RESEARCH OF THREE AND TWO-DIMENSIONAL NONLINEAR DYNAMICAL SYSTEMS DESCRIBING THE TRAINING OF SCIENTISTS

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Abstract

In the paper two and three-dimensional nonlinear dynamical systems are offered describing training of scientists. In two-level model two subjects are considered: research associates without degree and the scientists with doctor's degree, and in three-stage model three subjects: research associates without degree, candidates of science and the scientists with doctor's degree. For system of two or three nonlinear differential equations the Cauchy problem task is set. The mathematical model describes process of self-reproduction of research associates (training), their irreversible exit or transition from one category to another.

In a three-dimensional case singular points of dynamic system are found. By means of Routh-Hurwitz stability criterion singular points are investigated on stability. In the specific case, the first integral of three-dimensional dynamic system which in space of decisions represents some two-dimensional surface is found.

The two-level model actually comes down to the known model "the victim" (research associates without degree) - "a predator" (scientists with doctor's degree) taking into account the intraspecific competition (members with self-limitation of increase). The system of the nonlinear differential equations in the first closed quarter of the phase plane of decisions has three position of balance, and the balance position corresponding to the trivial decision is a saddle at any values of parameters of model, the second position of balance corresponding to extinction of "predators" and to an equilibrium condition of "the victims" in one case is a saddle, and in the second stable node.

Conditions on constant models for which the stationary decision, the third position of balance in an open first quarter of the phase plane of decisions (the only limit point of system of the differential equations) corresponding to equilibrium coexistence of "predators" and "the victims" asymptotically is stable (stable node or stable focus) are found.

Key words and phrases: Two or three-dimensional nonlinear dynamic systems, first integral, position of balance, saddle, stable node, limit point.

AMS subject classification: 00A71, 97M10, 97M70.

1 Introduction

Mathematical modeling has been widely recognized in such disciplines as sociology, history, political science, and others [1].

In [2, 3] quantities of information streams by means of new mathematical models of information warfare are studied. By information warfare the authors mean an antagonism by means of mass media (an electronic and printing press, the Internet) between the two states or the two associations of states, or the economic structures (consortiums) conducting purposeful misinformation, propagation against each other. It was shown that in case of high aggression of the contradictory countries, not preventive image the operating peacekeeping organizations won't be able to extinguish the expanding information warfare.

In [4, 5] linear and nonlinear mathematical models of information warfare, and also optimizing problems are considered.

In [6] the nonlinear mathematical model of information warfare with participation of interstate authoritative institutes is offered. The model is described by Cauchy's problem for nonlinear non-homogeneous system of the differential equations. Confronting sides in extend of provocative statements, the third side (the peacekeeping international organizations) extends of soothing statements, interstate authoritative institutes the peacekeeping statements call the sides for the termination of information warfare. In that specific case, modes of information warfare "aggressor- victim, for the third peacekeeping side are received exact analytical solutions, and functions defining a number of the provocative statements distributed by the antagonistic sides satisfy Cauchy's problems for Riccati certain equations which are solved by a numerical method. For the general model computer modeling is carried out and shown that irrespective of high aggression of confronting sides, interstate authoritative institutes will be able to extinguish information warfare and when for this purpose efforts of only the international organizations insufficiently.

These papers [7, 8] present the nonlinear mathematical model of the public or the administrative management (or the macro and micro model). The cases of both constant and variable pressure forces on freethinking people were analyzed. Exact analytical decisions which determine dynamics of a spirit both free-thinking people, and operated (conformists) of people by time are received. During this analyses various governance systems were considered: a liberal, democratic, semi dictatorial and dictatorial.

These works [9-13] considered a two or three-party (one pro-government and two opposition parties) nonlinear mathematical model of elections when coefficients are constant. The assumption was made that the number of voters remain the same between two consecutive elections (zero demographic factor of voters). The exact analytical solutions were received. The conditions under which opposition party can win the upcoming elections were established.

These works [14-15] considered a two-party (pro-government and opposition parties) nonlinear mathematical model of elections with variable coefficients.

In [16] proposed the nonlinear mathematical model with variable coefficients in the case of three-party elections, that describes the dynamics of the quantitative change of the votes of the pro-government and two opposition parties from election to election. The model takes into account the change in the total number of voters in the period from election to election, i.e. the so-called demographic factor during the elections is taken into account. The model considered the cases with variable coefficients. In the particular case exact analytical solutions are obtained. The conditions have been identified under which the opposition can win the forthcoming elections, and in some cases, the pro-government party can stay in power.

In [17] consider the nonlinear mathematical model of bilateral assimilation without demographic factor. It was shown that the most part of the population talking in the third language is assimilated by that widespread language which speaks a bigger number of people. Also it was shown that in case of zero demographic factor of all three subjects, the population with less widespread language completely assimilates the states with two various widespread languages, and the result of assimilation (redistribution of the assimilated population) is connected with initial quantities, technological and economic capabilities of the assimilating states.

