# THE PROBLEM OF FINDING AN EQUALLY STRONG CONTOUR FOR A RECTANGULAR PLATE WEAKENED BY A RECTILINEAR CUT

G. Kapanadze

I. Vekua Institute of Applied Mathematics of
I. Javakhishvili Tbilisi State University
2 University Str., Tbilisi 0186, Georgia
A. Razmadze Mathematical Institute of
I. Javakhishvili Tbilisi State University
6 Tamarashvili Str., Tbilisi 0186, Georgia

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*Abstract*

In the prezent work we consider the problem of finding an equally strong contour for a rectangular plate weakened by a rectilinear cut which ends are cut out by convex smooth arcs. It is assumed that absolutely smooth rigid punches are applied to every link of the rectangular. The punches are under the action of normal stretching forces with the given principal vectors and the internal part of the boundary is free from external forces. Our problem is to find an elastic equilibrium of the plate and analytic form of the unknown contour under the condition that the tangential normal stress on it takes constant value (the condition of the unknown contour full-strength). For solution of the problem using the method of complex analysis and Kolosov-Muskhelishvilis potentials and the equation of the equally strong contour are constructed effectively (analytically).

*Key words and phrases*: Kolosov-Muskhelishvili's formulas, Conformal mapping, The Riemann-Hilbert problem, The Keldysh-Sedov problem.

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## 1 Introduction

The boundary value problems of the plane theory of elasticity and plate bending with a partially unknown boundary (or, which is the same, the problems of finding an equally stable contour) was started [1-2] and always was in the focus of attention of many scientists. Different methods were introduced for researching these problems and among them one important is the method of complex analysis.

In the present paper we consider the problem of finding a partially unknown boundary of the plane theory of elasticity for a rectangular domain which is weakened by a rectilinear cut which ends are cut out by convex smooth arcs (the unknown part of the boundary).

Analogous problems of plane elasticity are considered in [3-10].

## 2 Statement of the Problem

Let a middle surface of the homogeneous Isotropic plate on a plane $z$ of a complex variable occupy a doubly-connected domain $S_0$ whose external boundary is a rectangle and internal boundary is a rectilinear cut which ends are cut out by convex smooth arcs (the unknown part of the boundary, see Fig. 1). It is assumed that the sides of the rectangle are under the action of constant normal tensile forces with a given principal vector $P$ and $Q$ (we consider the symmetrical case), and the interior boundary is free from stresses. Consider the problem: Find an elastic equilibrium of the

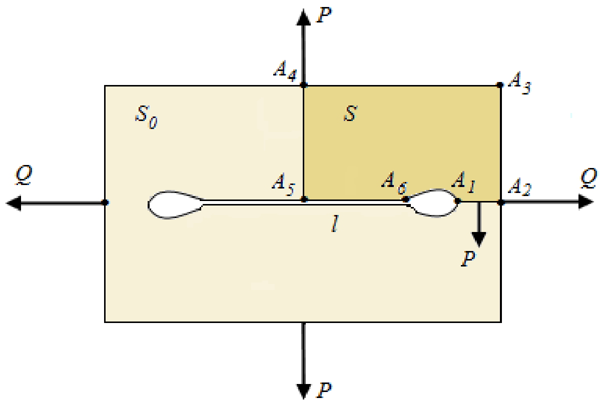


Fig. 1.

rectangular and analytic form of an unknown contour under the condition that the tangential normal stress takes on that contour value $\sigma_\vartheta = k_0 = const$.

## 3 Solution of the problem

Due to the axial symmetry, we restrict ourselves to the consideration of elastic equilibrium on a quarter of the plate only and denote it by $S$. By $L = L_1 \cup L_2$ we denote its boundary consisting of rectilinear segments $L_1 = \bigcup\limits_{k=1}^{5} L_k^{(1)} = \bigcup\limits_{k=1}^{5} A_k A_{k+1}$ and arc $L_2 = A_6 A_1$.

It is not difficult to see that in this case the tangential stresses $\tau_{ns} = 0$ on the whole boundary $L = L_1 \cup L_2$ of the domain $S$, the normal displacements $v_n(t) = const,\ t \in A_2A_3 \cup A_3A_4$ and $v_n(t) = 0$ on $L_1^{(1)} \cup L_1^{(4)} \cup L_2$.

