A CONDITIONS OF EXISTENCE OF NEUTRAL SURFACES IN THE SHELLS CONSISTING OF BINARY MIXTURES

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Abstract

In the paper the shells consisting of binary mixtures are considered. Based on I. Vekua's works, the question of existence of neutral surfaces in such shells is studied. By neutral surface is called a surface which belongs to a shell but is not subject to tensions and compressions by the deformation of the elastic body.

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1 Introduction

In chapter 5 of Vekua's monograph [1] (see also monograph [2]) the question of existence of neutral surfaces in an elastic shells was studied. By neutral surface I. Vekua calls a surface which belongs to a shell but is not subject to tensions and compressions by the deformation of the shell. In other words, under the deformation of the shell the linear elements of the neutral surface are unchanged. On the neutral surface the tangential components of the deformation tensor are zero. Our goal is to establish the conditions for the existence of neutral surfaces in shells consisting of a mixture of two isotropic elastic materials. In such shells on the neutral surface, in addition to the tangential components of the two deformation tensors the tangential components of the rotation tensor are also zero.

2 Tangential and transverse stress fields in shells consisting of a binary mixture

In this paper we consider the Green-Naghdi-Steel model of a mixture of two isotropic elastic materials [3-5]. In this case, the relations corresponding to the generalized Hooke's law have the form

$$P^{i\cdot}_{\cdot j} = \Lambda\theta g^i_j + 2M\varepsilon^i_j - 2\lambda_5 h^{i\cdot}_{\cdot j},\quad i,j=1,2,3, \tag{1}$$

where $P^{i\cdot}_{\cdot j} = (P^{/i\cdot}_{\cdot j}, P^{//i\cdot}_{\cdot j})^T$ is a column-matrix consisting of mixed components of stress tensors of two constituents of the mixture; $\varepsilon^i_j = (\varepsilon^{/i}_j, \varepsilon^{//i}_j)^T$ is a column-matrix consisting of mixed components of deformations tensors of two constituents of the mixture, which with the matrix of covariant components of the displacement vectors of the two constituents of the mixture $u_j = (u^/_j, u^{//}_j)^T$ is connected by the formula

$$\varepsilon^i_j = 0.5(\overset{\circ}{\nabla}{}^i u_j + \overset{\circ}{\nabla}_j u^i),$$

where $\overset{\circ}{\nabla}{}^i, \overset{\circ}{\nabla}_j$ are symbols of a spatial covariant and contravariant derivatives; $u^i = (u^{/i}, u^{//i})^T$ is a column-matrix of contravariant components of displacement vectors of two constituents of the mixture;

$$\theta \equiv \varepsilon^1_1 + \varepsilon^2_2 + \varepsilon^3_3 = \theta^* + \varepsilon^3_3;\quad \theta^* \equiv \varepsilon^1_1 + \varepsilon^2_2. \tag{2}$$

$h^{i\cdot}_{\cdot j} = (h^{/i\cdot}_{\cdot j}, h^{//i\cdot}_{\cdot j})^T$ are mixed components of rotation tensor of components of mixture

$$h^{/i\cdot}_{\cdot j} = -h^{//i\cdot}_{\cdot j} = 0.5(\overset{\circ}{\nabla}{}^i u^/_j - \overset{\circ}{\nabla}_j u^{/i} + \overset{\circ}{\nabla}_j u^{//i} - \overset{\circ}{\nabla}{}^i u^{//}_j),$$

$$\Lambda = \begin{pmatrix} \lambda_1 - \frac{\alpha\rho_2}{\rho} & \lambda_3 - \frac{\alpha\rho_1}{\rho} \\ \lambda_4 + \frac{\alpha\rho_2}{\rho} & \lambda_2 + \frac{\alpha\rho_1}{\rho} \end{pmatrix},\quad M = \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix};$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mu_1, \mu_2, \mu_3$ are the elasticity constants characterizing the mechanical properties of the mixture, $\alpha = \lambda_3 - \lambda_4$; ρ_1, ρ_2 are the densities of two mixture components, $\rho = \rho_1 + \rho_2$; g^i_j are mixed components of the metric tensor in space.

In the above formulas Latin indexes take the value 1,2,3. Below with respect we assume summation on the repeating indexes. Greek indexes will take the values 1,2.

