(1)}{u_3}=\chi(z,\bar z)-\frac{\lambda ha}{4(\lambda+\mu)}(\varphi'(z)+\overline{\varphi'(z)})+a_1\overset{(0)}{\widetilde{p}_1}+a_2\overset{(0)}{\widetilde{p}_2}, \tag{15}$$

where $\chi(z,\bar z)$ the general solution of the following Helmholtz equation

$$\Delta\chi-\frac{12(\lambda+\mu)}{(\lambda+2\mu)h^2}\chi=0;$$

$$a_\alpha=\frac{12\mu h}{9\lambda+6-\delta_{\alpha2}\zeta^2(\lambda+2\mu)h^2}\beta_\alpha^*,\quad \alpha=1,2;$$

Substituting equation (13) into (11) we obtain

$$2\mu\partial_z\overset{(0)}{u_+}+(\lambda+\mu)\frac{\mu h}{3\lambda}\Delta\overset{(1)}{u_3}-(3\lambda+2\mu)\frac{\mu}{\lambda h}\overset{(1)}{u_3}+\frac{\mu}{\lambda}\left(\beta_1^*\overset{(0)}{\widetilde{p}_1}+\beta_2^*\overset{(0)}{\widetilde{p}_2}\right)=a\varphi'(z).$$

In this last formula are substituting the expression (15) for $\overset{(1)}{u_3}$

$$2\mu\partial_z\overset{(0)}{u_+}=\frac{5\lambda+6\mu}{8(\lambda+\mu)}a\varphi'(z)-\frac{3\lambda+2\mu}{8(\lambda+\mu)}a\overline{\varphi'(z)}+\frac{\lambda h}{6(\lambda+\mu)}\partial_z\partial_{\bar z}\chi$$
$$+\partial_z\partial_{\bar z}\left(a_0\overset{(0)}{\widetilde{p}_1}+b_0\overset{(0)}{\widetilde{p}_2}\right), \tag{16}$$

where

$$a_0=\frac{4\mu h^2}{3(3\lambda+2\mu)}\beta_1^*,\quad b_0=\frac{4\mu h^2}{3+h^2\zeta^2}\frac{3-h^2\zeta^2}{9\lambda+6\mu-\zeta^2(\lambda+\mu)h^2}\beta_2^*.$$

Let $a=\frac{8(\lambda+\mu)}{3\lambda+2\mu}$, then integrating on z the above formula (16), we obtain

$$2\mu\overset{(0)}{u_+}=\varkappa^*\varphi(z)-z\overline{\varphi' z}-\overline{\psi(z)}+\frac{\lambda h}{6(\lambda+\mu)}\partial_{\bar z}\chi(z,\bar z)$$

$$+\partial_z\Big(a_0 \overset{(0)}{\widetilde{p}_1} + b_0 \overset{(0)}{\widetilde{p}_2}\Big), \tag{17}$$

where

$$\varkappa^* = \frac{5\lambda + 6\mu}{2\lambda + 2\mu}.$$

$\psi(z)$ are an arbitrary analytic function of z. Thus, the general solution of (6) and (8) are represented by formulas

$$\overset{(0)}{\widetilde{p}_1} = \overset{(0)}{\chi}_1(z,\bar{z}), \;\; \overset{(0)}{\widetilde{p}_2} = \overset{(0)}{\chi}_2(z,\bar{z}), \tag{18}$$

$$\begin{aligned} 2\mu \overset{(0)}{u}_+ &= \varkappa^*\varphi(z) - z\overline{\varphi' z} - \overline{\psi(z)} - \frac{\lambda h}{6(\lambda+\mu)}\partial_{\bar z}\chi(z,\bar z) \\ &+ \frac{4h^2}{3}a_0\partial_z \overset{(0)}{\chi}_1(z,\bar z) + \frac{4h^2}{3+\zeta^2 h^2} b_0 \partial_z \overset{(0)}{\chi}_2(z,\bar z), \end{aligned} \tag{19}$$

$$2\mu \overset{(1)}{u}_3 = \chi(z,\bar z) - \frac{2\lambda h}{3\lambda+2\mu}(\varphi'(z) + \overline{\varphi'(z)}) + a_1 \overset{(0)}{\chi}_1(z,\bar z) + a_2 \overset{(0)}{\chi}_2(z,\bar z), \tag{20}$$

where $\chi(z,\bar z)$, $\overset{(0)}{\chi}_1(z,\bar z)$, $\overset{(0)}{\chi}_2(z,\bar z)$ are the general solutions of the following Helmholtz equations

$$\Delta\chi - \eta^2\chi = 0, \;\; \Delta \overset{(0)}{\chi}_1 - \eta_1^2 \overset{(0)}{\chi}_1 = 0, \;\; \Delta \overset{(0)}{\chi}_2 - \eta_2^2 \overset{(0)}{\chi}_2 = 0,$$

$$\eta^2 = \frac{12(\lambda+\mu)}{(\lambda+2\mu)h^2}, \;\; \eta_1^2 = \frac{3}{h^2}, \;\; \eta_2^2 = \left(\frac{3}{h^2} + \zeta^2\right).$$

From the second equation of system (7)

$$2\partial_{\bar z} \overset{(0)}{u}_3 + \frac{1}{h} \overset{(1)}{u}_+ = \frac{i}{h}\partial_{\bar z}\omega, \tag{21}$$

where ω as yet unknown real function.