On the basis of the well-known Kolosov-Muskhelishvili's formulas [11] the problem under consideration is reduced to finding two holomorphic in $S$ functions $\varphi(z)$ and $\psi(z)$ with the following boundary conditions:

$$\mathrm{Re}\left[e^{-i\alpha(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})\right] = C(t), \quad t \in L_1, \tag{1}$$

$$\mathrm{Re}\left[e^{-i\alpha(t)}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}\right] = 2\mu v_n(t), \quad t \in L_1, \tag{2}$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = 0,\ t \in L_2, \tag{3}$$

$$\mathrm{Re}\left[\varphi'(t)\right] = \frac{k_0}{4}, \quad t \in L_2, \tag{4}$$

where $\alpha(t)$ is the angle lying between the $ox$-axis and the external normal to the boundary $L_1$ at the point $t \in L_1$, so $\alpha(t) = \alpha_k = -\dfrac{\pi}{2} + \dfrac{\pi(k-1)}{2}, \quad t \in L_k^{(1)}$ $(k = \overline{1,5})$. $C(t)$ and $v_n(t)$ are the piecewise constant functions

$$C(t) = \mathrm{Re}\int\limits_{A_1}^{t} i\sigma_n(s_0)\exp i[\alpha(t_0) - \alpha(t)]ds_0$$

$$= \sum_{j=1}^{k}\int\limits_{L_j^{(1)}} \sigma_n(s_0)\sin(\alpha_k - \alpha_j]ds_0 = \begin{cases} 0, \ \ t \in L_1^{(1)} \cup L_4^{(1)} \cup L_5^{(1)}, \\ \dfrac{P}{2}, \ \ t \in L_2^{(1)}, \\ -\dfrac{Q}{2}, \ \ t \in L_3^{(1)}. \end{cases}$$

$$v_n(t) = v_n^{(k)} = \mathrm{const}, \quad t \in L_1^{(k)}(k = 2,3)$$

$$v_n(t) = 0 \ \ t \in L_k^{(1)} \ \ (k = 1,4,5).$$

Summing up the equalities (1) and (2), differentiating with respect to the arc abscissa $s$ and taking into account the fact that the functions $c(t)$ and $v_n(t)$ are piecewise constants, we obtain

$$\mathrm{Im}\,\varphi'(t) = 0, \quad t \in L_1. \tag{5}$$

The conditions (4) and (5) are the Keldysh-Sedov problem (see [12], [13])

$$\mathrm{Re}\left[\varphi'(t) - \frac{k_0}{4}\right] = 0,\ t \in L_2; \quad \mathrm{Im}\left[\varphi'(t) - \frac{k_0}{4}\right] = 0 \ \ t \in L_1. \tag{6}$$

After the conformal mapping of the domain $S$ onto the circular ring, this problem for the function $\phi^0(\zeta) = \varphi'[\omega_0(\zeta)] - \frac{k_0}{4}$ reduces to the Riemann-Hilbert problem which has a unique solution

$$\varphi(z) = \frac{k_0}{4} z \tag{7}$$

(an arbitrary constant of integration is assumed to be equal to zero).

Let the function $z = \omega(\zeta)$ map conformally the unit semi-circle $D_0 = \{|\zeta| < 1,\ \mathrm{Im}\zeta > 0\}$ onto the region $S$. By $a_k,\ (k = 1, ..., 6)$ we denote preimages of the points $A_k$ and assume that $a_1 = 1,\ a_3 = i,\ a_5 = -1$, i.e. the contour $A_5A_1 = A_5A_6 \cup A_6A_1$ transforms into the segment $[-1,\ 1]$ and point $A_6$ into the point $-\delta_0 \in [-1,\ 1]$.

We easily observe that for $t \in L_k^{(1)}$ $(k = \overline{1,4})$ we have the equality

$$\mathrm{Re}[e^{-i\alpha(t)} t] = \mathrm{Re}[e^{-i\alpha(t)} A(t)], \tag{8}$$

where $A(t) = A_k,\ t \in L_k^{(1)},\ (k = \overline{1,4})$.