When $j = 3$ from (1) we have

$$P^{\alpha\cdot}_{\cdot 3} = 2M\varepsilon^\alpha_3 - 2\lambda_5 h^{\alpha\cdot}_{\cdot 3},\quad P^{3\cdot}_{\cdot\alpha} = 2M\varepsilon^3_\alpha - 2\lambda_5 h^{3\cdot}_{\cdot\alpha}, \tag{3}$$

$$P^3_3 = \Lambda\theta + 2M\varepsilon^3_3 = \Lambda\theta^* + (\Lambda + 2M)\varepsilon^3_3. \tag{4}$$

From (3) and (4)

$$\varepsilon^\alpha_3 = 0.25M^{-1}(P^{\alpha\cdot}_{\cdot 3} + P^{3\cdot}_{\cdot\alpha}), \tag{5}$$

$$\varepsilon_3^3 = (\Lambda + 2M)^{-1}(P_3^3 - \Lambda\theta^*). \tag{6}$$

By inserting expression (6) into (4), we obtain

$$\theta = \Lambda^{-1}\Lambda^*\theta^* + (\Lambda + 2M)^{-1}P_3^3, \tag{7}$$

where

$$\Lambda^* \equiv \Lambda - \Lambda(\Lambda + 2M)^{-1}\Lambda = 2\Lambda(\Lambda + 2M)^{-1}M. \tag{8}$$

Substituting expression (7) into (1) we get

$$P_{.j}^{i\cdot} = T_{.j}^{i\cdot} + Q_{.j}^{i\cdot}, \tag{9}$$

where

$$T_{.\beta}^{\alpha\cdot} = \Lambda^*\theta^* g_\beta^\alpha + 2M\varepsilon_\beta^\alpha - 2\lambda_5 h_{.\beta}^{\alpha\cdot}, \;\; T_{.3}^{i\cdot} = T_{.i}^{3\cdot} = 0, \tag{10}$$

$$Q_{.\beta}^{\alpha\cdot} = \Lambda(\Lambda + 2M)^{-1}P_3^3 g_\beta^\alpha, \;\; Q_{.3}^{i\cdot} = P_{.i}^{3\cdot}, \;\; Q_{3\cdot}^{\cdot i} = P_{3\cdot}^{\cdot i}. \tag{11}$$

From equalities (10) we easily derive relations

$$M\varepsilon_\beta^\alpha - \lambda_5 h_{.\beta}^{\alpha\cdot} = 0.5T_{.\beta}^{\alpha\cdot} - 0.25\Lambda^*(\Lambda^* + 2M)^{-1}T_\gamma^\gamma g_\beta^\alpha, \tag{12}$$

$$\theta^* = 0.5(\Lambda^* + M)^{-1}T_\gamma^\gamma. \tag{13}$$

If we now insert (1) into formula [1]

$$\boldsymbol{P}^i = P_{.j}^{i\cdot}\boldsymbol{R}^j \tag{14}$$

we obtain

$$\boldsymbol{P}^\alpha = \boldsymbol{T}^\alpha + \boldsymbol{Q}^\alpha \;\; \boldsymbol{P}^3 = \boldsymbol{Q}^3, \tag{15}$$

where

$$\boldsymbol{T}^\alpha = T_{.\beta}^{\alpha\cdot}\boldsymbol{R}^\beta, \;\; \boldsymbol{T}^3 = \boldsymbol{0}, \tag{16}$$

$$\boldsymbol{Q}^i = Q_{.j}^{i\cdot}\boldsymbol{R}^j. \tag{17}$$

The stress tensor is thus represented as the sum

$$\boldsymbol{P} = \boldsymbol{T} + \boldsymbol{Q}, \tag{18}$$

where

$$\boldsymbol{T} = \boldsymbol{T}^\alpha \otimes \boldsymbol{R}_\alpha, \;\; \boldsymbol{Q} = \boldsymbol{Q}^i \otimes \boldsymbol{R}_i. \tag{19}$$

Formula (16) implies that the tensor at any point of the shell satisfies the condition

$$\boldsymbol{n}\boldsymbol{T}^\alpha = 0, \;\; i.e. \;\; \boldsymbol{n}\boldsymbol{T} = 0. \tag{20}$$

In view of equalities (20) the stress forces $\boldsymbol{P}_{(n)}$ acting on longitudinal areas with the normal $\boldsymbol{n}$ are given by the formula

$$\boldsymbol{P}_{(n)} = \boldsymbol{P}^3 \equiv \boldsymbol{Q}^3.$$

The tensor $\boldsymbol{Q}$ according to formulae (11) and (19) is expressed by

$$\boldsymbol{Q} = Q^{\alpha\cdot}_{\cdot\beta}\boldsymbol{R}_\alpha \otimes \boldsymbol{R}^\beta + Q^{i\cdot}_{\cdot 3}\boldsymbol{R}_i \otimes \boldsymbol{n} + Q^{3\cdot}_{\cdot\alpha}\boldsymbol{n} \otimes \boldsymbol{R}^\alpha$$

$$= \Lambda(\Lambda + 2M)^{-1}P^3_3\boldsymbol{R}_\alpha \otimes \boldsymbol{R}^\alpha + P^{\alpha\cdot}_{\cdot 3}\boldsymbol{R}_\alpha \otimes \boldsymbol{n} + P^3_3\boldsymbol{n} \otimes \boldsymbol{n} + P^{3\cdot}_{\cdot\alpha}\boldsymbol{n} \otimes \boldsymbol{R}_\alpha. \quad (21)$$

$\boldsymbol{T}$ and $\boldsymbol{Q}$ are the tangential and normal stress fields, respectively.