In [18] mathematical modeling of nonlinear process of assimilation taking into account demographic factor is offered. In the considered model taking into account demographic factor natural decrease in the population of the assimilating states and a natural increase of the population which has undergone bilateral assimilation is supposed. At some ratios between coefficients of natural change of the population of the assimilating states, and also assimilation coefficients, for the nonlinear system of three differential equations the two first integral are received. Cases of two powerful states assimilating the population of small state formation (autonomy), with different number of the population, both with identical and with various economic and technological capabilities are considered. It is shown that in the first case the problem is actually reduced to nonlinear system of two differential equations describing the classical model "predator - the victim", thus, naturally a role of the victim plays the population which has undergone assimilation, and a predator role the population of one of the assimilating states. The population of the second assimilating state in the first case changes in proportion (the coefficient of proportionality is equal to the relation of the population of assimilators in an initial time point) to the population of the first assimilating side. In the second case the problem is actually reduced to nonlinear system of two differential equations describing type model "a predator the victim", with the closed integrated curves on the phase plane. In both cases there is no full assimilation of the population to the less widespread language. Intervals of change of number of the population of all three objects of model are found. The considered mathematical models which in some approach can model real situations, with the real assimilating countries and the state formations (an autonomy or formation with the unrecognized status), undergone to bilateral assimilation, show that for them the only possibility to avoid from assimilation is the natural demographic increase in population and hope for natural decrease in the population of the assimilating states.

In [19] mathematical modeling of nonlinear process of the assimilation taking into account positive demographic factor which underwent bilateral assimilation of the side and zero demographic factor of the assimilating sides is considered. In model three objects are considered: the population and government institutions with the widespread first language, influencing by means of state and administrative resources on the population of the third state formation for the purpose of their assimilation; the population and government institutions with the widespread second language, influencing by means of state and administrative resources on the population of the third state formation for the purpose of their assimilation; population of the third state formation which is exposed to bilateral assimilation from two powerful states or the coalitions. For nonlinear system of three differential equations of the first order the two first integrals are received. Special cases of two powerful states assimilating the population of small state formation (autonomy), with different initial number of the population, both with identical and with various economic and technological capabilities are considered. It is shown that in all cases there is a full assimilation of the population to a less widespread language. Thus, proportions in which assimilate the powerful states the population of small state formation are found.

In [20, 21] model it is supposed that the powerful state with a widespread state language carries out assimilation of the population of less powerful state and the third population talking in two languages, different in prevalence. Carries out assimilation of the population of the state formation with the least widespread language to the turn, less powerful state. Not triviality of model assumes negative demographic factor of the powerful state-assimilating and positive demographic factor of the state formation which is under bilateral assimilation. For some ratios between demographic factors of the sides and coefficients of assimilations, for the nonlinear system of three differential equations with the corresponding conditions of Cauchy the first integrals are found. In particular, in the first case the first integral in space of required functions represents a hyperbolic paraboloid, and in the second case a cone. In these cases, the nonlinear system of three differential equations is reduced to the nonlinear system of two differential equations for which the second first integrals are found and in the phase plane of decisions are investigated behavior of integrated curves. In a more general case with application of a criteria of Bendixson the possibility of existence of the closed integrated curves is proved that indicates a possibility of a survival of the population finding under double assimilation.

One of the perspective and quick fields of application of mathematical modeling is dynamics of innovative processes. Researches in this area show that the crisis phenomena have not the casual, but systematic character defined by the determined mechanisms. Therefore many features of behavior of innovative processes can be described within the determined systems of the differential equations. The difficult behavior of these systems, including self-organization processes, gives in to the description thanks to existence of the nonlinear members who are present at mathematical models of dynamic systems. Research of mathematical models of innovative processes in scientific and educational areas is of a great interest.

In [22] the nonlinear mathematical model of dynamics of processes of cooperation interaction in innovative system: fundamental researches applied researches developmental works innovations is offered.

In [23] the new nonlinear mathematical model of interaction of fundamental and applied researches is considered.

In [24-26] the new nonlinear continuous mathematical model of interference of fundamental and applied researches on the example of one, perhaps closed for external customers, of scientifically - research institute (micromodel) is considered. For a special case, Cauchy's problem for nonlinear system of differential equations of first order is definitely decided analytically. In a more general case based on Bendixson's criteria the theorem of not existence in the first quarter of the phase plane of solutions of closed integral curves is proved. Conditions on model parameters in case of which existence of limited solutions of system of nonlinear differential equations is possible are found.

2 Nonlinear three-dimensional dynamical system describing training of the scientists

In three-stage mathematical model of training of scientists three subjects (three categories of research associates) are considered: research associates without degree (the first category), candidates of science (the second category) and the scientists with doctor's degree (the third category).