The boundary conditions (1) and (3) with respect to the function $\omega(\zeta)$ and (7), (8), have the following forms

$$\mathrm{Re}[e^{-i\alpha(\sigma)} \omega(\sigma)] = \mathrm{Re}[e^{-i\alpha(\sigma)} A(\sigma)],\ \ \sigma \in l_1, \tag{9}$$

$$\mathrm{Re}[e^{-i\alpha(\sigma)} \overline{\psi_0(\sigma)}] = C(\sigma) - \frac{k_0}{2} \mathrm{Re}[e^{-i\alpha(\sigma)} A(\sigma)],\ \ \sigma \in l_1, \tag{10}$$

$$\frac{k_0}{2} \omega(\sigma) = \psi_0(\sigma),\quad \sigma \in l_2 \tag{11}$$

where $l_1 = \bigcup\limits_{k=1}^{4} l_1^{(k)}$ ($l_1^{(k)}$ are preimages of the segments $L_k^{(1)}$ and $l_2 = [-1, 1]$ is preimages of the segment $A_5A_1$), $\psi_0(\zeta) = \psi[\omega(\zeta)]$.

Consider the fuction

$$W(\zeta) = \begin{cases} \frac{k_0}{2} \omega(\zeta), & |\zeta| < 1,\quad \mathrm{Im}\zeta > 0, \\ -\psi_{0*}(\zeta), & |\zeta| < 1,\ \ \mathrm{Im}\zeta < 0, \end{cases} \tag{12}$$

where $\psi_{0*}(\zeta) = \overline{\psi_0(\overline{\zeta})}$.

On the basis of (11) we conclude that $W^+(\sigma) = W^-(\sigma),\ \sigma \in l_2$ and so the function $W(\zeta)$ is holomorphic in the circle $D = \{|\zeta| < 1\}$, continuously extendable up to the boundary $l = \{|\sigma| = 1\}$ and satisfies the boundary conditions

$$\begin{aligned} &\mathrm{Re}[iW(\sigma)] = 0,\ \ \sigma \in l_1^{(1)};\quad \mathrm{Re}[iW(\sigma)] = 0,\ \ \sigma \in l_{1*}^{(1)}; \\ &\mathrm{Re}[W(\sigma)] = \frac{k_0}{2} a,\ \ \sigma \in l_2^{(1)};\quad \mathrm{Re}[W(\sigma)] = \frac{P - k_0 a}{2},\ \ \sigma \in l_{2*}^{(1)}; \\ &\mathrm{Re}[iW(\sigma)] = -\frac{k_0}{2} b,\ \ \sigma \in l_3^{(1)};\quad \mathrm{Re}[iW(\sigma)] = \frac{Q + k_0 b}{2},\ \ \sigma \in l_{3*}^{(1)}; \\ &\mathrm{Re}[W(\sigma)] = 0,\ \ \sigma \in l_4^{(1)};\quad \mathrm{Re}[iW(\sigma)] = 0,\ \ \sigma \in l_{4*}^{(1)}, \end{aligned} \tag{13}$$

where $2a$ and $2b$ are the length of the sides of the rectangle $S_0$; $l_{j*}^{(1)}$ $(j = \overline{1,4})$ are the image of the arc $l_j^{(1)}$ under the mapping $\zeta = \bar{\zeta}$.

We will seek for a bounded at points $a_k$ and $\bar{a}_k$ $(k = \overline{1,4})$ solution of the problem (13) of the class $h(a_1, ..., \bar{a}_4)$. The indices of this problems of the given class are equal to $-2$ (see [12]).

Consider the functions $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ defined by the formulas

$$\Phi_1(\zeta) = \frac{1}{2}\left[(\zeta) + W_*(\zeta)\right]; \quad \Phi_2(\zeta) = \frac{i}{2}[W(\zeta) - W_*(\zeta)], \tag{14}$$

where $W_*(\zeta) = \overline{W(\bar{\zeta})}$.

It is easily seen that the functions $\Phi_j(\zeta)$ $(j = 1, 2)$ satisfy the condition

$$\Phi_j(\zeta) = \Phi_{j*}(\zeta), \quad (j = 1, 2), \tag{15}$$

and the function $W(\zeta)$ is defined through the above functions by the formula

$$W(\zeta) = \Phi_1(\zeta) - i\Phi_2(\zeta). \tag{16}$$

From (13) the basis of (11) we conclude that the function $\Phi_j(\zeta)$ $(j = 1, 2)$ are holomorphic in the circle $D$, continuously extendable up to the boundary $l$ and satisfies the boundary conditions