From (21) we have

$$\overset{(1)}{u}_+ = -2h\partial_{\bar z} \overset{(0)}{u}_3 + i\partial_{\bar z}\omega. \tag{22}$$

Substituting (22) and $\vartheta = -h\Delta \overset{(0)}{u}_3$ in the first equation of system (9) and integrating on z we obtain the equation

$$-(\lambda+2\mu)h\Delta \overset{(0)}{u}_3 + i\mu\left(\Delta\omega - \frac{3}{h^2}\omega\right) - 2\Big(\beta_1^* \overset{(1)}{\widetilde{p}_1} + \beta_2^* \overset{(1)}{\widetilde{p}_2}\Big) = a\overline{f'(z)}, \tag{23}$$

where $f'(z)$ are an arbitrary analytic function of z; a are arbitrary nonzero constant. Summing equation (23) with the conjugate equation, we will obtain

$$-4(\lambda+2\mu)h\Delta \overset{(0)}{u}_3 = a(f'(z) + \overline{f'(z)}) + 4\Big(\beta_1^* \overset{(1)}{\widetilde{p}_1} + \beta_2^* \overset{(1)}{\widetilde{p}_2}\Big).$$

From the last equation follows

$$\overset{(0)}{u_3} = -\frac{a}{16(\lambda+2\mu)}(\bar{z}f(z) + z\overline{f(z)}) + \mathrm{g}(z) + \overline{\mathrm{g}(z)}$$

$$-\frac{1}{4(\lambda+2\mu)}\int\int\Big(\beta_1^*\overset{(1)}{\widetilde{p}_1} + \beta_2^*\overset{(1)}{\widetilde{p}_2}\Big)dzd\bar{z}, \qquad (24)$$

where $\mathrm{g}(z)$ is an arbitrary analytic function of z.

Considering the imaginary part of the equation (23), we obtain the equation

$$2\mu\Big(\Delta\omega - \frac{3}{h^2}\omega\Big) = -i(f'(z) - \overline{f'(z)}),$$

which general solution is represented as follows

$$\omega = \tau(z,\bar{z}) + i\frac{h^2}{6\mu}(f'(z) - \overline{f'(z)}), \qquad (25)$$

where $\tau(z,\bar{z})$ the general solution of the following Helmholtz equation

$$\Delta\tau(z,\bar{z}) - \frac{3}{h^2}\tau(z,\bar{z}) = 0,$$

Substituting formulas (24) and (25) in the (22), we obtain ($a = 8(\lambda+2\mu)$)

$$\overset{(1)}{u_+} = i\partial_{\bar{z}}\tau(z,\bar{z}) + \frac{4h^2(\lambda+2\mu)}{3\mu}\overline{f''(z)} + z\overline{f'(z)} + f(z) - 2h\overline{\mathrm{g}'(z)}$$

$$+\frac{1}{2(\lambda+2\mu)h}\int\Big(\beta_1^*\overset{(1)}{\widetilde{p}_1} + \beta_2^*\overset{(1)}{\widetilde{p}_2}\Big)dz. \qquad (26)$$

Taking into account the taken value of a constant a we will rewrite a formula (21)

$$\overset{(0)}{u_3} = -\frac{1}{2h}(\bar{z}f(z) + z\overline{f(z)}) + \mathrm{g}(z) + \overline{\mathrm{g}(z)}$$

$$-\frac{1}{4(\lambda+2\mu)}\int\int\Big(\beta_1^*\overset{(1)}{\widetilde{p}_1} + \beta_2^*\overset{(1)}{\widetilde{p}_2}\Big)dzd\bar{z}. \qquad (27)$$

Thus, the general solution of (7) and (9) are represented by formulas

$$\overset{(1)}{\widetilde{p}_1} = \overset{(1)}{\chi_2}(z,\bar{z}), \quad \overset{(1)}{\widetilde{p}_1} = \overset{(1)}{\chi_1}(z,\bar{z}), \qquad (28)$$

$$\begin{aligned}\overset{(1)}{u_+} &= i\partial_{\bar{z}}\tau(z,\bar{z}) + \frac{4h^2(\lambda+2\mu)}{3\mu}\overline{f''(z)} + z\overline{f'(z)} + f(z) - 2h\overline{\mathrm{g}'(z)}\\ &+\frac{1}{2(\lambda+2\mu)h}\left(\frac{4h^2}{15}\beta_1^*\partial_{\bar{z}}\overset{(1)}{\chi_1}(z,\bar{z}) + \frac{4h^2}{15+\zeta^2h^2}\beta_2^*\partial_{\bar{z}}\overset{(1)}{\chi_2}(z,\bar{z})\right),\end{aligned} \qquad (29)$$

$$\overset{(0)}{u}_3 = -\frac{1}{2h}(\bar{z}f(z) + z\overline{f(z)}) + \mathrm{g}(z) + \overline{\mathrm{g}(z)}$$
$$-\frac{1}{4(\lambda+2\mu)}\left(\frac{4h^2}{15}\beta_1^* \overset{(1)}{\chi}_1(z,\bar{z}) + \frac{4h^2}{15+\zeta^2 h^2}\beta_2^* \overset{(1)}{\chi}_2(z,\bar{z})\right) \tag{30}$$

where $\tau(z,\bar{z})$, $\overset{(1)}{\chi}_1(z,\bar{z})$, $\overset{(1)}{\chi}_2(z,\bar{z})$ are the general solutions of the following Helmholtz equations

$$\Delta\tau - \gamma^2\tau = 0, \quad \Delta \overset{(1)}{\chi}_1 - \eta_3^2 \overset{(1)}{\chi}_1 = 0, \quad \Delta \overset{(1)}{\chi}_2 - \eta_4^2 \overset{(1)}{\chi}_2 = 0,$$

$$\gamma^2 = \frac{3}{h^2}, \quad \eta_3^2 = \frac{15}{h^2}, \quad \eta_4^2 = \left(\frac{15}{h^2} + \zeta^2\right).$$

The constructed general solution enables one to solve analytically a sufficiently wide class of boundary value problems of the elastic equilibrium of porous plates with double porosity.

4 A problem for a circular ring

In this section, we solve a concrete boundary value problem for a concentric circular ring with radius R_1 and R_2 (see fig. 1). On the boundary of the considered domain the values of pressures p_1 and p_2 and the displacement vector are given.

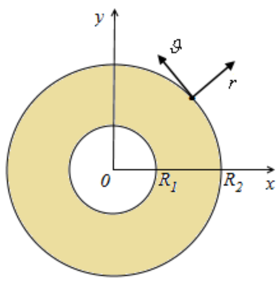

Fig. 1.

