ON DETERMINING THE COEFFICIENT OF A MULTI-DIMENSIONAL HYPERBOLIC EQUATION WITH INTEGRAL OVERDETERMINATION CONDITION

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Abstract

In the paper, the problem of determination of the coefficient at the first time derivative of a hyperbolic equation with integral overdetermination condition is reduced to an optimal control problem. A theorem on the existence of optimal control is proved, the gradient of the functional is calculated and the necessary optimality condition is derived in the form of an integral inequality.

Key words and phrases: Coefficient, integral condition, optimal control problem, necessary condition.

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1 Introduction

As is known, inverse coefficient problems is one of the important sections of theory of inverse problems for partial differential equations [1-8]. Such problems appear in various fields of physics, geophysics, seismology, etc. Usually the properties of the researched environment (coefficients of equations) are unknown. Then there appear inverse problems where in by information about the solution of the direct problem it is required to determine the coefficients of equations. In many cases these problems are ill-posed. It should be noted that the problem in [1-8] were studied by means of the methods of the theory of inverse problems. In works [1,2], in some cases the inverse problems were reduced to an operator equation and then quadratic functional corresponding to this operator is constructed and studied. The find minimum of the functional is studied by using optimization methods. In this paper we consider such a problem where it is required to determine the coefficient at the first time derivative of the hyperbolic equation, in availability of additional information in the form of integral overdetermination condition. This problem is reduced to an optimal control problem and is studied by the methods of optimal control theory.

2 Problem statement

Let Ω be a bounded domain in R^n with the smooth boundary Γ, $T > 0$ be a given number, $Q = \Omega \times (0,T)$ be a cylinder, $S = \Gamma \times (0,T)$ be a lateral surface of the cylinder Q. The following problem is stated: find the functions $u\,(x,t)$ and $\upsilon\,(t)$ which are connected in Q with the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \upsilon\,(t)\,\frac{\partial u}{\partial t} = f\,(x,t)\,, \tag{2.1}$$

when for the function $u\,(x,t)$ the following initial conditions

$$u\,|_{t=0} = u_0\,(x)\,,\, \frac{\partial u}{\partial t}\,|_{t=0} = u_1\,(x)\,, x \in \Omega \tag{2.2}$$

the boundary condition

$$u\,|_S = 0, \tag{2.3}$$

and also the overdetemination condition is satisfied

$$\int_\Omega K\,(x,t)\,u\,(x,t)\,dx = g\,(t)\,\,,\,\, t \in [0,T]\,, \tag{2.4}$$

where $f\,(x,t), u_0\,(x)$,$u_1\,(x)$, $K\,(x,t)$, $g\,(t)$ are the given functions, Δ is Laplace operator with respect to x. For the given function $\upsilon\,(t)$ problem (2.1)-(2.3) is a direct problem in the domain Q. But not always all the data of the problem are determined. There arise situations when they should be determined by some additional information. Such problems are said to be inverse problems [2]. In this paper as the additional information the overdetermination condition (2.4) is taken.

Thus we consider an inverse problem in the following statement: by the known functions

$$f\,(x,t) \in L_2\,(Q)\,,\;\; u_0\,(x) \in \overset{0}{W}{}_2^1\,(\Omega)\,,\;\; u_1\,(x) \in L_2\,(\Omega)\,,$$

$$K\,(x,t) \in L_\infty\,(Q)\,,\; g\,(t) \in L_2\,(0,T)$$

to find the pair of functions $\{u\,(x,t)\,,\;\; \upsilon\,(t)\} \in W_2^1\,(Q) \times V$ so that conditions (2.1)-(2.3) and additional condition (2.4) should be fulfilled, where,

$$\begin{gathered} V = \left\{\upsilon = \upsilon(t)\,|\,\upsilon(t) \in W_2^1\,[0,T]\,, \right. \\ \left. |\upsilon\,(t)| \le \mu_1,\, |\upsilon'\,(t)| \le \mu_2\, a.e.on\,\, [0,T] \right\} \end{gathered} \tag{2.5}$$

where $\mu_1 > 0$, $\mu_2 > 0$ are the given numbers.