$$\begin{aligned}
&\Phi_1(\sigma) - \Phi_1\left(\frac{1}{\sigma}\right) = 0, \quad \sigma \in l_1^{(1)};\\
&\Phi_1(\sigma) + \Phi_1\left(\frac{1}{\sigma}\right) = H_2^{(1)}, \ \sigma \in l_2^{(1)};\\
&\Phi_1(\sigma) - \Phi_1\left(\frac{1}{\sigma}\right) = H_3^{(1)}, \quad \sigma \in l_3^{(1)};\\
&\Phi_1(\sigma) + \Phi_1\left(\frac{1}{\sigma}\right) = 0, \quad \sigma \in l_4^{(1)}.
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\Phi_2(\sigma) + \Phi_2\left(\frac{1}{\sigma}\right) = 0, \ \ \sigma \in l_1^{(1)};\\
&\Phi_2(\sigma) - \Phi_2\left(\frac{1}{\sigma}\right) = H_2^{(2)}, \quad \sigma \in l_2^{(1)};\\
&\Phi_2(\sigma) + \Phi_2\left(\frac{1}{\sigma}\right) = H_3^{(2)}, \quad \sigma \in l_3^{(1)};\\
&\Phi_2(\sigma) - \Phi_2\left(\frac{1}{\sigma}\right) = 0, \quad \sigma \in l_4^{(1)}.
\end{aligned} \tag{18}$$

where $H_2^{(1)} = \dfrac{-P + 2k_0 a}{2}$; $H_3^{(1)} = -i\dfrac{Q}{2}$; $H_2^{(2)} = i\dfrac{P}{2}$; $H_3^{(2)} = \dfrac{Q - 2k_0 b}{2}$.

The problems (17) and (18) are of the same type. For the solution of these problems we use the method of conformal sewing (see [14]). Under the sewing function we mean Zhukovski's function $\xi = \zeta + \frac{1}{\zeta}$ which maps the circle D onto the plane with a cut along the segment $I = [-2;\ 2]$ of the real axis in such a way that the upper semicircle $l_1$ is mapped onto the upper contour and the lower semicircle $l_{1*}$ onto the lower contour of the segment $I$. The positive direction on $I$ is assumed to coincide with that of the real axis.

We introduce the functions

$$\Phi_j(\xi) = \Psi_j[\zeta(\xi)] = \Psi_j[(\xi - \sqrt{\xi^2 - 4})/2], \quad (j = 1, 2), \tag{19}$$

where under the square root is understood that branch which is positive on the real axis outside the segment $I$. It is easily seen that points $\tau \in I$, we have

$$\zeta^+(\tau) = \frac{1}{2}(\tau - \sqrt{\tau^2 - 4}) = \sigma, \quad \sigma \in l_1,$$
$$\zeta^-(\tau) = \frac{1}{2}(\tau + \sqrt{\tau^2 - 4}) = \frac{1}{\sigma}, \quad \sigma \in l_1,$$

and so we obtain

$$\begin{aligned} &\Phi_j(\sigma) = \Psi_j\left[\frac{1}{2}\left(\tau - \sqrt{\tau^2 - 4}\right)\right] = \Psi_j[\xi^+(\tau)] = \Psi_j^+(\tau), \\ &\Phi_j\left(\frac{1}{\sigma}\right) = \Psi_j\left[\frac{1}{2}\left(\tau + \sqrt{\tau^2 - 4}\right)\right] = \Psi_j[\xi^-(\tau)] = \Psi_j^-(\tau), \\ &(j = 1, 2). \end{aligned} \tag{20}$$

From (20) and boundary conditions (17), (18) for the function $\Psi_j(\xi)$ $(j = 1, 2)$ we obtain the Keldysh-Sedov problems for the segment $I$:

$$\begin{aligned} &\Psi_1^+(\tau) - \Psi_1^-(\tau) = 0, \quad \tau \in [\delta_2; 2]; \\ &\Psi_1(\tau) + \Psi_1^-(\tau) = H_2^{(1)}, \quad \tau \in [0; \delta_2]; \\ &\Psi_1^+(\tau) - \Psi_1^-(\tau) = H_3^{(1)}, \quad \tau \in [-\delta_4; 0]; \\ &\Psi_1^+(\tau) + \Psi_1^-(\tau) = 0, \quad \tau \in [-2; -0; \delta_4], \end{aligned} \tag{21}$$