We consider the following problem

$$\overset{(0)}{\widetilde{p}}_1 = \begin{cases} \sum\limits_{-\infty}^{+\infty} A'_{1n} e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} A''_{1n} e^{in\vartheta}, & |z| = R_2, \end{cases} \qquad \overset{(0)}{\widetilde{p}}_2 = \begin{cases} \sum\limits_{-\infty}^{+\infty} B'_{1n} e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} B''_{1n} e^{in\vartheta}, & |z| = R_2, \end{cases} \tag{31}$$

$$\overset{(0)}{u}_{+} = \begin{cases} \sum\limits_{-\infty}^{+\infty} C'_{1n} e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} C''_{1n} e^{in\vartheta}, & |z| = R_2, \end{cases} \qquad \overset{(1)}{u}_{3} = \begin{cases} \sum\limits_{-\infty}^{+\infty} D'_{1n} e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} D''_{1n} e^{in\vartheta}, & |z| = R_2, \end{cases} \tag{32}$$

$$\overset{(1)}{\widetilde{p}}_{1} = \begin{cases} \sum\limits_{-\infty}^{+\infty} A'_{2n} e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} A''_{2n} e^{in\vartheta}, & |z| = R_2, \end{cases} \qquad \overset{(1)}{\widetilde{p}}_{2} = \begin{cases} \sum\limits_{-\infty}^{+\infty} B'_{2n} e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} B''_{2n} e^{in\vartheta}, & |z| = R_2, \end{cases} \tag{33}$$

$$\overset{(1)}{u}_{+} = \begin{cases} \sum\limits_{-\infty}^{+\infty} C'_{2n} e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} C''_{2n} e^{in\vartheta}, & |z| = R_2, \end{cases} \qquad \overset{(0)}{u}_{3} = \begin{cases} \sum\limits_{-\infty}^{+\infty} D'_{2n} e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} D''_{2n} e^{in\vartheta}, & |z| = R_2, \end{cases} \tag{34}$$

The metaharmonic function $\overset{(0)}{\chi}_1(z,\bar{z})$ and $\overset{(0)}{\chi}_2(z,\bar{z})$ are represented as a series

$$\begin{aligned} \overset{(0)}{\chi}_1(z,\bar{z}) &= \sum_{-\infty}^{+\infty} (\alpha'_{1n} I_n(\eta_1 r) + \alpha'_{2n} K_n(\eta_1 r)) e^{in\vartheta}, \\ \overset{(0)}{\chi}_2(z,\bar{z}) &= \sum_{-\infty}^{+\infty} (\beta'_{1n} I_n(\eta_2 r) + \beta'_{2n} K_n(\eta_2 r)) e^{in\vartheta}, \end{aligned} \tag{35}$$

where $I_n(\cdot)$ and $K_n(\cdot)$ are modified Bessel function of n-th order, $z = re^{i\vartheta}$, and are substituted in the boundary conditions (31) we have

$$\sum_{-\infty}^{+\infty} (\alpha'_{1n} I_n(\eta_1 R_1) + \alpha'_{2n} K_n(\eta_1 R_1)) e^{in\vartheta} = \sum_{-\infty}^{+\infty} A'_{1n} e^{in\vartheta},$$

$$\sum_{-\infty}^{+\infty} (\alpha'_{1n} I_n(\eta_1 R_2) + \alpha'_{2n} K_n(\eta_1 R_2)) e^{in\vartheta} = \sum_{-\infty}^{+\infty} A''_{1n} e^{in\vartheta},$$

$$\sum_{-\infty}^{+\infty} (\beta'_{1n} I_n(\eta_2 R_1) + \beta'_{2n} K_n(\eta_2 R_1)) e^{in\vartheta} = \sum_{-\infty}^{+\infty} B'_{1n} e^{in\vartheta},$$

$$\sum_{-\infty}^{+\infty} (\beta'_{1n} I_n(\eta_2 R_2) + \beta'_{2n} K_n(\eta_2 R_2)) e^{in\vartheta} = \sum_{-\infty}^{+\infty} B''_{1n} e^{in\vartheta},$$

Compare the coefficients at identical degrees. We obtain the following systems of equations

$$\begin{aligned} I_n(\eta_1 R_1)\alpha'_{1n} + K_n(\eta_1 R_1)\alpha'_{2n} &= A'_{1n}, \\ I_n(\eta_1 R_2)\alpha'_{1n} + K_n(\eta_1 R_2)\alpha'_{2n} &= A''_{1n}, \\ I_n(\eta_2 R_1)\beta'_{1n} + K_n(\eta_2 R_1)\beta'_{2n} &= B'_{1n}, \\ I_n(\eta_2 R_2)\beta'_{1n} + K_n(\eta_2 R_2)\beta'_{2n} &= B''_{1n}. \end{aligned} \tag{36}$$

From (36) for coefficients α'_{1n}, α'_{2n}, β'_{1n} and β'_{2n} we have:

$$\alpha'_{1n} = \frac{A'_{1n}K_n(\eta_1 R_2) - A''_{1n}K_n(\eta_1 R_1)}{I_n(\eta_1 R_1)K_n(\eta_1 R_2) - I_n(\eta_1 R_2)K_n(\eta_1 R_1)},$$

$$\alpha'_{2n} = \frac{A'_{1n}I_n(\eta_1 R_2) - A''_{1n}I_n(\eta_1 R_1)}{I_n(\eta_1 R_1)K_n(\eta_1 R_2) - I_n(\eta_1 R_2)K_n(\eta_1 R_1)},$$

$$\beta'_{1n} = \frac{B'_{1n}K_n(\eta_2 R_2) - B''_{1n}K_n(\eta_2 R_1)}{I_n(\eta_2 R_1)K_n(\eta_2 R_2) - I_n(\eta_2 R_2)K_n(\eta_2 R_1)},$$

$$\beta'_{2n} = \frac{B'_{1n}I_n(\eta_2 R_2) - B''_{1n}I_n(\eta_2 R_1)}{I_n(\eta_2 R_1)K_n(\eta_2 R_2) - I_n(\eta_2 R_2)K_n(\eta_2 R_1)}.$$