Differentiating the vector $\boldsymbol{u} = u_i\boldsymbol{R}^i$ with respect to Gaussian coordinates x^α when $x^3 = const$, we have

$$\partial_\alpha \boldsymbol{u} = (\hat{\nabla}_\alpha u_\beta - \hat{b}_{\alpha\beta}u_3)\boldsymbol{R}^\beta + (\hat{\nabla}_\alpha u_3 - \hat{b}^\beta_\alpha u_\beta)\boldsymbol{n}, \quad (22)$$

where

$$\hat{\nabla}_\alpha u_\beta = \partial_\alpha u_\beta - \hat{\Gamma}^\nu_{\alpha\beta}u_\nu, \;\; \hat{\nabla}_\alpha u_3 = \partial_\alpha u_3. \quad (23)$$

Here $\hat{\Gamma}^\nu_{\alpha\beta}$ are second-order Christoffel's symbols of the coordinate surface $\hat{S} : x^3 = comst$. $\hat{b}_{\alpha\beta}$, $\hat{b}^\beta_\alpha$ are covariant and mixed coefficients of the second fundamental quadratic form of this surface;

$$\hat{\Gamma}^\nu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} + A^{\nu\cdot}_{\cdot\gamma}\nabla_\alpha A^{\gamma\cdot}_{\beta\cdot},$$

where $\Gamma^\nu_{\alpha\beta}$ are second-order Christoffel's symbols of the surface $S : x^3 = 0$;

$$A^{i\cdot}_{\cdot j} = \vartheta^{-1}[(1 - 2Hx^3)a^i_j + x^3b^i_j + (x^3)^2a^i_3a^3_jK],$$

$$\vartheta = 1 - 2Hx^3 + K(x^3)^2,$$

$H = 0.5b^\gamma_\gamma$, $K = b^1_1b^2_2 - b^2_1b^1_2$ are the mean and Gaussian curvatures of the surface S respectively. Then for the components ε^i_j and $h^{i\cdot}_{\cdot j}$ we have

$$\varepsilon^i_j = 0.5(\boldsymbol{R}^i\partial_j\boldsymbol{u} + \boldsymbol{R}_j\partial^i\boldsymbol{u}), \;\; h^{i\cdot}_{\cdot j} = 0.5S(\boldsymbol{R}_j\partial^i\boldsymbol{u} - \boldsymbol{R}^i\partial_j\boldsymbol{u}), \quad (24)$$

where $\partial^i \equiv g^{ik}\partial_k$, $S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

From (22) we obtain

$$\varepsilon^\alpha_\beta = 0.5(\hat{\nabla}^\alpha u_\beta + \hat{\nabla}_\beta u^\alpha) - \hat{b}^\alpha_\beta u_3, \;\; h^{\alpha\cdot}_{\cdot\beta} = 0.5S(\hat{\nabla}^\alpha u_\beta - \hat{\nabla}_\beta u^\alpha). \quad (25)$$

This gives

$$\theta^* = \varepsilon^\alpha_\alpha = \hat{\nabla}_\alpha u^\alpha - 2\hat{H}u_3. \quad (26)$$

Inserting expressions (25) and (26) into (10) we have

$$T^{\alpha\cdot}_{\cdot\beta} = \Lambda^*(\hat{\nabla}_\gamma u^\gamma - 2\hat{H}u_3)g^\alpha_\beta + (M-\lambda_5 S)\hat{\nabla}^\alpha u_\beta + (M+\lambda_5 S)\hat{\nabla}_\beta u^\alpha - 2M\hat{b}^\alpha_\beta u_3, \tag{27}$$

$$\hat{\nabla}_\alpha u^\beta = \partial_\alpha u^\beta + \hat{\Gamma}^\beta_{\alpha\gamma}u^\gamma, \;\; \hat{\nabla}_\alpha u_\beta = \partial_\alpha u_\beta - \hat{\Gamma}^\gamma_{\alpha\gamma}u_\gamma, \;\; \hat{\nabla}^\alpha = g^{\alpha\gamma}\hat{\nabla}_\gamma. \tag{28}$$

Formula (27) yields

$$T^\gamma_\gamma = 2(\Lambda^* + M)\hat{\nabla}_\gamma u^\gamma - 4(\Lambda^* + M)\hat{H}u_3. \tag{29}$$

If $\hat{H} \neq 0$ then from (29) we get

$$u_3 = \frac{1}{2\hat{H}}\hat{\nabla}_\gamma u^\gamma - \frac{1}{4\hat{H}}(\Lambda^* + M)^{-1}T^\gamma_\gamma. \tag{30}$$