$$\begin{aligned} &\Psi_2^+(\tau) + \Psi_2^-(\tau) = 0, \quad \tau \in [\delta_2; 2]; \\ &\Psi_2^+(\tau) - \Psi_2^-(\tau) = H_2^{(2)}, \quad \tau \in [0; \delta_2]; \\ &\Psi_2^+(\tau) + \Psi_2^-(\tau) = H_3^{(2)}, \quad \tau \in [-\delta_4; 0]; \\ &\Psi_2^+(\tau) - \Psi_2^-(\tau) = 0, \quad \tau \in [-2; -\delta_4], \end{aligned} \tag{22}$$

where $(-\delta_4)$ and $\delta_2$ are the images of the points $a_4$ and $a_2$, respectively, under the mapping $\xi = \zeta + \dfrac{1}{\zeta}$.

We will seek for bounded at infinity solutions of problems (21) and (22) of the class $h(-2; -\delta_4; 0; \delta_2; 4)$, satisfying the condition

$$\Psi_j(\xi) = \overline{\Psi_j(\bar{\xi})}, \quad (j = 1, 2). \tag{23}$$

The canonical functions of problems (21) and (22) this class have the following form (see [11], [12], [13])

$$\begin{aligned} \chi_1(\xi) &= \sqrt{(\xi + \delta_4)\xi(\xi - \delta_2)(\xi + 2)}, \\ \chi_2(\xi) &= \sqrt{(\xi + \delta_4)\xi(\xi - \delta_2)(\xi - 2)}. \end{aligned} \tag{24}$$

The necessary and sufficient condition for the solvability of the problem (21) has the form

$$H_3^{(1)} \int_{-\delta_4}^{0} \frac{d\tau}{\chi_1(\tau)} + H_2^{(1)} \int_{0}^{\delta_2} \frac{d\tau}{\chi_1(\tau)} = 0, \tag{25}$$

and the solution itself is given by the formula

$$\Psi_1(\xi) = \frac{\chi_1(\xi)}{2\pi i} \left[ H_3^{(1)} \int_{-\delta_4}^{0} \frac{d\tau}{\chi_1(\tau)(\tau - \xi)} + H_2^{(1)} \int_{0}^{\delta_2} \frac{d\tau}{\chi_1(\tau)(\tau - \xi)} \right]. \tag{26}$$

Analogous the necessary and sufficient condition for the solvability of problem (22) has the form

$$H_3^{(2)} \int_{-\delta_4}^{0} \frac{d\tau}{\chi_2(\tau)} + H_2^{(2)} \int_{0}^{\delta_2} \frac{d\tau}{\chi_2(\tau)} = 0, \tag{27}$$

and the solution itself is given by the formula

$$\Psi_2(\xi) = \frac{\chi_2(\xi)}{2\pi i} \left[ H_3^{(2)} \int_{-\delta_4}^{0} \frac{d\tau}{\chi_2(\tau)(\tau - \xi)} + H_2^{(2)} \int_{0}^{\delta_2} \frac{d\tau}{\chi_2(\tau)(\tau - \xi)} \right]. \tag{28}$$

It is easy verify that the functions $\Psi_j(\xi)$ $(j = 1, 2)$ satisfy the condition (23).

Having found the functions $\Psi_j(\xi)$ $(j = 1, 2)$, from (12) and (16) functions $\omega_0(\xi) = \omega[\zeta(\xi)]$ and $\psi_0(\xi) = \Psi[\omega(\zeta(\xi))]$ are represented by the formulas

$$\omega_0(\xi) = \frac{2}{k_0}[\Psi_1(\xi) - i\Psi_2(\xi)] \quad \psi_0(\xi) = \Psi_1(\xi) + i\Psi_2(\xi), \tag{29}$$

where $\Psi_1(\xi)$ and $\Psi_2(\xi)$ are represented by the formulas (26) and (28), respectively.

From (29) the equation for part $A_6A_1$ of the unknown contour can be obtained from the image of the function $\omega(\zeta)$ for $\xi \in \left[-\infty; -\dfrac{\delta_0^2+1}{\delta_0} \cup [2;\infty)\right]$.