Let us introduce the functions $\varphi(z)$, $\psi(z)$, $\chi(z,\bar{z})$, by the series:

$$\varphi(z) = \alpha \ln z + \sum_{-\infty}^{+\infty} a_n z^n \quad \psi(z) = \beta \ln z + \sum_{-\infty}^{+\infty} b_n z^n,$$

$$\chi = \sum_{-\infty}^{\infty} \left(\alpha'_n I_n(\eta r) + \alpha''_n K_n(\eta r)\right) e^{in\theta},$$

and are substituted in the boundary conditions (32) we have

$$\sum_{-\infty}^{\infty} (\varkappa^* a_n r^n e^{in\vartheta} - n\bar{a}_n r^n e^{-i(n-2)\vartheta} - \bar{b}_n r^n e^{-in\vartheta})$$

$$-\frac{\lambda\eta h}{12(\lambda+\mu)} \sum_{-\infty}^{+\infty} (\alpha'_n I_{n+1}(r\zeta) - \alpha''_n K_{n+1}(r\zeta)) e^{i(n+1)\vartheta}$$

$$(\varkappa^*\alpha - \bar{\beta})\ln r + (\varkappa^*\alpha + \bar{\beta}) i\vartheta - \bar{\alpha} e^{2i\vartheta} = \begin{cases} \sum\limits_{-\infty}^{+\infty} E'_n e^{in\vartheta}, \ |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} E''_n e^{in\vartheta}, \ |z| = R_2, \end{cases}$$

$$-\frac{2\lambda h}{3\lambda + 2\mu}\left(\frac{\alpha}{r} e^{-i\vartheta} + \frac{\bar{\alpha}}{r} e^{i\vartheta} + \sum_{-\infty}^{\infty} n r^{n-1}\left(a_n e^{i(n-1)\vartheta} + \bar{a}_n e^{-i(n-1)\vartheta}\right)\right)$$

$$+\sum_{-\infty}^{+\infty} (\alpha'_n I_n(r\zeta) + \alpha''_n K_n(r\zeta)) e^{in)\vartheta} = \begin{cases} \sum\limits_{-\infty}^{+\infty} F'_n e^{in\vartheta}, \ |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} F''_n e^{in\vartheta}, \ |z| = R_2, \end{cases}$$

where

$$B_n = D_n - \frac{\mu(\beta_1 + \beta_2 + \beta_3)}{\lambda + 2\mu}\left((n+1) r^n c_{n+1} - (n-1) r^{-n} \bar{c}_{1-n}\right)$$

$$
\begin{gathered}
-\frac{4\mu}{\lambda+2\mu}\left[\frac{\delta_1\kappa_1}{2}\left(\alpha_{n-1}I_n(\kappa_1 r)-\beta_{n-1}K_n(\kappa_1 r)\right)\right. \\
\left.+\frac{\delta_2\kappa_2}{2}\left(\gamma_{n-1}I_n(\kappa_2 r)-\delta_{n-1}K_n(\kappa_2 r)\right)\right], \\
(n=\pm 1,\ -2,\ \pm 3,...), \\
B_1=D_1-\frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}\left(2rc_2+\frac{\alpha}{r}\right)-\frac{4\mu}{\lambda+2\mu} \\
\times\left[\frac{\delta_1\kappa_1}{2}\left(\alpha_0 I_1(\kappa_1 r)-\beta_0 K_1(\kappa_1 r)\right)+\frac{\delta_2\kappa_2}{2}\left(\gamma_0 I_1(\kappa_2 r)-\delta_0 K_1(\kappa_2 r)\right)\right],
\end{gathered}
$$

and $C_n=E_n$.

From the condition of displacement uniqueness it follows that

$$
\varkappa^*\alpha+\bar{\beta}=0.
$$

Comparison of terms independent of ϑ gives

$$
\begin{aligned}
&2\varkappa^*\ln R_1\alpha-2R_1^2\bar{a}_2-\frac{\lambda\eta h}{12(\lambda+\mu)}\big(\alpha'_{-1}I_0(\eta R_1)-\alpha''_{-1}K_0(\eta R_1)\big)\\
&+\varkappa^* a_0-\bar{b}_0=E'_0,\\
&2\varkappa^*\ln R_2\alpha-2R_2^2\bar{a}_2-\frac{\lambda\eta h}{12(\lambda+\mu)}\big(\alpha'_{-1}I_0(\eta R_2)-\alpha''_{-1}K_0(\eta R_2)\big)\\
&+\varkappa^* a_0-\bar{b}_0=E''_0.
\end{aligned}
\tag{37}
$$

Comparison of terms involving $e^{in\vartheta}$ for $n=\pm 1,\pm 2,...$ gives

$$
\begin{aligned}
&\varkappa^* R_1^2 a_2-\bar{\alpha}-R_1^{-2}\bar{b}_{-2}-\frac{\lambda\eta h}{12(\lambda+\mu)}\big(\alpha'_1 I_2(\eta R_1)-\alpha''_1 K_2(\eta R_1)\big)=E'_2,\\
&\varkappa^* R_2^2 a_2-\bar{\alpha}-R_2^{-2}\bar{b}_{-2}-\frac{\lambda\eta h}{12(\lambda+\mu)}\big(\alpha'_1 I_2(\eta R_2)-\alpha''_1 K_2(\eta R_2)\big)=E''_2
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&\varkappa^* R_1^n a_n+(n-2)R_1^{2-n}\bar{a}_{2-n}-R_1^{-n}\bar{b}_{-n}\\
&-\frac{\lambda\eta h}{12(\lambda+\mu)}\big(\alpha'_{n-1}I_n(\eta R_1)-\alpha''_{n-1}K_n(\eta R_1)\big)=E'_n,\\
&\varkappa^* R_2^n a_n+(n-2)R_2^{2-n}\bar{a}_{2-n}-R_2^{-n}\bar{b}_{-n}\\
&-\frac{\lambda\eta h}{12(\lambda+\mu)}\big(\alpha'_{n-1}I_n(\eta R_2)-\alpha''_{n-1}K_n(\eta R_2)\big)=E''_n,\\
&(n=\pm 1,\ -2,\ \pm 3,...),
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&\alpha'_1 I_1(\eta R_1)+\alpha''_1 K_1(\eta R_1)-\frac{2\lambda h}{3\lambda+2\mu}\left(2R_1 a_2+\frac{\bar{\alpha}}{R_1}\right)=F'_1,\\
&\alpha'_1 I_1(\eta R_2)+\alpha''_1 K_1(\eta R_2)-\frac{2\lambda h}{3\lambda+2\mu}\left(2R_2 a_2+\frac{\bar{\alpha}}{R_2}\right)=F''_1,
\end{aligned}
\tag{40}
$$