3 Reducing the problem to an optimal control problem and existence of the solution of a new problem

We reduce the problem to the following optimal control problem: in the class V it is required to minimize the functional

$$J\left(\upsilon\right)=\frac{1}{2}\int_{0}^{T}\left(\int_{\Omega}K\left(x,t\right)u\left(x,t;\upsilon\right)dx-g\left(t\right)\right)^{2}dt\,, \tag{3.1}$$

where $u\left(x,t;\upsilon\right)$ is the solution of problem (2.1)-(2.3), which corresponds to the function $\upsilon=\upsilon\left(t\right)$. Let us call the function $\upsilon\left(t\right)$a control , the class V a class of admissible controls. If we find an admissible control that delivers to functional (3.1) the zero value, then additional condition (2.4) is fulfilled.

as the generalized solution from $W_2^1\left(Q\right)$ of boundary value problem (2.1)-(2.3) for the given $\upsilon\in V$ we take the function $u=u\left(x,t;\upsilon\right)$ from $W_{2,0}^1\left(Q\right)$, which equals $u_0\left(x\right)$ at $t=0$ and satisfies the integral identity

$$\int_{Q}\left[-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t}+\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_i}+\upsilon\left(t\right)\frac{\partial u}{\partial t}\eta\right]dxdt$$

$$-\int_{\Omega}u_1\left(x\right)\eta\left(x,0\right)dx=\int_{Q}f\eta dxdt \tag{3.2}$$

for all $\eta=\eta\left(x,t\right)$ from $W_{2,0}^1\left(Q\right)$, which equals to zero at $t=T$.

From the results of the paper [9, p. 209-215] it follows that under the above adopted assumptions, for fixed $\upsilon\in V$ problem (2.1)-(2.3) has a unique generalized solution from $W_2^1\left(Q\right)$ and the estimation

$$\|u\|_{W_2^1(Q)}\leq c\left[\|u_0\|_{W_2^1(\Omega)}+\|u_1\|_{L_2(\Omega)}+\|f\|_{L_2(Q)}\right], \tag{3.3}$$

is valid. Here and in the sequel by c we will denote various constants independent of estimated variables and at admissible controls.

Theorem 3.1 *Let the conditions adopted at the statement of problem* (2.1)-(2.3), (2.5), (3.1) *be fulfilled. Then the set of optimal controls of this problem* $V_*=\left\{\upsilon_*\in V:J\left(\upsilon_*\right)=J_*=\inf\left\{J\left(\upsilon\right):\upsilon\in V\right\}\right\}$ *is non empty, weakly-compact in* $W_2^1\left[0,T\right]$ *and any minimizing sequence* $\left\{\upsilon^{(m)}\right\}$ *weakly in* $W_2^1\left[0,T\right]$ *converges to the set* V_*.

Proof. It is clear that the set V, determined by relation (2.5) is weakly compact in the Hilbert space $W_2^1\left[0,T\right]$. Show that the functional (3.1) is weakly continuous by in $W_2^1\left[0,T\right]$ on the set V. Let $\upsilon\in V$ be some element and $\left\{\upsilon^{(m)}\right\}\subset V$ be an arbitrary sequence such that $\upsilon^{(m)}\to\upsilon$ weakly in $W_2^1\left[0,T\right]$ at $m\to\infty$.

Hence, from the compactness of the embedding $W_2^1[0,T] \to C[0,T]$ [9, p. 84] it follows that

$$\upsilon^{(m)} \to \upsilon \quad \text{strongly in} \quad C[0,T]. \tag{3.4}$$

By unique solvability of boundary value problem (2.1)-(2.3), to each control $\upsilon^{(m)} \in V$ there corresponds a unique solution $u^{(m)} = u\left(x,t;\upsilon^{(m)}\right)$ of problem (2.1)-(2.3) and the following estimation is valid:

$$\left\|u^{(m)}\right\|_{W_2^1(Q)} \leq c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)}\right],$$

$\forall m = 1,2,...,$ i.e. The sequence $\left\{u^{(m)}\right\}$ is uniformly bounded in the norm of the space $W_2^1(Q)$. Then from the embedding theorem [11. p. 116] it follows that from a sequence one can choose a subspace $\left\{u^{(m_k)}\right\}$ such that as $k \to \infty$

$$u^{(m_k)} \to u \quad \text{strongly in } L_2(Q), \tag{3.5}$$

$$\frac{\partial u^{(m_k)}}{\partial x_i} \to \frac{\partial u}{\partial x_i} \ \left(i=\overline{1,n}\right), \frac{\partial u^{(m_k)}}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in} \quad L_2(Q), \tag{3.6}$$

where $u = u(x,t) \in W_{2,0}^1(Q)$ is some element.
Show that $u(x,t) = u(x,t;\upsilon)$, i.e. the function is the solution of problem (2.1)-(2.3) corresponding to $\upsilon \in V$. It is clear that the identities are valid

$$\int_Q \left[-\frac{\partial u^{(m_k)}}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^n \frac{\partial u^{(m_k)}}{\partial x_i}\frac{\partial \eta}{\partial x_i} + \upsilon^{(m_k)}\frac{\partial u^{(m_k)}}{\partial t}\eta\right] dxdt$$

$$-\int_\Omega u_1(x)\,\eta(x,0)\,dx = \int_Q f\eta dxdt \tag{3.7}$$

are valid for all $\eta = \eta(x,t) \in W_{2,0}^1(Q)$ which equal to zero at $t = T$.
Passing to limit (3.7) as $k \to \infty$, and using (3.4),(3.6) we get that the function $u(x,t)$ is equal to $u_0(x)$ at $t = 0$ and satisfies identity (3.2). Hence and from the uniqueness of the solution of problem (2.1)-(2.3) corresponding to the control $\upsilon \in V$ it follows that $u(x,t) = u(x,t;\upsilon)$.