The integrals appearing in formulas (25)-(28) are the first and third kind elliptic integrals (see [15]).

$$\int\limits_{-\delta_4}^{0} \frac{d\tau}{\chi_1(\tau)} = \frac{2}{\sqrt{2(\delta_2+\delta_4)}} F\left[\frac{\pi}{2};\ K_1^{(2)}\right];$$

$$\int\limits_{0}^{\delta_2} \frac{d\tau}{\chi_1(\tau)} = \frac{-2i}{\sqrt{2(\delta_2+\delta_4)}} F\left[\frac{\pi}{2};\ K_1^{(1)}\right];$$

$$\int\limits_{-\delta_4}^{0} \frac{d\tau}{\chi_2(\tau)} = \frac{-2i}{\sqrt{2(\delta_2+\delta_4)}} F\left[\frac{\pi}{2};\ K_2^{(2)}\right];$$

$$\int\limits_{0}^{\delta_2} \frac{d\tau}{\chi_2(\tau)} = \frac{2}{\sqrt{2(\delta_2+\delta_4)}} F\left[\frac{\pi}{2};\ K_2^{(1)}\right];$$

$$\int\limits_{-\delta_4}^{0} \frac{dt}{\chi_1(t)(t-\xi)} = -\frac{2}{(\xi+\delta_4)(\xi+2)\sqrt{2(\delta_2+\delta_4)}}$$

$$\times\left\{(2-\delta_4)\Pi\left[\frac{\pi}{2};\ n_1^{(2)}(\xi);\ K_1^{(2)}\right] + (\xi+\delta_4)F\left[\frac{\pi}{2};\ K_1^{(2)}\right]\right\};$$

$$\int\limits_{0}^{\delta_2} \frac{dt}{\chi_1(t)(t-\xi)} = \frac{2i}{(\xi-\delta_2)(\xi+2)\sqrt{2(\delta_2+\delta_4)}}$$

$$\times\left\{(\delta_2+2)\Pi\left[\frac{\pi}{2};\ n_1^{(1)}(\xi);\ K_1^{(1)}\right] + (\xi-\delta_2)F\left[\frac{\pi}{2};\ K_1^{(1)}\right]\right\};$$

$$\int\limits_{-\delta_4}^{0} \frac{dt}{\chi_2(t)(t-\xi)} = \frac{2i}{(\xi-2)(\xi+\delta_4)\sqrt{2(\delta_2+\delta_4)}}$$

$$\times\left\{-(\delta_4+2)\Pi\left[\frac{\pi}{2};\ n_2^{(2)}(\xi);\ K_2^{(2)}\right] + (\xi+\delta_4)F\left[\frac{\pi}{2};\ K_2^{(2)}\right]\right\};$$

$$\int\limits_{0}^{\delta_2} \frac{dt}{\chi_2(t)(t-\xi)} = -\frac{2}{(\xi-2)(\xi-\delta_2)\sqrt{2(\delta_2+\delta_4)}}$$

$$\times\left\{(\delta_2-2)\Pi\left[\frac{\pi}{2};\ n_2^{(1)}(\xi);\ K_2^{(1)}\right]+(\xi-\delta_2)F\left[\frac{\pi}{2};\ K_2^{(1)}\right]\right\},$$

where

$$K_1^{(2)}=\sqrt{\frac{\delta_4(\delta_2+2)}{2(\delta_2+\delta_4)}};\ K_1^{(1)}=\sqrt{\frac{\delta_2(2-\delta_4)}{2(\delta_2+\delta_4)}};\ K_2^{(2)}=\sqrt{\frac{\delta_4(2-\delta_2)}{2(\delta_2+\delta_4)}};$$

$$K_2^{(1)}=\sqrt{\frac{\delta_2(2+\delta_4)}{2(\delta_2+\delta_4)}};\ n_1^{(2)}(\xi)=\frac{\delta_4(\xi+2)}{2(\xi+\delta_4)};\ n_1^{(1)}\xi=\frac{-\delta_2(\xi+2)}{2(\xi-\delta_2)};$$

$$n_2^{(2)}(\xi)=\frac{-\delta_4(\xi-2)}{2(\xi+\delta_4)}\ \ n_2^{(1)}(\xi)=\frac{\delta_2(\xi-2)}{2(\xi-\delta_2)},$$

$F[\varphi;k]=\int\limits_0^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$ is the elliptic integral of the first kind.

$\Pi[\varphi;n;k]=\int\limits_0^{\pi}\frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}$ is the elliptic integral of the third kind.