$$\begin{aligned}
&\alpha_n' I_n(\eta R_1) + \alpha_n'' K_n(\eta R_1) \\
&-\frac{2\lambda h}{3\lambda+2\mu}\left[(n+1)R_1^n a_{n+1} - (n-1)R_1^{-n}\bar{a}_{1-n}\right] = F_n', \\
&\alpha_n' I_n(\eta R_2) + \alpha_n'' K_n(\eta R_2) \\
&-\frac{2\lambda h}{3\lambda+2\mu}\left[(n+1)R_2^n a_{n+1} - (n-1)R_2^{-n}\bar{a}_{1-n}\right] = F_n'', \\
&(n = 0,\ -1,\ \pm 2,\ \pm 3, \ldots).
\end{aligned} \tag{41}$$

From (38) and (40) we have

$$a_2 = \frac{2\varkappa^* \ln R_2/R_1 L_2 + (R_2^2 - R_1^2)L_1}{2\varkappa^* \ln R_2/R_1 L_3 + (R_2^2 - R_1^2)L_4}$$

where

$$\begin{aligned}
L_1 &= 2(R_1^2 - R_2^2) - \frac{\lambda^2 \eta h^2 (I_0(\eta R_2) - I_0(\eta R_1))(R_2 K_1(\eta R_1) - R_1 K_1(\eta R_2))}{3(\lambda+\mu)(3\lambda+2\mu)(I_1(\eta R_2)K_1(\eta R_1) - I_1(\eta R_1)K_1(\eta R_2))} \\
&+\frac{\lambda^2 \eta h^2 (K_0(\eta R_2) - K_0(\eta R_1))(R_2 I_1(\eta R_1) - R_1 I_1(\eta R_2))}{3(\lambda+\mu)(3\lambda+2\mu)(I_1(\eta R_2)K_1(\eta R_1) - I_1(\eta R_1)K_1(\eta R_2))}, \\
L_2 &= \frac{\lambda^2 \eta h^2 (R_2^2 K_2(\eta R_2) - R_1^2 K_2(\eta R_1))(R_2 I_1(\eta R_1) - R_1 I_1(\eta R_2))}{3(\lambda+\mu)(3\lambda+2\mu)(I_1(\eta R_2)K_1(\eta R_1) - I_1(\eta R_1)K_1(\eta R_2))} \\
&+\varkappa^*(R_1^4 - R_2^4) - \frac{\lambda^2 \eta h^2 (R_2^2 I_2(\eta R_2) - R_1^2 I_2(\eta R_1))(R_2 K_1(\eta R_1) - R_1 K_1(\eta R_2))}{3(\lambda+\mu)(3\lambda+2\mu)(I_1(\eta R_2)K_1(\eta R_1) - I_1(\eta R_1)K_1(\eta R_2))}, \\
L_3 &= E_0'' - E_0' + \frac{\lambda \eta h (I_0(\eta R_2) - I_0(\eta R_1))(F_1'' K_1(\eta R_1) - F_1' K_1(\eta R_2))}{12(\lambda+\mu)(I_1(\eta R_2)K_1(\eta R_1) - I_1(\eta R_1)K_1(\eta R_2))} \\
&-\frac{\lambda \eta h (K_0(\eta R_2) - K_0(\eta R_1))(F_1'' I_1(\eta R_1) - F_1' I_1(\eta R_2))}{12(\lambda+\mu)(I_1(\eta R_2)K_1(\eta R_1) - I_1(\eta R_1)K_1(\eta R_2))}, \\
L_4 &= -\frac{\lambda \eta h (R_2^2 K_2(\eta R_2) - R_1^2 K_2(\eta R_1))(F_1'' I_1(\eta R_1) - F_1' I_1(\eta R_2))}{12(\lambda+\mu)(I_1(\eta R_2)K_1(\eta R_1) - I_1(\eta R_1)K_1(\eta R_2))} \\
&+R_2^2 E_2'' - R_1^2 E_2' + \frac{\lambda \eta h (R_2^2 I_2(\eta R_2) - R_1^2 I_2(\eta R_1))(F_1'' K_1(\eta R_1) - F_1' K_1(\eta R_2))}{12(\lambda+\mu)(I_1(\eta R_2)K_1(\eta R_1) - I_1(\eta R_1)K_1(\eta R_2))}.
\end{aligned}$$

The coefficients $\alpha,\ \beta,\ b_{-2},\ \alpha_1',\ \alpha_1'',\ \varkappa a_0 - b_0$ are found by solving (37), (38), (40).

From (41), dividing the first equation of (39) by R_1^n, and second by R_2^n, and subtracting, one obtains the first of the following formulas:

$$\begin{cases}
T_n a_n + S_n \bar{a}_{-n+2} = G_n, \\
S_{-n+2} a_n + T_{-n+2} \bar{a}_{-n+2} = \bar{G}_{-n+2},
\end{cases} \tag{42}$$

where

$$\begin{aligned}
G_n &= R_2^n E_n'' - R_1^n E_n' \\
&+\frac{\lambda \eta h (R_2^n I_n(\eta R_2) - R_1^n I_n(\eta R_1))(F_{n-1}'' K_{n-1}(\eta R_1) - B_{n-1}' K_{n-1}(\eta R_2))}{12(\lambda+\mu)(I_{n-1}(\eta R_2)K_{n-1}(\eta R_1) - I_{n-1}(\eta R_1)K_{n-1}(\eta R_2))} \\
&-\frac{\lambda \eta h (R_2^n K_n(\eta R_2) - R_1^n K_n(\eta R_1))(F_{n-1}'' I_{n-1}(\eta R_1) - B_{n-1}' I_{n-1}(\eta R_2))}{12(\lambda+\mu)(I_{n-1}(\eta R_2)K_{n-1}(\eta R_1) - I_{n-1}(\eta R_1)K_{n-1}(\eta R_2))},
\end{aligned}$$