In view of (12) we may write relation (25) in the form

$$\begin{aligned} 0.5[(M-\lambda_5 S)\hat{\nabla}_\alpha u_\beta + (M+\lambda_5 S)\hat{\nabla}_\beta u_\alpha] - \hat{b}_{\alpha\beta}Mu_3 \\ = 0.5T^{\alpha\cdot}_{\cdot\beta} - 0.25\Lambda^*(\Lambda^* + 2M)^{-1}T^\gamma_\gamma g_{\alpha\beta}. \end{aligned} \tag{31}$$

3 Conditions for the existence of a neutral surface of a shell consisting of binary mixture

We now assume that on some coordinate surface $\hat{S} : x^3 = const$, , belonging to the shell Ω the tangential stress field is identically zero, i.e.

$$T_{\alpha\beta} = 0 \;\; (\hat{S} : x^3 = const). \tag{32}$$

Then relations (31) imply that the displacement field satisfies the system of equations

$$\begin{cases} 0.5(\hat{\nabla}_\alpha u_\beta + \hat{\nabla}_\beta u_\alpha) - \hat{b}_{\alpha\beta}u_3 = 0, \\ 0.5S(\hat{\nabla}_\alpha u_\beta - \hat{\nabla}_\beta u_\alpha) = 0. \end{cases} \tag{33}$$

The first two equations (33) are the equations of infinitesimal bendings of the coordinate surface $\hat{S} : x^3 = const$. The third equation has the form

$$\hat{\nabla}_1 u'_2 - \hat{\nabla}_2 u'_1 + \hat{\nabla}_2 u''_1 - \hat{\nabla}_1 u''_2 = 0.$$

The system (33) can be written in an expanded form

$$\begin{cases} 0.5(\hat{\nabla}_\alpha u'_\beta + \hat{\nabla}_\beta u'_\alpha) - \hat{b}_{\alpha\beta}u'_3 = 0, \\ 0.5(\hat{\nabla}_\alpha u''_\beta + \hat{\nabla}_\beta u''_\alpha) - \hat{b}_{\alpha\beta}u''_3 = 0, \\ \hat{\nabla}_1 u'_2 - \hat{\nabla}_2 u'_1 + \hat{\nabla}_2 u''_1 - \hat{\nabla}_1 u''_2 = 0. \end{cases}$$

Conversely, if equations (33) hold on the coordinate surface $\hat{S} : x^3 = const$ then, according to (27), the tangential stress field in the shell consisting of binary mixture identically vanishes on this surface.

Thus, the coordinate surface $\hat{S} : x^3 = const$ for which this conditions holds, is a neutral surfaces of the shell consisting of binary mixture.

We write the system of equilibrium equations for shells consisting of binary mixture in the vector form

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\boldsymbol{P}^i) + \boldsymbol{\Phi} = \boldsymbol{0}, \tag{34}$$

where g is the discriminant of the relative metric square form.

In view of (15) system (34) may be written as

$$\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}\boldsymbol{T}^\alpha) + \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\boldsymbol{Q}^i) + \boldsymbol{\Phi} = \boldsymbol{0}, \tag{35}$$

where

$$\overset{0}{\boldsymbol{T}}{}^\alpha = (\boldsymbol{T}^\alpha)_{x^3=0}, \quad \overset{0}{\boldsymbol{Q}}{}^\alpha = (\boldsymbol{Q}^\alpha)_{x^3=0}. \tag{36}$$

Then on the middle surfaces $S : x^3 = 0$ equations (35) becomes

$$\frac{1}{\sqrt{a}}\partial_\alpha\left(\sqrt{a}\,\overset{0}{\boldsymbol{T}}{}^\alpha\right) + \frac{1}{\sqrt{a}}\partial_\alpha\left(\sqrt{a}\,\overset{0}{\boldsymbol{Q}}{}^\alpha\right) + \frac{1}{\sqrt{g}}\partial_3\left(\sqrt{g}\boldsymbol{P}^3\right)_{x^3=0} + \overset{0}{\boldsymbol{\Phi}} = \boldsymbol{0}, \tag{37}$$

where

$$\overset{0}{\boldsymbol{\Phi}} = (\boldsymbol{\Phi})_{x^3=0}. \tag{38}$$

Let the middle surface $S : x^3 = 0$ be the neutral surface of a shell. Then

$$\overset{0}{\boldsymbol{T}}{}^{\boldsymbol{\alpha}} = \boldsymbol{0} \;\; i.e.\; \overset{0}{T}{}^{\alpha\beta} = 0 \;\; (onS), \tag{39}$$

And equation (37) becomes

$$\frac{1}{\sqrt{a}}\partial_\alpha\left(\sqrt{a}\,\overset{0}{\boldsymbol{Q}}{}^\alpha\right) + \frac{1}{\sqrt{g}}\partial_3\left(\sqrt{g}\boldsymbol{P}^3\right)_{x^3=0} + \overset{0}{\boldsymbol{\Phi}} = \boldsymbol{0}, \tag{40}$$

Thus, the satisfaction of this equation is the condition necessary for the surface $S : x^3 = 0$ to be neutral.