If the approximations

$$F[\varphi;k]=\int\limits_0^{\varphi}\left(1+\frac{1}{2}k^2\sin^2\varphi\right)d\varphi=\frac{\pi}{2}\frac{4+k^2}{4};$$

$$\Pi[\varphi;n;k]=\int\limits_0^{\varphi}\left[1+\left(\frac{k^2}{2}+n\right)\sin^2\varphi\right]d\varphi=\frac{\pi}{2}\left[1+\frac{k^2+2n}{4}\right],$$

are satisfied, then the equation of the unknown contour has the from

$$\omega_0(\sigma)=\frac{\chi_1(\sigma)}{\sqrt{2(\delta_2+\delta_4)k\pi i}}\left\{\frac{-H_3^{(1)}}{(\sigma+\delta_4(\sigma+2)}\left[(2-\delta_4)\left(1+\frac{\delta_4(\delta_2+2)}{8(\delta_2+\delta_4)}\right.\right.\right.$$

$$\left.\left.+\ \frac{\delta_4(\sigma+2)}{4(\sigma+\delta_4)}\right)+(\sigma+\delta_4)\frac{8\delta_2+10\delta_4+\delta_2\delta_4}{8(\delta_2+\delta_4)}\right]$$

$$+\ \frac{iH_2^{(1)}}{(\sigma-\delta_2)(\sigma+2)}\left[(\delta_2+2)\left(1+\frac{\delta_2(2-\delta_4)}{8(\delta_2+\delta_4)}-\frac{\delta_2(\sigma+2)}{4(\sigma-\delta_2)}\right)\right.$$

$$\left.\left.+\ (\sigma-\delta_2)\frac{10\delta_2+8\delta_4-\delta_2\delta_4}{8(\delta_2+\delta_4)}\right]\right\}$$

$$+\ \frac{\chi_2(\sigma)}{\sqrt{2(\delta_2+\delta_4)k\pi i}}\left\{\frac{H_3^{(2)}}{(\sigma-2)(\sigma+\delta_4)}\left[-(\delta_4+2)\left(1+\frac{\delta_4(2-\delta_2)}{8(\delta_2+\delta_4)}\right.\right.\right.$$

$$- \left.\frac{\delta_4(\sigma-2)}{4(\sigma+\delta_4)}\right) + \frac{(\sigma+\delta_4)(8\delta_2+10\delta_4-\delta_2\delta_4)}{8(\delta_2+\delta_4)}\Bigg]$$

$$+ \frac{iH_2^{(2)}}{(\sigma-2)(\sigma-\delta_2)}\left[(\delta_2-2)\left(1+\frac{\delta_2(2+\delta_4)}{8(\delta_2+\delta_4)}+\frac{\delta_2(\sigma-2)}{4(\sigma-\delta_2)}\right)\right.$$

$$\left.\left. + (\sigma-\delta_2)\frac{10\delta_2+8\delta_4+\delta_2\delta_4}{8(\delta_2+\delta_4)}\right]\right\},$$

and the conditions (25) and (27) take the form

$$H_3^{(1)}(4+K_1^{2(2)}) - iH_2^{(1)}(4+K_1^{2(1)}) = 0,$$

$$H_3^{(2)}(4+K_2^{2(2)}) + iH_2^{(2)}(4+K_1^{2(2)}) = 0.$$

Also one condition for determination of the parameters $\delta_2,\ \delta_4,\ k_0$, we obtain from condition $\omega_0(-2)=0$, which from $\Pi[\varphi;0;k]=F[\varphi;k+0]$ has the form

$$\frac{H_3^{(2)}}{2-\delta_4}\left[(\delta_4+2)\left(1+\frac{\delta_4(2-\delta_2)}{8(\delta_2+\delta_4)}-\frac{\delta_4}{2-\delta_4}\right)+(2-\delta_4)\,\frac{8\delta_2+10\delta_4-\delta_2\delta_4}{8(\delta_2+\delta_4)}\right.+$$

$$\frac{iH_2^{(2)}}{2-\delta_2}\left[(2-\delta_2)\left(1+\frac{\delta_2(2-\delta_4)}{8(\delta_2+\delta_4)}+\frac{\delta_2}{2+\delta_2}\right)+(2+\delta_2)\frac{10\delta_2+8\delta_4+\delta_2\delta_4}{8(\delta_2+\delta_4)}\right]=0.$$

From the condition $\omega_0\left(-\frac{\delta_0^2+1}{\delta_0}=l\right)$ we can found $\delta_0$, where $2l$ is a length of the rectilinear cut.

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