$$T_n = \varkappa^*(R_2^{2n} - R_1^{2n})$$
$$-\frac{\lambda^2\eta h^2 n(R_2^n I_n(\eta R_2) - R_1^n I_n(\eta R_1))(R_2^{n-1}K_{n-1}(\eta R_1) - R_1^{n-1}K_{n-1}(\eta R_2))}{6(\lambda+\mu)(3\lambda+2\mu)(I_{n-1}(\eta R_2)K_{n-1}(\eta R_1) - I_{n-1}(\eta R_1)K_{n-1}(\eta R_2))}$$
$$+\frac{\lambda^2\eta h^2 n(R_2^n K_n(\eta R_2) - R_1^n K_n(\eta R_1))(R_2^{n-1}I_{n-1}(\eta R_1) - R_1^{n-1}I_{n-1}(\eta R_2))}{6(\lambda+\mu)(3\lambda+2\mu)(I_{n-1}(\eta R_2)K_{n-1}(\eta R_1) - I_{n-1}(\eta R_1)K_{n-1}(\eta R_2))},$$

$$S_n = (n-2)\Big[(R_2^2 - R_1^2)$$
$$+\frac{\lambda^2\eta h^2(R_2^n I_n(\eta R_2) - R_1^n I_n(\eta R_1))(R_2^{1-n}K_{n-1}(\eta R_1) - R_1^{1-n}K_{n-1}(\eta R_2))}{6(\lambda+\mu)(3\lambda+2\mu)(I_{n-1}(\eta R_2)K_{n-1}(\eta R_1) - I_{n-1}(\eta R_1)K_{n-1}(\eta R_2))}$$
$$-\frac{\lambda^2\eta h^2(R_2^n K_n(\eta R_2) - R_1^n K_n(\eta R_1))(R_1^{1-n}I_{n-1}(\eta R_1) - R_2^{1-n}I_{n-1}(\eta R_2))}{6(\lambda+\mu)(3\lambda+2\mu)(I_{n-1}(\eta R_2)K_{n-1}(\eta R_1) - I_{n-1}(\eta R_1)K_{n-1}(\eta R_2))}\Big].$$

The second equation (42) is obtained from the first by replacing n by $-n+2$ and going the conjugate complex expression.

From (42) we have

$$a_n = \frac{T_{2-n}G_n - S_n C_{2-n}}{T_n T_{2-n} - S_n S_{2-n}}, \quad (n = \pm 1,\ -2,\ \pm 3, ...).$$

The coefficients α_n', α_n'', b_n are found by solving (39), (41).

The metaharmonic function $\overset{(1)}{\chi}_1(z,\bar z)$ and $\overset{(1)}{\chi}_2(z,\bar z)$ are represented as a series

$$\begin{aligned}\overset{(1)}{\chi}_1(z,\bar z) &= \sum_{-\infty}^{+\infty}(\alpha_{1n}'' I_n(\eta_3 r) + \alpha_{2n}'' K_n(\eta_3 r))e^{in\vartheta},\\ \overset{(1)}{\chi}_2(z,\bar z) &= \sum_{-\infty}^{+\infty}(\beta_{1n}'' I_n(\eta_4 r) + \beta_{2n}'' K_n(\eta_4 r))e^{in\vartheta},\end{aligned} \tag{43}$$

and are substituted in the boundary conditions (33) we have

$$\sum_{-\infty}^{+\infty}(\alpha_{1n}'' I_n(\eta_3 R_1) + \alpha_{2n}'' K_n(\eta_3 R_1))e^{in\vartheta} = \sum_{-\infty}^{+\infty} A_{2n}' e^{in\vartheta},$$

$$\sum_{-\infty}^{+\infty}(\alpha_{1n}'' I_n(\eta_3 R_2) + \alpha_{2n}'' K_n(\eta_3 R_2))e^{in\vartheta} = \sum_{-\infty}^{+\infty} A_{2n}'' e^{in\vartheta},$$

$$\sum_{-\infty}^{+\infty}(\beta_{1n}'' I_n(\eta_4 R_1) + \beta_{2n}'' K_n(\eta_4 R_1))e^{in\vartheta} = \sum_{-\infty}^{+\infty} B_{2n}' e^{in\vartheta},$$

$$\sum_{-\infty}^{+\infty}(\beta_{1n}'' I_n(\eta_4 R_2) + \beta_{2n}'' K_n(\eta_4 R_2))e^{in\vartheta} = \sum_{-\infty}^{+\infty} B_{2n}'' e^{in\vartheta},$$

Compare the coefficients at identical degrees. We obtain the following systems of equations

$$\begin{array}{l} I_n(\eta_3 R_1)\alpha''_{1n} + K_n(\eta_3 R_1)\alpha''_{2n} = A'_{2n}, \\ I_n(\eta_3 R_2)\alpha''_{1n} + K_n(\eta_3 R_2)\alpha''_{2n} = A''_{2n}, \\ I_n(\eta_4 R_1)\beta''_{1n} + K_n(\eta_4 R_1)\beta''_{2n} = B'_{2n}, \\ I_n(\eta_4 R_2)\beta''_{1n} + K_n(\eta_4 R_2)\beta''_{2n} = B''_{2n}. \end{array} \tag{44}$$

From (36) for coefficients α''_{1n}, α''_{2n}, β''_{1n} and β''_{2n} we have:

$$\alpha''_{1n} = \frac{A'_{2n}K_n(\eta_3 R_2) - A''_{2n}K_n(\eta_3 R_1)}{I_n(\eta_3 R_1)K_n(\eta_3 R_2) - I_n(\eta_3 R_2)K_n(\eta_3 R_1)},$$

$$\alpha''_{2n} = \frac{A'_{2n}I_n(\eta_3 R_2) - A''_{2n}I_n(\eta_3 R_1)}{I_n(\eta_3 R_1)K_n(\eta_3 R_2) - I_n(\eta_3 R_2)K_n(\eta_3 R_1)},$$

$$\beta''_{1n} = \frac{B'_{2n}K_n(\eta_4 R_2) - B''_{2n}K_n(\eta_4 R_1)}{I_n(\eta_4 R_1)K_n(\eta_4 R_2) - I_n(\eta_4 R_2)K_n(\eta_4 R_1)},$$

$$\beta''_{2n} = \frac{B'_{2n}I_n(\eta_4 R_2) - B''_{2n}I_n(\eta_4 R_1)}{I_n(\eta_4 R_1)K_n(\eta_4 R_2) - I_n(\eta_4 R_2)K_n(\eta_4 R_1)}.$$