The mathematical model of nonlinear three-dimensional dynamical system has an appearance:

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1 u - \varepsilon_1 u^2 - \beta_1 uv - \gamma_1 uw, \\ \dfrac{dv(t)}{dt} = -\alpha_2 v - \varepsilon_2 v^2 - \gamma_2 v + \beta_1 uv + \gamma_1 uw - \beta_2 vw, \\ \dfrac{dw(t)}{dt} = -\alpha_3 w - \varepsilon_3 w^2 + \gamma_2 v + \beta_2 vw, \end{cases} \tag{2.1}$$

$$u(0) = u_0, \quad v(0) = v_0, \quad w(0) = w_0. \tag{2.2}$$

The mathematical model (2.1), (2.2) describes processes of reproduction of scientific research associates (scientists), their irrevocable leaving and also transitions of one category in another. In her the following designations are entered:

$u(t)$- the number of scientific research associates without degree in time point t;

$v(t)$ - the number of candidates of science in time point t;

$w(t)$ - the number of doctors of science in time point t;

$\alpha_1 u(t)$ - reproduction of research associates without academic degree (the difference between their preparation and leaving that isn't connected with transition to category of candidates of science in unit of time);

$\beta_1 u(t)v(t)$ - intensity of training of candidates of science from among research associates without degree $u(t)$ candidates of science $v(t)$;

$\gamma_1 u(t)w(t)$ - intensity of training of candidates of science from among research associates without degree $u(t)$ doctors of science $w(t)$;

$\alpha_2 v(t)$ - intensity of leaving of candidates of science from research associates without transition to category of doctors of science (leaving at the expense of mortality, intellectual migration, transition to other field of activity);

$\gamma_2 v(t)$ - intensity of self-training of candidates of science (without scientific consultant of the doctor of science) to the level of doctors of science;

$\beta_2 w(t)v(t)$ -intensity of training of doctors of science from among candidates sciences $v(t)$ doctors of science $w(t)$ (scientific consultants of the doctor of science);

$\alpha_3 w(t)$ - intensity of leaving of doctors of science from research associates (leaving at the expense of mortality, intellectual migration the abroad, transition to other field of activity);

$\varepsilon_1 u^2(t),\ \varepsilon_2 v^2(t),\ \varepsilon_3 w^2(t)$ - the members describing the intra group competition in the categories (the members who are responsible for growth self-restriction);

$\alpha_1,\ \alpha_2,\ \alpha_3,\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \beta_1,\ \beta_2,\ \gamma_1,\ \gamma_2$ - positive parameters of model.

For a case, when $\gamma_2 = 0$ (what, as a rule, corresponds to modern practice when doctors of science are trained only with participation of the scientific consultants who are doctors of science) the system of the equations (2.1) will take a form:

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1 u - \varepsilon_1 u^2 - \beta_1 uv - \gamma_1 uw, \\ \dfrac{dv(t)}{dt} = -\alpha_2 v - \varepsilon_2 v^2 + \beta_1 uv + \gamma_1 uw - \beta_2 vw, \\ \dfrac{dw(t)}{dt} = -\alpha_3 w - \varepsilon_3 w^2 + \beta_2 vw, \end{cases} \tag{2.3}$$

Singular points of the system (2.3) have an appearance:

$$O(0,0,0), \quad M_1\left(\frac{\alpha_1}{\varepsilon_1}, 0, 0\right), \quad M_2(u_\bullet, v_\bullet, w_\bullet),$$

$$u_* = \frac{\alpha_1\varepsilon_2 + \alpha_2\beta_1}{\varepsilon_1\varepsilon_2 + \beta_1^2}, \quad v_* = \frac{\alpha_1\beta_1 - \alpha_2\varepsilon_1}{\varepsilon_1\varepsilon_2 + \beta_1^2}, \quad w_* = 0, \tag{2.4}$$

$$M_3(u_{1\bullet}, v_{1\bullet}, w_{1\bullet}), \quad M_4(u_{2\bullet}, v_{2\bullet}, w_{2\bullet}),$$

$$v_{1\bullet},\ v_{2\bullet}: \quad a v_\bullet^2 - b v_\bullet + c = 0, \tag{2.5}$$

$$a = \varepsilon_1\varepsilon_2\varepsilon_3^2 + \beta_2^2\varepsilon_1\varepsilon_3 + \gamma_1^2\beta_2^2 + \gamma_1\beta_1\beta_2\varepsilon_3 + \beta_1^2\varepsilon_3^2 + \beta_1\varepsilon_3\gamma_1\beta_2 > 0,$$

$$c = \gamma_1\alpha_3(\alpha_1\varepsilon_3 + \gamma_1\alpha_3) > 0,$$

$$b = (\beta_1\alpha_1 - \varepsilon_1\alpha_2)\varepsilon_3^2 + \beta_1\gamma_1\varepsilon_3\alpha_3 + \gamma_1\beta_2\alpha_1\varepsilon_3$$
$$+2\gamma_1^2\beta_2\alpha_3 + \beta_1\varepsilon_3\gamma_1\alpha_3 + \beta_2\varepsilon_1\varepsilon_3\alpha_3.$$