In the sequel we consider thin shells of constant thickness $2h$. Denote the stress forces acting of the face surfaces S^+ and S^- by $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$

$$\overset{(+)}{\boldsymbol{P}} = -(\boldsymbol{P}^3)_{x^3=h}, \quad \overset{(-)}{\boldsymbol{P}} = (\boldsymbol{P}^3)_{x^3=-h}. \tag{41}$$

If we approximately represent the transverse stress force by the formula

$$\boldsymbol{P}^3(x^1,x^2,x^3) \cong \overset{0}{\boldsymbol{P}^3}(x^1,x^2) + x^3 \overset{1}{\boldsymbol{P}^3}(x^1,x^2), \tag{42}$$

where

$$\overset{0}{\boldsymbol{P}^3} = (\boldsymbol{P}^3)_{x^3=0} = 0, \quad \overset{1}{\boldsymbol{P}^3} = (\partial_3 \boldsymbol{P}^3)_{x^3=0}. \tag{43}$$

Due to (41) we have

$$\overset{0}{\boldsymbol{P}^3} = 0.5(\overset{(-)}{\boldsymbol{P}} - \overset{(+)}{\boldsymbol{P}}), \quad \overset{1}{\boldsymbol{P}^3} = -0.5h^{-1}(\overset{(+)}{\boldsymbol{P}} + \overset{(-)}{\boldsymbol{P}}). \tag{44}$$

Consequently

$$\frac{1}{\sqrt{g}}\partial_3(\sqrt{g}\boldsymbol{P}^3)_{x^3=0} \cong H(\overset{(+)}{\boldsymbol{P}} - \overset{(-)}{\boldsymbol{P}}) - 0.5h^{-1}(\overset{(+)}{\boldsymbol{P}} + \overset{(-)}{\boldsymbol{P}})$$

$$= 0.5h^{-1}(2hH-1)\overset{(+)}{\boldsymbol{P}} - 0.5h^{-1}(2hH+1)\overset{(-)}{\boldsymbol{P}}. \tag{45}$$

Here we have made use of the formula

$$\frac{1}{\sqrt{g}}\partial_3(\sqrt{g})_{x^3=0} = (\partial_3 \ln\vartheta)_{x^3=0} = -2H \tag{46}$$

and the boundary conditions (41).

In view of equalities (21) and (44) we have

$$\overset{0}{\boldsymbol{Q}^\alpha} = \Lambda(\Lambda+2M)^{-1}P^{33}\boldsymbol{r}^\alpha + P^{3\cdot}_{\cdot\alpha}\boldsymbol{n}$$

$$\cong -0.5\Lambda(\Lambda+2M)^{-1}(\overset{(+)}{P^3} - \overset{(-)}{P^3})\boldsymbol{r}^\alpha - 0.5(\overset{(+)}{P^\alpha} - \overset{(-)}{P^\alpha})\boldsymbol{n}, \tag{47}$$

where $\overset{(+)}{P^i} = (\overset{(+)}{P^{/i}}, \overset{(+)}{P^{//i}})^T$ and $\overset{(-)}{P^i} = (\overset{(-)}{P^{/i}}, \overset{(-)}{P^{//i}})^T$ are the contravariant components of the vectors $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$, respectively. In view of equalities (47) and (44) we may write equation (40) as

$$\frac{1}{\sqrt{a}}\partial_\alpha[\sqrt{a}\Lambda(\Lambda+2M)^{-1}(\overset{(+)}{P^3} - \overset{(-)}{P^3})\boldsymbol{r}^\alpha + 0.5(\overset{(+)}{P^\alpha} - \overset{(-)}{P^\alpha})]\boldsymbol{n}$$

$$+h^{-1}(1-2hH)\overset{(+)}{\boldsymbol{P}} + 0.5h^{-1}(1+2hH)\overset{(-)}{\boldsymbol{P}} - 2\overset{0}{\boldsymbol{\Phi}} = \boldsymbol{0}. \tag{48}$$

where a is the discriminant of a quadratic form of the middle surface.

Thus, if the middle surface of the thin shell consisting of a binary mixture is neutral then the stresses $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$, applied to the face surfaces of a shell, must satisfy the system of equations (48). It will be shown below that if one of these vectors is given, the other may be defined by equation (48).

In future therefore only one of these forces is assumed to be prescribed, for example $\overset{(-)}{\boldsymbol{P}}$.