Let us introduce the functions $f(z)$, $g(z)$, $\tau(z,\bar{z})$, by the series:

$$f(z) = \sum_{-\infty}^{+\infty} c_n z^n \quad g(z) = \delta \ln z + \sum_{-\infty}^{+\infty} d_n z^n,$$

$$\tau = \sum_{-\infty}^{\infty} \left(\beta'_n I_n(\gamma r) + \beta''_n K_n(\gamma r)\right) e^{in\theta},$$

and are substituted in the boundary conditions (34) we have

$$\begin{aligned} &\frac{i\gamma}{2}\sum_{-\infty}^{\infty}\left(\beta'_n I_{n+1}(\gamma r) - \beta''_n K_{n+1}(\gamma r)\right) e^{i(n+1)\vartheta} - 2h\sum_{-\infty}^{\infty} n\bar{d}_n r^{n-1} e^{i(1-n)\vartheta} \\ &+\sum_{-\infty}^{\infty} n\bar{c}_n r^n e^{i(2-n)\vartheta} + \frac{4(\lambda+2\mu)h^2}{3\mu}\sum_{-\infty}^{\infty} n(n-1)\bar{c}_n r^{n-2} e^{i(2-n)\vartheta} \\ &+\sum_{-\infty}^{\infty} c_n r^n e^{in\vartheta} - \frac{2\bar{\delta}h}{r}e^{i\vartheta} = \begin{cases} \sum\limits_{-\infty}^{+\infty} G'_n e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} G''_n e^{in\vartheta}, & |z| = R_2, \end{cases} \end{aligned}$$

$$\begin{aligned} &\sum_{-\infty}^{\infty}\left(d_n e^{in\vartheta} + \bar{d}_n e^{-in\vartheta}\right) r^n - \frac{1}{2h}\sum_{-\infty}^{\infty}\left(c_n e^{i(n-1)\vartheta} + \bar{c}_n e^{-i(n-1)\vartheta}\right) r^{n+1} \\ &+(\delta+\bar{\delta})\ln r + (\delta-\bar{\delta})i\vartheta = \begin{cases} \sum\limits_{-\infty}^{+\infty} F'_n e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} F''_n e^{in\vartheta}, & |z| = R_2, \end{cases} \end{aligned}$$

where

$$B_n = D_n - \frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}\big((n+1)r^n c_{n+1} - (n-1)r^{-n}\bar{c}_{1-n}\big)$$

$$-\frac{4\mu}{\lambda+2\mu}\left[\frac{\delta_1\kappa_1}{2}\left(\alpha_{n-1}I_n(\kappa_1 r) - \beta_{n-1}K_n(\kappa_1 r)\right)\right.$$

$$\left.+\frac{\delta_2\kappa_2}{2}\left(\gamma_{n-1}I_n(\kappa_2 r) - \delta_{n-1}K_n(\kappa_2 r)\right)\right],$$

$$(n = \pm 1,\ -2,\ \pm 3, ...),$$

$$B_1 = D_1 - \frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}\left(2rc_2 + \frac{\alpha}{r}\right) - \frac{4\mu}{\lambda+2\mu}$$

$$\times\left[\frac{\delta_1\kappa_1}{2}\left(\alpha_0 I_1(\kappa_1 r) - \beta_0 K_1(\kappa_1 r)\right) + \frac{\delta_2\kappa_2}{2}\left(\gamma_0 I_1(\kappa_2 r) - \delta_0 K_1(\kappa_2 r)\right)\right],$$

and $C_n = E_n$.

From the condition of displacement uniqueness it follows that

$$\delta - \bar{\delta} = 0.$$

Comparison of terms involving $e^{in\vartheta}$ gives

$$\begin{aligned}
&\frac{i\gamma}{2}\big(\beta_0' I_1(\gamma R_1) - \beta_0'' K_1(\gamma R_1)\big) + R_1(c_1+\bar{c}_1) - \frac{2h\delta}{R_1} = G_1',\\
&\frac{i\gamma}{2}\big(\beta_0' I_1(\gamma R_2) - \beta_0'' K_1(\gamma R_2)\big) + R_1(c_1+\bar{c}_1) - \frac{2h\delta}{R_2} = G_1'',
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\frac{i\gamma}{2}\big(\beta_{n-1}' I_n(\gamma R_1) - \beta_{n-1}'' K_n(\gamma R_1)\big) + R_1^n c_n - (n-2)R_1^{2-n}\bar{c}_{2-n}\\
&+\frac{4(\lambda+2\mu)h^2}{3\mu}(n-1)(n-2)R_1^{-n}\bar{c}_{2-n} + 2(n-1)hR_1^{-n}\bar{d}_{1-n} = G_n',\\
&\frac{i\gamma}{2}\big(\beta_{n-1}' I_n(\gamma R_2) - \beta_{n-1}'' K_n(\gamma R_2)\big) + R_2^n c_n - (n-2)R_2^{2-n}\bar{c}_{2-n}\\
&+\frac{4(\lambda+2\mu)h^2}{3\mu}(n-1)(n-2)R_2^{-n}\bar{c}_{2-n} + 2(n-1)hR_2^{-n}\bar{d}_{1-n} = G_n'',
\end{aligned} \tag{46}$$

$$\begin{aligned}
&d_0 + \bar{d}_0 - \frac{R_1^2}{2h}(c_1+\bar{c}_1) + 2\delta\ln R_1 = Q_0',\\
&d_0 + \bar{d}_0 - \frac{R_2^2}{2h}(c_1+\bar{c}_1) + 2\delta\ln R_2 = Q_0'',
\end{aligned} \tag{47}$$

$$\begin{aligned}
&R_1^n d_n + R_1^{-n}\bar{d}_{-n} - \frac{1}{2h}\big(R_1^{n+2}c_{n+1} + R_1^{-n+2}\bar{c}_{-n+1}\big) = Q_n',\\
&R_2^n d_n + R_2^{-n}\bar{d}_{-n} - \frac{1}{2h}\big(R_2^{n+2}c_{n+1} + R_2^{-n+2}\bar{c}_{-n+1}\big) = Q_n''.
\end{aligned} \tag{48}$$