Taking into account uniqueness of the solution of problem (2.1)-(2.3) corresponding to the control $\upsilon \in V$ it is easy to verify that relations (3.5), (3.6) are valid not only for the subsequence $\left\{u^{(m_k)}\right\}$ but for all sequences $\left\{u^{(m)}\right\}$ as well. Therefore, in particular, the limit relation $u^{(m)} \to u$ strongly in $L_2(Q)$ as $m \to \infty$ is valid. Using this relation, from (3.1) we get $J\left(\upsilon^{(m)}\right) \to J(\upsilon)$ as $m \to \infty$ i.e. $J(\upsilon)$ weakly in $W_2^1[0,T]$ is continuous on the set V. Then by theorem 2 and 4 from [13, p. 49, p. 51] we get that all the statements of Theorem 3.1 are valid. Theorem 3.1 is proved.

4 Study of differentiability of functional (3.1)

Let $\psi = \psi(x,t;\upsilon)$ be a generalized solution from $W_2^1(Q)$ of the adjoint problem

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi - \frac{\partial}{\partial t}(\upsilon\psi) = -K(x,t)\left(\int_\Omega K(x,t)\,u(x,t;\upsilon)\,dx - g(t)\right),\ (x,t)\in Q, \tag{4.1}$$

$$\psi|_{t=T} = 0, \frac{\partial\psi}{\partial t}|_{t=T} = 0, x\in\Omega, \psi|_S = 0. \tag{4.2}$$

As the generalized solution of boundary value problem (4.1), (4.2) for each fixed control $\upsilon\in V$ we will take the function $\psi = \psi(x,t;\upsilon)$ from $W_{2,0}^1(Q)$, which equals to zero at $t=T$ and satisfies the integral identity

$$\int_Q\left[\frac{\partial\psi}{\partial t}\frac{\partial\mu}{\partial t} - \sum_{i=1}^n\frac{\partial\psi}{\partial x_i}\frac{\partial\mu}{\partial x_i} - \upsilon\psi\frac{\partial\mu}{\partial t}\right]dxdt$$

$$-\int_Q K(x,t)\left(\int_\Omega K(x,t)\,u(x,t)\,dx - g(t)\right)\mu(x,t)\,dxdt = 0 \tag{4.3}$$

for all $\mu = \mu(x,t)\in W_{2,0}^1(Q)$, which equal to zero for $t=0$.
From the results of [9, p.209-215] it follows that for each fixed $\upsilon\in V$ boundary value problem (4.1), (4.2) has a unique generalized solution from $W_2^1(Q)$, and taking into account estimation (3.1) we have that the estimation

$$\|\psi\|_{W_2^1(Q)} \le c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)} + \|g\|_{L_2(0,T)}\right] \tag{4.4}$$

is valid.

For the given $\upsilon\in V$ we introduce the following boundary value problem [12] on definition of the function $\psi_1 = \psi_1(t;\upsilon)$ from the conditions

$$-\frac{d^2\psi_1}{dt^2} + \psi_1 = \int_\Omega\frac{\partial u}{\partial t}\psi dx,\ 0<t<T, \tag{4.5}$$

$$\frac{d\psi_1}{dt}|_{t=0} = \frac{d\psi_1}{dt}|_{t=T} = 0. \tag{4.6}$$

As the generalized generalized solution from $W_2^1[0,T]$ of boundary value problem (4.5), (4.6) for the given $\upsilon\in V$ we will taken the function $\psi_1 = \psi_1(t;\upsilon)$ from $W_2^1[0,T]$ satisfying the integral identity

$$\int_0^T\left[\frac{d\psi_1}{dt}\frac{d\eta}{dt} + \psi_1\eta\right]dt = \int_0^T\left(\int_\Omega\frac{\partial u}{\partial t}\psi dx\right)\eta dt \tag{4.7}$$

for all $\eta = \eta(t)$ from $W_2^1[0,T]$.

The conditions of the Lax-Millgram lemma [10, p. 39] are fulfilled for problem (4.5), (4.6) and therefore, for the given $\upsilon \in V$ this problem has a unique solution from $W_2^1[0,T]$.