At the same time, if it is executed

$$\alpha_1\beta_1 > \alpha_2\varepsilon_1. \tag{2.6}$$

that takes place

$$M_2 \in R_+^3.$$

If it is executed (2.6), then $b > 0$ and at

$$D = b^2 - 4ac \geq 0, \tag{2.7}$$

$$v_{1\bullet} \in \mathrm{Re}, \quad v_{2\bullet} \in \mathrm{Re}, \quad v_{1\bullet} > 0, \quad v_{2\bullet} > 0,$$

$$(D = 0,\ v_{1\bullet} = v_{2\bullet}),$$

$$u_{1\bullet} = \frac{1}{\varepsilon_1\varepsilon_3}[\alpha_1\varepsilon_3 + \gamma_1\alpha_3 - (\beta_1\varepsilon_3 + \gamma_1\beta_2)v_{1\bullet}] > 0,$$

$$u_{2\bullet}=\frac{1}{\varepsilon_1\varepsilon_3}[\alpha_1\varepsilon_3+\gamma_1\alpha_3-(\beta_1\varepsilon_3+\gamma_1\beta_2)v_{2\bullet}]>0,$$

$$w_{1\bullet}=\frac{\beta_2 v_{1\bullet}-\alpha_3}{\varepsilon_3}>0,\quad w_{2\bullet}=\frac{\beta_2 v_{2\bullet}-\alpha_3}{\varepsilon_3}>0,$$

that takes place

$$M_3\in R_+^3,\quad M_4\in R_+^3.$$

Let us investigate the singular point $M_2(u_\bullet, v_\bullet, 0)$ of the system (2.3) for stability:

We will enter transformation

$$\begin{cases} u_1=u-u_\bullet,\\ v_1=v-v_\bullet,\\ w_1=w. \end{cases} \tag{2.8}$$

Substituting (2.8) in (2.3) we will receive

$$\begin{cases} \dfrac{du_1}{dt}=(\alpha_1-2\varepsilon_1u_\bullet-\beta_1v_\bullet)u_1-\beta_1u_\bullet v_1-\gamma_1u_\bullet w_1\\ -\varepsilon_1u_1^2-\beta_1u_1v_1-\gamma_1u_1w_1,\\ \dfrac{dv_1}{dt}=\beta_1v_\bullet u_1+(-\alpha_2-2\varepsilon_2v_\bullet+\beta_1u_\bullet)v_1+(\gamma_1u_\bullet-\beta_2v_\bullet)w_1\\ -\varepsilon_2v_1^2+\beta_1u_1v_1+\gamma_1u_1w_1-\beta_2v_1w_1,\\ \dfrac{dw_1)}{dt}=(-\alpha_3+\beta_2v_\bullet)w_1-\varepsilon_3w_1^2+\beta_2v_1w_1. \end{cases} \tag{2.9}$$

Leaving only linear members in system (2.9), we will receive the linear system of the differential equations

$$\begin{cases} \dfrac{du_1}{dt}=(\alpha_1-2\varepsilon_1u_\bullet-\beta_1v_\bullet)u_1-\beta_1u_\bullet v_1-\gamma_1u_\bullet w_1,\\ \dfrac{dv_1}{dt}=\beta_1v_\bullet u_1+(-\alpha_2-2\varepsilon_2v_\bullet+\beta_1u_\bullet)v_1+(\gamma_1u_\bullet-\beta_2v_\bullet)w_1,\\ \dfrac{dw_1}{dt}=(-\alpha_3+\beta_2v_\bullet)w_1. \end{cases} \tag{2.10}$$

Thus, the nonlinear autonomous system (2.3), with the help of linearization(2.8) near a singular point $M_2(u_\bullet, v_\bullet, 0)$ is reduced to a linear autonomous system (2.10), the stability or instability of its equilibrium position $(0,0,0)$ is determined by the signs of the real values (parts) of the eigenvalues (numbers) of the following matrix A

$$A=\begin{pmatrix} \alpha_1-2\varepsilon_1u_\bullet-\beta_1v_\bullet & -\beta_1u_\bullet & -\gamma_1u_\bullet\\ \beta_1v_\bullet & -\alpha_2+\beta_1u_\bullet-2\varepsilon_2v_\bullet & \gamma_1u_\bullet-\beta_2v_\bullet\\ 0 & 0 & -\alpha_3+\beta_2v_\bullet \end{pmatrix} \tag{2.11}$$