Then to define the vector field $\overset{(+)}{\boldsymbol{P}}$ we have the equation

$$\frac{1}{\sqrt{a}}[\sqrt{a}(\Lambda(\Lambda+2M)^{-1}p\boldsymbol{r}^{\alpha}+p^{\alpha}\boldsymbol{n})]+h^{-1}(1-2hH)(p\boldsymbol{n}+p^{\alpha}\boldsymbol{r}_{\alpha})+\tilde{\boldsymbol{\Phi}}=\boldsymbol{0}, \quad (49)$$

where

$$p=(p^{/},p^{//})^{T},\ \ p=\overset{(+)}{P^{3}}-\overset{(-)}{P^{3}},\ \ p^{\alpha}=\overset{(+)}{P^{\alpha}}-\overset{(-)}{P^{\alpha}}, \quad (50)$$

$$\tilde{\boldsymbol{\Phi}}=-2\overset{0}{\boldsymbol{\Phi}}+2h^{-1}\overset{(-)}{\boldsymbol{P}}. \quad (51)$$

Equation (49) is equivalent to the system of equations

$$\Lambda(\Lambda+2M)^{-1}\partial_{\alpha}p+h^{-1}[(1-2hH)a_{\alpha\beta}-hb_{\alpha\beta}]p^{\beta}+\tilde{\Phi}_{\alpha}=0, \quad (52)$$

$$\frac{1}{\sqrt{a}}\partial_{\alpha}(\sqrt{a}p^{\alpha})+h^{-1}[I-2hH(I-\Lambda(\Lambda+2M)^{-1})]p+\tilde{\Phi}_{3}=0, \quad (53)$$

where $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$;

$$\tilde{\Phi}_{\beta}=-2\overset{0}{\Phi}_{\beta}+2h^{-1}\overset{(-)}{P}_{\beta}, \quad (54)$$

$$\tilde{\Phi}_{3}=-2\overset{0}{\Phi}_{3}+2h^{-1}\overset{(-)}{P}_{3}. \quad (55)$$

We assume that, on the face surface S^{-} , only normal forces act, of the form

$$\overset{(-)}{\boldsymbol{P}}=q\boldsymbol{n}, \quad (56)$$

where $q=(q^{/},q^{//})^{T}$ is a column-matrix consisting of some scalar functions of the point of the surface S^{-} . Then formulas (54), (55) take the form

$$\tilde{\Phi}_{\beta}=-2\overset{0}{\Phi}_{\beta},\ \ \tilde{\Phi}_{3}=-2\overset{0}{\Phi}_{3}+2h^{-1}q. \quad (57)$$

From the system of equations (52) it is easy to derive the formula

$$p^{\alpha}=\overset{(+)}{P^{\alpha}}-\overset{(-)}{P^{\alpha}}=-\Lambda(\Lambda+2M)^{-1}\tilde{d}^{\alpha\beta}\partial_{\beta}p-\tilde{d}^{\alpha\beta}\tilde{\Phi}_{\beta}, \quad (58)$$

where

$$\tilde{d}^{\alpha\beta} = \frac{h[a^{\alpha\beta}(1-4hH)+hb^{\alpha\beta}]}{1-2hH+Kh^2+4hH(2hH-1)}. \tag{59}$$

Inserting (58) into (53) we obtain the equation

$$\Lambda(\Lambda+2M)^{-1}\nabla_\alpha(\tilde{d}^{\alpha\beta}\nabla_\beta p)-h^{-1}[I-2hH(I-\Lambda(\Lambda+2M)^{-1})]p+\Phi=0, \tag{60}$$

where

$$\Phi = -\tilde{\Phi}_3+\nabla_\alpha(\tilde{d}^{\alpha\beta}\tilde{\Phi}_\beta)$$
$$= -2h^{-1}\overset{(-)}{P^3}+2h^{-1}\nabla_\alpha(\tilde{d}^{\alpha\beta}\overset{(-)}{P^\beta})+2\overset{0}{\Phi}_3-2\nabla_\alpha(\tilde{d}^{\alpha\beta}\overset{0}{\Phi}_\beta). \tag{61}$$

If the stresses S^- on have the form (56) then, in view of (57), equality (61) takes the form

$$\Phi = -2h^{-1}q+2\overset{0}{\Phi}_3-2\nabla_\alpha(\tilde{d}^{\alpha\beta}\overset{0}{\Phi}_\beta). \tag{62}$$

Thus, to determine the $\overset{(+)}{P^3}=p+\overset{(-)}{P^3}$ we have the fourth-order system of partial differential equations (60).

If the solution $p=(p^{/},p^{//})^T$ of system (60), then by virtue of the formulae

$$\overset{(+)}{P^3}=p+\overset{(-)}{P^3} \tag{63}$$

and

$$\overset{(+)}{P^\alpha}=\overset{(-)}{P^\alpha}-\Lambda(\Lambda+2M)^{-1}\tilde{d}^{\alpha\beta}\partial_\beta p-\tilde{d}^{\alpha\beta}\tilde{\Phi}_\beta \tag{64}$$

we find the normal and tangential components of the unknown stress field.