From (45) and (47) we have

$$c_1 + \bar{c}_1 = \frac{R_2 \mathrm{Re} G_1'' - R_1 \mathrm{Re} G_1'}{R_2^2 - R_1^2},$$

$$\beta_0' = \frac{2\gamma(K_1(\gamma R_1)\mathrm{Im} G_1'' - K_1(\gamma R_2)\mathrm{Im} G_1')}{3(K_1(\gamma R_1)I_1(\gamma R_2) - K_1(\gamma R_2)I_1(\gamma R_1))}.$$

The coefficients $d_0 + \bar{d}_0,\ \beta_0''$ are found by solving (45), (47).
From (48)

$$\begin{aligned} d_{1-n} = &\frac{(R_2^{4-2n} - R_1^{4-2n})c_{2-n} + (R_2^2 - R_1^2)c_n}{2h(R_2^{2-2n} - R_1^{2-2n})} \\ +&\frac{2R_2^{1-n}Q_{n-1}'' - 2R_1^{1-n}Q_{n-1}'}{2(R_2^{2-2n} - R_1^{2-2n})}. \end{aligned} \tag{49}$$

Substituting from (49) in (46) we have

$$\begin{aligned} &\frac{i\gamma}{2} I_n(\gamma R_1)\beta_{n-1}' - \frac{i\gamma}{2} K_n(\gamma R_1)\beta_{n-1}'' + \left[R_1^n - \frac{(n-1)R_1^n(R_2^2 - R_1^2)}{R_2^{-2n+2} - R_1^{-2n+2}}\right] c_n \\ &+\frac{4(\lambda+2\mu)h^2}{3\mu}(R_1^{-n}(n-1) - R_1^{-n+2})(n-2)\bar{c}_{-n+2} \\ &+\frac{(n-1)hR_1^n(R_2^{-2n+4} - R_1^{-2n+4})}{R_2^{-2n+2} - R_1^{-2n+2}}\bar{c}_{-n+2} \\ &= G_n' - \frac{2(n-1)hR_1^n(R_2^{-n+1}\overline{Q}_{-n+1}'' - R_1^{-n+1}\overline{Q}_{-n+1}')}{R_2^{-2n+2} - R_1^{-2n+2}}, \end{aligned} \tag{50}$$

$$\begin{aligned} &\frac{i\gamma}{2} I_n(\gamma R_2)\beta_{n-1}' - \frac{i\gamma}{2} K_n(\gamma R_2)\beta_{n-1}'' + \left[R_2^n - \frac{(n-1)R_2^n(R_2^2 - R_1^2)}{R_2^{-2n+2} - R_1^{-2n+2}}\right] c_n \\ &+\frac{4(\lambda+2\mu)h^2}{3\mu}(R_2^{-n}(n-1) - R_2^{-n+2})(n-2)\bar{c}_{-n+2} \\ &+\frac{(n-1)hR_2^n(R_2^{-2n+4} - R_1^{-2n+4})}{R_2^{-2n+2} - R_1^{-2n+2}}\bar{c}_{-n+2} \\ &= G_n'' - \frac{2(n-1)hR_2^n(R_2^{-n+1}\overline{Q}_{-n+1}'' - R_1^{-n+1}\overline{Q}_{-n+1}')}{R_2^{-2n+2} - R_1^{-2n+2}}, \end{aligned} \tag{51}$$

$$\begin{aligned} &-\frac{i\gamma}{2} I_{n-2}(\gamma R_1)\beta_{n-1}' + \frac{i\gamma}{2} K_{n-2}(\gamma R_1)\beta_{n-1}'' \\ &+\left[R_1^{-n+2} + \frac{(n-1)R_1^{-n+2}(R_2^2 - R_1^2)}{R_2^{2n-2} - R_1^{2n-2}}\right]\bar{c}_{-n+2} \\ &+\left[\frac{4(\lambda+2\mu)h^2}{3\mu}(R_1^{n-2}(n-1) + R_1^n)n + \frac{(n-1)hR_1^{2-n}(R_2^{2n} - R_1^{2n})}{R_2^{2n-2} - R_1^{2n-2}}\right] c_n \\ &= \overline{G}_{-n+2}' - \frac{2(n-1)hR_1^{2-n}(R_2^{n-1}\overline{Q}_{n-1}'' - R_1^{n-1}\overline{Q}_{n-1}')}{R_2^{2-2n} - R_1^{2-2n}}, \end{aligned} \tag{52}$$

$$
\begin{aligned}
&-\frac{i\gamma}{2}I_{n-2}(\gamma R_2)\beta'_{n-1}+\frac{i\gamma}{2}K_{n-2}(\gamma R_2)\beta''_{n-1}\\
&+\left[R_2^{-n+2}+\frac{(n-1)R_2^{-n+2}(R_2^2-R_1^2)}{R_2^{2n-2}-R_1^{2n-2}}\right]\bar{c}_{-n+2}\\
&+\left[\frac{4(\lambda+2\mu)h^2}{3\mu}(R_2^{n-2}(n-1)+R_2^n)n+\frac{(n-1)hR_2^{2-n}(R_2^{2n}-R_1^{2n})}{R_2^{2n-2}-R_1^{2n-2}}\right]c_n\\
&=\overline{G}'_{-n+2}-\frac{2(n-1)hR_2^{2-n}(R_2^{n-1}\overline{Q}''_{n-1}-R_1^{n-1}\overline{Q}'_{n-1})}{R_2^{2-2n}-R_1^{2-2n}}.
\end{aligned}
\tag{53}
$$

The coefficients $\beta'_n,\ \beta''_n,\ c_n,\ d_n$ may be found from (49)-(53).

It is easy to prove the absolute and uniform convergence of the series obtained in the circular ring (including the contours) when the functions set on the boundaries have sufficient smoothness.

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