which are found from the characteristic equation

$$|A - \lambda E| = 0,$$

which has the form

$$\begin{aligned}&\lambda^3 + (\alpha_3 - \beta_2 v_\bullet + \varepsilon_1 u_\bullet + \varepsilon_2 v_\bullet)\lambda^2\\ &+[(\varepsilon_1\varepsilon_2 + \beta_1^2)v_\bullet u_\bullet + (\alpha_3 - \beta_2 v_\bullet)(\varepsilon_1 u_\bullet + \varepsilon_2 v_\bullet)]\lambda\\ &+(\alpha_3 - \beta_2 v_\bullet)(\varepsilon_1\varepsilon_2 + \beta_1^2)v_\bullet u_\bullet = 0.\end{aligned} \tag{2.12}$$

The Routh-Hurwitz stability criterion consisting of the positivity of all the coefficients of the cubic polynomial in (2.12) and the positivity of all the diagonal minors of the Hurwitz matrix leads in inequality for the parameters of the model

$$\alpha_3(\varepsilon_1\varepsilon_2 + \beta_1^2) - \beta_2\alpha_1\beta_1 + \alpha_2\beta_2\varepsilon_1 > 0. \tag{2.13}$$

It is not difficult to show that the other two singular points $O(0,0,0)$, $M_1\left(\frac{\alpha_1}{\varepsilon_1}, 0, 0\right)$ are unstable.

For example, in the case of a singular point $O(0,0,0)$, one of the three real eigenvalues of the linearized matrix is positive, and in the case of a singular point $M_1\left(\frac{\alpha_1}{\varepsilon_1}, 0, 0\right)$, not all coefficients of a cubic polynomial

$$\varepsilon_1\lambda^3 - (\alpha_1\varepsilon_1 + \alpha_3\varepsilon_1 + \alpha_2\varepsilon_1 - \beta_1\alpha_1)\lambda^2$$

$$+[\alpha_1\alpha_3\varepsilon_1 - (\alpha_1 + \alpha_3)(\beta_1\alpha_1 - \alpha_2\varepsilon_1)]\lambda$$

$$+\alpha_1\alpha_3(\alpha_2\varepsilon_1 - \beta_1\alpha_1) = 0$$

is positive and the Routh-Hurwitz stability criterion is not satisfied.

3 Special case. First integral of the three-dimensional dynamical system

We will consider a special case of system (2.1) when

$$\varepsilon_i = \gamma_2 = 0, \quad i = 1, 2, 3;$$

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1 u - \beta_1 uv - \gamma_1 uw,\\ \dfrac{dv(t)}{dt} = -\alpha_2 v + \beta_1 uv + \gamma_1 uw - \beta_2 vw,\\ \dfrac{dw(t)}{dt} = -\alpha_3 w + \beta_2 vw. \end{cases} \tag{3.1}$$

Add up the second and third equations of system (3.1)

$$\frac{d(v(t)+w(t))}{dt} = -\alpha_2 v - \alpha_3 w + \beta_1 uv + \gamma_1 uw. \tag{3.2}$$

We will assume that equalities take place

$$\alpha_2 = \alpha_3, \quad \beta_1 = \gamma_1. \tag{3.3}$$

Then from (3.2), (3.3) we will receive

$$\frac{d(v(t)+w(t))}{dt} = (v+w)(\beta_1 u - \alpha_2). \tag{3.4}$$

Add up all three equations of system (3.1), taking into account (3.3), we will receive

$$\frac{d(u(t)+v(t)+w(t))}{dt} = \alpha_1 u - \alpha_2(v+w). \tag{3.5}$$

We will enter designation

$$\psi \equiv v + w. \tag{3.6}$$

Then from (3.4)–(3.6), we will receive

$$\begin{cases} \dfrac{d\psi}{dt} = \psi(\beta_1 u - \alpha_2), \\ \dfrac{du}{dt} + \dfrac{d\psi}{dt} = \alpha_1 u - \alpha_2 \psi. \end{cases} \tag{3.7}$$

From system (3.7) it is easy to receive the following system

$$\begin{cases} \dfrac{d\psi}{dt} = \psi(\beta_1 u - \alpha_2), \\ \dfrac{du}{dt} = u(\alpha_1 - \beta_1 \psi). \end{cases} \tag{3.8}$$

From system (3.8), taking into account (2.2), it is easy to receive its first integral

$$\alpha_1 \ln \frac{\psi}{\psi_0} - \beta_1(\psi - \psi_0) = \beta_1(u - u_0) - \alpha_2 \ln \frac{u}{u_0}. \tag{3.9}$$

Taking into account designation (3.6), from (3.9) we will receive the first integral of system (3.1), taking into account (2.2), (3.3)

$$\alpha_1 \ln \frac{v+w}{v_0+w_0} = \beta_1(u+v+w-u_0-v_0-w_0) - \alpha_2 \ln \frac{u}{u_0}. \tag{3.10}$$

At last, if we assume

$$\alpha_1 = \alpha_2 = \beta_1.$$

Then the first integral (3.10), will take the form

$$\frac{u(v+w)}{u_0(v_0+w_0)} = \exp(u-u_0)\exp(v+w-v_0-w_0). \tag{3.11}$$

The first integral (3.11) in phase space $(O, u(t), v(t), w(t))$ of solutions of system (2.1), (2.2) represents some two-dimensional surface.