Let the shell be subject to bush constraints and the stress field $\overset{(-)}{\boldsymbol{P}}$ on the face surface S^- be assigned beforehand. Let us assume that the normal and tangential components of the stress field $\overset{(+)}{\boldsymbol{P}}$ on the surface S^+ are expressed by (63), (64), where $p=(p^{/},p^{//})^T$ is some solution of the non-homogeneous equation (60). Then the middle surface of the shell consisting of binary mixtures is neutral, i.e. it may be subject only to infinitesimal bendings, if on it the boundary value problem

$$\frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}\,\overset{0}{\boldsymbol{T}^\alpha})\equiv\frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}\,\overset{0}{T^{\alpha\beta}}\boldsymbol{r}_\beta)=\boldsymbol{0}\ \ (on\ S) \tag{65}$$

and

$$\overset{0}{P_{ls}}=\overset{0}{T^{\alpha\beta}}l_\alpha l_\beta=0\ \ (on\ S)$$

has only the trivial solution $\overset{0}{T^{\alpha\beta}}=0$.

If the shell in this case does not experience an infinitesimal bending, then its middle surface is rigid.

4 Closed convex shells consisting of binary mixtures

If Ω is a closed shell, then S is an ovaloid. We prove that in this case system (60) may have only the unique regular solution, and hence, the corresponding homogeneous equation has no non-zero solution on S .

By the regular solution (60) we understand the continuous and continuously differentiable solution in the domain under consideration. The second derivatives of the regular solution may, in general, exist only in the generalized sense, and the equation is satisfied in the domain under question almost everywhere.

We write the homogeneous equation, corresponding to equation (60) in the form

$$\nabla_\alpha(\tilde{d}^{\alpha\beta}\nabla_\beta u) - Cu = 0, \tag{66}$$

where C denotes the matrix

$$C = h^{-1}\Lambda^{-1}(\Lambda + 2M)[I - \Lambda(\Lambda + 2M)^{-1}]. \tag{67}$$

Let $u = (u^{/}, u^{//})^T$ be the regular solution of equation (66) on S , i.e. is the continuous function of the point of the surface and has continuous partial derivatives with respect to Gaussian coordinates of this surface. We represent the surface S as $S = S_1 \cup S_2$, where S_1 and S_2 are parts of the surface with no common points $S_1 \cap S_2 = \varnothing$. Let L be the common boundary of S_1 and S_2 . Denote the tangential normal to L by l directed to S_1 . If we transpose (66) and multiplying both sides of transpose equation by u, we may rewrite it as

$$\nabla_\alpha(\tilde{d}^{\alpha\beta}(\nabla_\beta u)^T u) - \tilde{d}^{\alpha\beta}(\nabla_\alpha u)^T\nabla_\beta u - (Cu)^T u = 0. \tag{68}$$

Integrating this equality with respect to the surfaces S_1 and S_2, and then applying Green's formula, we have

$$\int_L \tilde{l}_\alpha \tilde{d}^{\alpha\beta}(\nabla_\beta u)^T u ds - \int\int_{S_1} \left(\tilde{d}^{\alpha\beta}(\nabla_\alpha u)^T\nabla_\alpha u + u^T Cu\right) dS_1 = 0, \tag{69}$$

$$-\int_L \tilde{l}_\alpha \tilde{d}^{\alpha\beta}(\nabla_\beta u)^T u ds - \int\int_{S_1} \left(\tilde{d}^{\alpha\beta}(\nabla_\alpha u)^T\nabla_\alpha u + u^T Cu\right) dS_2 = 0. \tag{70}$$

$C = C^T$ is a symmetric matrix

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}.$$

Let's say $detC > 0$ and $c_{11} > 0$. By adding equalities (69) and (70) we obtain

$$\int\int\limits_{S_1} \left(\tilde{d}^{\alpha\beta} (\nabla_\alpha u)^T \nabla_\beta u + u^T C u \right) dS = 0. \tag{71}$$

Since $\tilde{d}^{\alpha\beta} (\nabla_\alpha u)^T \nabla_\beta u \geq 0$, $u^T C u \geq 0$, from (71) it follows that $u \equiv 0$ which was to be proved.

The problem under consideration in thus reduced to the determination of the globally regular particular solution of the non-homogeneous equation (60).

It remains to show that if equation (60) has a globally regular solution, then the middle surface $S : x^3 = 0$ of the shell is neutral. To do this we have to show first that the tangential stress field vanishes on S , i.e. it should be shown that the equation

$$\frac{1}{\sqrt{a}} \partial_\alpha (\sqrt{a}\, \overset{0}{\boldsymbol{T}}{}^\alpha) \equiv \frac{1}{\sqrt{a}} \partial_\alpha (\sqrt{a}\, \overset{0}{T}{}^{\alpha\beta}\, \boldsymbol{r}_\beta) = \mathbf{0}. \tag{72}$$

has no globally regular solution, except trivial $\overset{0}{T}{}^{\alpha\beta}$. It is evident since, with respect to isometric-conjugate coordinates x, y, equation (72) is equivalent to the homogeneous generalized Cauchy-Riemann equation [1, Section 3.2.3]

$$\partial_{\bar{z}} \omega^/ - \overline{B} \overline{\omega}^/ = 0, \quad z = x + iy \ (i^2 = -1), \tag{73}$$

where

$$\omega^/ = 0.5 a K^{\frac{1}{4}} (\overset{0}{T}{}^{11} - \overset{0}{T}{}^{22} - i(\overset{0}{T}{}^{12} + \overset{0}{T}{}^{21})), \tag{74}$$

$$\overset{0}{T}{}^{11} + \overset{0}{T}{}^{22} + i(\overset{0}{T}{}^{12} - \overset{0}{T}{}^{21}) = 0.$$