4 Nonlinear two-dimensional dynamical system describing training of the scientists

In two-stage mathematical model of training of scientists two subjects (two categories of research associates) are considered: research associates without degree (the first category) and the scientists with doctor's degree (the second category).

The mathematical model of nonlinear dynamical system has an appearance:

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1 u - \varepsilon_1 u^2 - \beta_1 uv = u(\alpha_1 - \varepsilon_1 u - \beta_1 v), \\ \dfrac{dv(t)}{dt} = -\alpha_2 v - \varepsilon_2 v^2 + \beta_2 uv = v(-\alpha_2 - \varepsilon_2 v + \beta_2 u). \end{cases} \tag{4.1}$$

$$u(0) = u_0, \quad v(0) = v_0. \tag{4.2}$$

The mathematical model (4.1), (4.2) describes processes of reproduction of scientific research associates (scientists), their irrevocable leaving and also transitions of one category in secondary. In her the following designations are entered:

$u(t)$ - the number of research associates without degree in time point t;

$v(t)$ - the number of research associates with doctor degree in time point t;

$\alpha_1 u(t)$ - reproduction of research associates without academic degree (the difference between their preparation and leaving that isn't connected with transition to secondary category in unit of time);

$\beta_1 u(t)v(t)$, $\beta_2 u(t)v(t)$ - intensity of training of doctors from among research associates without degree;

$\alpha_2 v(t)$ - intensity of leaving of doctors from research associates without transition to secondary category (leaving at the expense of mortality, intellectual migration, transition to other field of activity);

$\varepsilon_1 u^2(t)$, $\varepsilon_2 v^2(t)$ - the members describing the intra group competition in the categories (the members who are responsible for growth self-restriction);

$\alpha_1,\ \alpha_2,\ \varepsilon_1,\ \varepsilon_2,\ \beta_1$ - positive parameters of model.

In the first closed quarter of the phase plane of decisions R^2_+ system (4.1) has three equilibrium positions

$$O(0,0),\ \ N_1\left(\frac{\alpha_1}{\varepsilon_1},0\right),\ \ N_2\left(\frac{\alpha_1\varepsilon_2+\alpha_2\beta_1}{\varepsilon_1\varepsilon_2+\beta_1\beta_2},\frac{\alpha_1\beta_2-\alpha_2\varepsilon_1}{\varepsilon_1\varepsilon_2+\beta_1\beta_2}\right). \tag{4.3}$$

It is clear, that

$$N_2 \in R^2_+,$$

if

$$\alpha_1\beta_2 > \alpha_2\varepsilon_1. \tag{4.4}$$

Jacobi's matrix of system (4.1) has an appearance:

$$J(u,v)=\begin{pmatrix} \alpha_1-2\varepsilon_1 u-\beta_1 v & -\beta_1 u \\ \beta_2 v & -\alpha_2-2\varepsilon_2 v+\beta_2 u \end{pmatrix} \tag{4.5}$$

We will consider the two-dimensional dynamical system

$$\begin{cases} \dfrac{du(t)}{dt}=f(u,v), \\ \dfrac{dv(t)}{dt}=g(u,v), \end{cases} \tag{4.6}$$

where $u(t),v(t)\in R$.

Let $(u,v)=(0,0)$ be position of balance of task (4.6). Jacobi's matrix has an appearance

$$J(u,v)=\begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix}.$$

Accordingly, there are two eigenvalues $\lambda_1,\ \lambda_2$, which are the roots of the characteristic equation

$$\lambda^2-\lambda trJ+\det J=0,$$

$$\lambda_{1,2}=\frac{trJ\pm\sqrt{(trJ)^2-4\det J}}{2}.$$

There are three topological classes of hyperbolic equilibrium positions (the equilibrium position of a dynamical system is said to be hyperbolic if there are no eigenvalues located on the imaginary axis, a hyperbolic equilibrium position is called a hyperbolic saddle if there exists at least one eigenvalue with a positive and negative real part) in the plane :stable node (focuses), saddles, unstable node (focuses). The first class of equilibrium positions is asymptotically stable (the equilibrium position is a particular case of the more general concept of an attractor), the saddle is unstable, the node (focuses) are unstable, which are particular cases of repellers. In describing the behavior of a dynamical system in terms of the phase space (the space of state variables of a mathematical model), a steady state corresponds to an object (state / set of states / some trajectory) that "attracts" trajectories from a certain neighborhood. Such an object (or geometric image of steady self-oscillations in the phase space of the system) is called the attractor. A dynamic system can have several attractors with the same values of control parameters.