As has been proved above [1, Section 3.3.1], the complex stresses function $\omega^/$ is continuous on the whole plane E of the complex variable $z = x+iy$ and at infinity satisfies the condition

$$\omega^/ = O(|z|^{-4}).$$

This implies, in view of the generalized Liuville theorem [2], that $\omega^/ = 0$. Then from (74) it follows that $\overset{0}{T}{}^{\alpha\beta} \equiv 0$, which was to be proved. The system of equations (33) therefore becomes

$$\begin{cases} \nabla_\alpha u_\beta + \nabla_\beta u_\alpha - 2 b_{\alpha\beta} u_3 = 0, \\ S(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha) = 0. \end{cases} \tag{75}$$

However, according to the Gaussian theorem, the ovaloid is rigid and system (75) therefore has only the solution of the form

$$\boldsymbol{u}^{/} = \boldsymbol{c}^{/} \times \boldsymbol{R} + \boldsymbol{c}_0^{/}, \ \ \boldsymbol{u}^{//} = \boldsymbol{c}^{//} \times \boldsymbol{R} + \boldsymbol{c}_0^{//},$$

where $\boldsymbol{c}^{/} = \boldsymbol{c}^{//}$ (This follows from the second equation of system (75)), $\boldsymbol{U} = (\boldsymbol{u}^{/}, \boldsymbol{u}^{//})^T$, $\boldsymbol{C} = (\boldsymbol{c}^{/}, \boldsymbol{c}^{//})^T$, $\boldsymbol{C}_0 = (\boldsymbol{c}_0^{/}, \boldsymbol{c}_0^{//})^T$

$$\boldsymbol{U} = \boldsymbol{C} \times \boldsymbol{R} + \boldsymbol{C}_0, \tag{76}$$

where $\boldsymbol{c}^{/}, \boldsymbol{c}_0^{/}, \boldsymbol{c}_0^{//}$ are the constant vector fields.

Thus the surface S allows only the rigid motion. It is therefore proved that S is a rigid neutral surface.

Assume that the free term Φ of equation (60) is zero. Then from (61) we have

$$\overset{(-)}{P^3} - \nabla_\alpha(\tilde{d}^{\alpha\beta} \overset{(-)}{P^\beta}) - h \overset{0}{\Phi}_3 - h\nabla_\alpha(\tilde{d}^{\alpha\beta} \overset{0}{\Phi}_\beta) = 0. \tag{77}$$

This equality may be regarded as the condition which must be satisfied by the stress force $\overset{(-)}{\boldsymbol{P}}$ applied to the face surface S^-. If condition (77) is satisfied, we obtain the homogeneous equation for p, which, as was shown above has zero solution $p = 0$. Therefore, in view of (77) and (58)

$$\overset{(+)}{P^3} = \overset{(-)}{P^3} = \nabla_\alpha(\tilde{d}^{\alpha\beta} \overset{(-)}{P_\beta}) + h \overset{0}{\Phi}_3 - h\nabla_\alpha(\tilde{d}^{\alpha\beta} \overset{0}{\Phi}_\beta) = 0, \tag{78}$$

$$\overset{(+)}{P^\alpha} = \overset{(-)}{P^\alpha} - \tilde{d}^{\alpha\beta} \overset{0}{\Phi}_\beta \equiv \overset{(-)}{P^\alpha} - 2h^{-1}\tilde{d}^{\alpha\beta} \overset{(-)}{P}_\beta + 2\tilde{d}^{\alpha\beta} \overset{0}{\Phi}_\beta \, . \tag{79}$$

In particular, if the normal stresses act on the inner face surface, then $\overset{(-)}{P^\alpha} \equiv 0$ and (77) takes the form

$$\overset{(-)}{P^3} = h \overset{0}{\Phi}_3 - h\nabla_\alpha(\tilde{d}^{\alpha\beta} \overset{0}{\Phi}_\beta). \tag{80}$$

Then the stress force acts on the external surface whose components are defined by the equalities

$$\overset{(+)}{P^3} = \overset{(-)}{P^3} = h \overset{0}{\Phi}_3 - h\nabla_\alpha(\tilde{d}^{\alpha\beta} \overset{0}{\Phi}_\beta). \tag{81}$$

$$\overset{(+)}{P^\alpha} = 2\tilde{d}^{\alpha\beta} \overset{0}{\Phi}_\beta \, . \tag{82}$$

Thus, the normal stresses on face surfaces S^+ and S^- are equal and are given by (80), the tangential stresses on S^+ are not zero and are expressed by (82).

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