The attractor is asymptotically the stable solution of the closed system, and the repeller, in turn, is the area of phase space rejecting phase trajectories of the movement of system.

A detailed analysis of the values of the eigenvalues of the Jacobi matrix (4.5) in points of position of balance (4.3) shows that $O(0,0)$ a saddle at any values of parameters, $N_1\left(\frac{\alpha_1}{\varepsilon_1},0\right)$ the saddle if is executed (4.4) and stable node if

$$\alpha_1\beta_2 < \alpha_2\varepsilon_1. \tag{4.7}$$

Let it is executed (4.4).

Then the solution of system

$$\begin{cases} \varepsilon_1 u + \beta_1 v = \alpha_1, \\ \beta_2 u - \varepsilon_2 v = \alpha_2. \end{cases}$$

has the form

$$\begin{aligned} u = u_* &= \frac{\alpha_1\varepsilon_2 + \alpha_2\beta_1}{\varepsilon_1\varepsilon_2 + \beta_1\beta_2}, \\ v = v_* &= \frac{\alpha_1\beta_2 - \alpha_2\varepsilon_1}{\varepsilon_1\varepsilon_2 + \beta_1\beta_2}. \end{aligned} \tag{4.8}$$

Then, it agrees (4.5), (4.8), we will receive

$$\begin{aligned} &trJ(u_*,v_*) = \alpha_1 - 2\varepsilon_1 u_* - \beta_1 v_* - \alpha_2 - 2\varepsilon_2 v_* + \beta_2 u_* \\ &= -\varepsilon_1 u_* - \varepsilon_2 v_* < 0, \end{aligned} \tag{4.9}$$

$$\det J(u_*,v_*) = u_* v_*(\varepsilon_1\varepsilon_2 + \beta_1\beta_2) > 0.$$

Therefore balance position $N_2\left[\frac{\alpha_1\varepsilon_2 + \alpha_2\beta_1}{\varepsilon_1\varepsilon_2 + \beta_1\beta_2}, \frac{\alpha_1\beta_2 - \alpha_2\varepsilon_1}{\varepsilon_1\varepsilon_2 + \beta_1\beta_2}\right]$ asymptotically it is stable(stable node $((trJ(u_*,v_*))^2 \geq 4\det J(u_*,v_*))$ or stable focus $((trJ(u_*,v_*))^2 < 4\det J(u_*,v_*))$).

Unlike the classical Lotka-Volterra model, the variables in the trajectory equation (4.1) are not separated, so for the global analysis of the phase flow we apply the null-isocline method (lines where one of the components of the vector field is zero).

The key idea consists in dividing a set R_+^2 into areas, in which $\frac{du}{dt}, \frac{dv}{dt}$ have a certain sign and to use the following statement.

Statement. Let $\varphi(t) = (u(t), v(t))$ be the solution of continuous dynamic system on the plane. We will assume that U is an open domain, and her closure $\overline{U}$ is compact. If $u(t), v(t)$ are strongly monotonous in U, that or $\varphi(t)$ reaches U domain boundary for some finite time $t = t_0$ or meets to equilibrium position $(u_*, v_*) \in \overline{U}$.

We will consider a case (4.7). Areas in which $\frac{du}{dt}, \frac{dv}{dt}$ have a certain sign are divided by straight lines

$$L_1 = \{(u(t), v(t)) :\ \varepsilon_1 u + \beta_1 v = \alpha_1\},$$

We will designate areas on which a set R_+^2 breaks straight lines L_1 and L_2 as (from left to right). We will assume that the trajectory begins in a point $(u_0, v_0) \in U_3$. Then, to draw a conclusion that trajectories surely get to the area U_2 through L_2, we will add restriction $u < u_0$. The trajectories beginning in U_2, meet to position of equilibrium $N_1\Big(\frac{\alpha_1}{\varepsilon_1}, 0\Big)$ or, crossing L_1, get in U_1. At last, if the trajectory begins in U_1, that the only opportunity the aspiration is to $N_1\Big(\frac{\alpha_1}{\varepsilon_1}, 0\Big)$. Thus, it is proved that any trajectory beginning int $R_+^2 = \{(u, v) :\ u > 0,\ v > 0\}$ meets to equilibrium position $N_1\Big(\frac{\alpha_1}{\varepsilon_1}, 0\Big)$.

Let inequality be executed now (4.4). Then equilibrium $N_1\Big(\frac{\alpha_1}{\varepsilon_1}, 0\Big)$ - hyperbolic saddle, point $N_2 \in R_+^2$ - asymptotically stable equilibrium position. Straight lines L_1 and L_2 also break space of states into four areas. As well as earlier, it is possible to prove that trajectories pass through these areas in the following order: $U_4 \to U_3 \to U_2 \to U_1 \to U_4$ if the orbits do not converge to the position of equilibrium. The main difference from the previous case consists in a possibility (basic) of existence of periodic trajectories as the orbit beginning in U_4 again can get to this area.

Let the dynamical system be set

$$\frac{dw}{dt} = f(w), \quad w \in R^n, \quad f : R^n \to R^n \tag{4.10}$$

and differentiable function $V : R^n \to R$. We will remind that in each point of phase space the vector $f(w)$ sets the direction, tangent to phase trajectories if $f(w) \neq 0$. We will consider the speed of change of function $V(w)$ in the direction of a vector $f(w)$ (a derivative in the direction $f(w)$).

By definition of a derivative in the direction of a vector $f(w)$ we have:

$$\frac{\partial V}{\partial f} = \sum_{i=1}^{n} \frac{\partial V}{\partial w_i} f_i(w) = (gradV, f)\,.$$

Definition 1. A derivative of Lie or a derivative along the trajectory of system (4.10) is the expression

$$L_t V = \left(gradV, \frac{dw}{dt} \right) = \sum_{i=1}^{n} \frac{\partial V}{\partial w_i} \frac{dw_i}{dt} = (gradV, f) \tag{4.11}$$

As is known, the following theorem holds [27].

Theorem 1. Let system

$$\frac{dw}{dt} = f(w), \;\; w \in U \subset R^n, \;\; f : U \to R^n \tag{4.12}$$

it is defined on some set $U \subseteq R^n$. Let function $V : U \to R$ it is continuously differentiated. If for some decision $w(t; w_0)$, belonging U for all $t \geq 0$, derivative $L_t V$ (4.11) owing to system (4.12) satisfies to inequality $L_t V \leq 0$ (or $L_t V \geq 0$), that $\omega(w_0) \cap U(\alpha(w_0) \cap U)$ contains in a set $\{w \in U : \; L_t V = 0\}$.

Definition 2. The point w is called a positive limit point of a trajectory of the system (4.12) corresponding to the decision $w(t; w_0)$, if there is a sequence $\{t_k\}$, $t_k \to \infty$ such that $w(t_k; w_0) \to w$.

Definition 3. The set of all positive limit points of the trajectory answering $w(t; w_0)$ is called an omega - a limit (alpha and limit) set and is designated $\omega(w_0)(\alpha(w_0))$.

Theorem 2. If inequality (4.4) is satisfied, then the point $N_2(u_*, v_*)$ is a limit point for the system (4.1).

Proof of the theorem. To prove the absence of periodic trajectories of the system (4.1), we consider the function

$$V(u, v) = \beta_2(u_\bullet \ln u - u) + \beta_1(v_\bullet \ln v - v) \tag{4.13}$$

where $u_\bullet$, $v_\bullet$-the coordinates N_2, determined in (4.8).

The derivative $L_t V(u, v)$ (4.11) along a trajectory of system (4.1) has the form:

$$L_t V(u, v) = \beta_2 \left(\frac{u_\bullet}{u} \frac{du}{dt} - \frac{du}{dt} \right) + \beta_1 \left(\frac{v_\bullet}{v} \frac{dv}{dt} - \frac{dv}{dt} \right)$$

$$= \beta_2(u_\bullet - u)(\alpha_1 - \varepsilon_1 u - \beta_1 v) + \beta_1(v_\bullet - v)(-\alpha_2 - \varepsilon_2 v + \beta_2 u),$$

$$L_t V(u, v) = \beta_2 \varepsilon_1 (u_\bullet - u)^2 + \beta_1 \varepsilon_2 (v_\bullet - v)^2 \geq 0. \tag{4.14}$$

The expression (4.14) is nonnegative, and becomes equal to zero when $u = u_\bullet,\ v = v_\bullet$.

Applying the theorem 1, we will receive that the point $N_2(u_\bullet, v_\bullet)$ is a limit point for system (4.1). Thus, the system (4.1) allows existence of two topological not equivalent phase portraits. If $\alpha_1\beta_2 < \alpha_2\varepsilon_1$, that a global attractor is $N_1\left(\frac{\alpha_1}{\varepsilon_1}, 0\right)$ (”predators” die out, and population of ”victims” is in an equilibrium state). If $\alpha_1\beta_2 > \alpha_2\varepsilon_1$, that appears asymptotically a stable position of equilibrium $N_2\left(u_\bullet, v_\bullet\right)$ (equilibrium coexistence of” predators” and ”victims”).

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