Assuming in (4.7) $\eta = \psi_1$ we get

$$\|\psi_1\|_{W_2^1[0,T]} \leq c\left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q)} \cdot \|\psi\|_{L_2(Q)} \ . \tag{4.8}$$

Theorem 4.1 *Let the conditions of theorem 3.1 be fulfilled. Then functional* (3.1) *is continuously, Frechet differentiable on* V *and its gradient at the point* $\upsilon \in V$ *is determined by the expression*

$$J'(\upsilon) = \psi_1(t;\upsilon),\ t \in [0,T]. \tag{4.9}$$

Proof. Let υ, $\upsilon + \delta\upsilon \in V$ be arbitrary controls, $\delta u(x,t) = u(x,t;\upsilon+\delta\upsilon)$ $-u(x,t;\upsilon)$ where $\delta\upsilon \in W_2^1[0,T]$. From conditions (2.1)-(2.3) it follows that $\delta u(x,t)$ is the generalized solution from $W_{2,0}^1(Q)$ of the boundary value problem

$$\frac{\partial^2 \delta u}{\partial t^2} - \Delta\delta u + (\upsilon + \delta\upsilon)\frac{\partial \delta u}{\partial t} = -\delta\upsilon\frac{\partial u}{\partial t},\ (x,t) \in Q, \tag{4.10}$$

$$\delta u\,|_{t=0} = 0, \frac{\partial \delta u}{\partial t}\,|_{t=0} = 0, x \in \Omega, \delta u\,|_S = 0\ . \tag{4.11}$$

From [9, p.215] it follows that for the solution of problem (4.10), (4.11) the estimation

$$\|\delta u\|_{W_2^1(Q)} \leq c\left\|\delta\upsilon\frac{\partial u}{\partial t}\right\|_{L_2(Q)} \leq c\left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q)} \cdot \|\delta\upsilon\|_{C[0,T]}$$

is valid.

By boundedues of the embedding $W_2^1[0,T] \to C[0,T]$ [9, p. 84] and estimation (3.3), we have

$$\|\delta u\|_{W_2^1(Q)} \leq c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)}\right]\|\delta\upsilon\|_{W_2^1[0,T]}\,. \tag{4.12}$$

The increment of functional (3.1) at the point $\upsilon \in V$ has the form:

$$\begin{gathered}\Delta J(\upsilon) = J(\upsilon+\delta\upsilon) - J(\upsilon) = \textstyle\int_0^T \left(\int_\Omega K(x,t)u(x,t;\upsilon)dx - g(t)\right) \\ \times \displaystyle\int_\Omega K(x,t)\,\delta u(x,t)\,dxdt + \frac{1}{2}\int_0^T\left(\int_\Omega K(x,t)\,\delta u(x,t)\,dx\right)^2 dt.\end{gathered} \tag{4.13}$$

With the help of the solution of boundary value problems (4.1),(4.2) and (4.5),(4.6), we transform expression (4.13). It is clear that the solution of boundary value problem (4.10), (4.11) satisfies the identity

$$\int_Q \left[-\frac{\partial \delta u}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^{n} \frac{\partial \delta u}{\partial x_i}\frac{\partial \eta}{\partial x_i} + (\upsilon + \delta\upsilon)\frac{\partial \delta u}{\partial t}\eta \right] dxdt$$
$$= -\int_Q \delta\upsilon \frac{\partial u}{\partial t}\eta dxdt \tag{4.14}$$

for all $\eta = \eta(x,t) \in W_{2,0}^1(Q)$, which equal to zero at $t = T$. If in (4.3) we assume $\mu = \delta u$, in (4.14) $\eta = \psi$ and sum up the obtained relations, we get

$$\int_Q K(x,t)\left(\int_\Omega K(x,t)\,u(x,t)\,dx - g(t)\right)\delta u(x,t)\,dxdt$$
$$= \int_0^T \left(\int_\Omega \frac{\partial u}{\partial t}\psi dx\right)\delta\upsilon(t)\,dt + \int_Q \delta\upsilon\frac{\partial \delta u}{\partial t}\psi dxdt.$$

Taking into account this equality in (4.13), we have

$$\Delta J(\upsilon) = \int_Q \frac{\partial u}{\partial t}\psi\delta\upsilon dxdt + R, \tag{4.15}$$

where

$$R = \int_Q \delta\upsilon\frac{\partial \delta u}{\partial t}\psi dxdt + \frac{1}{2}\int_0^T \left(\int_\Omega K(x,t)\,\delta u(x,t)\,dx\right)^2 dt. \tag{4.16}$$

Assuming in (4.7) $\eta = \delta\upsilon$, we get

$$\int_0^T \left[\frac{d\psi_1}{dt}\frac{d\delta\upsilon}{dt} + \psi_1\delta\upsilon\right] dt = \int_Q \frac{\partial u}{\partial t}\psi\delta\upsilon dxdt.$$

Then taking into account this equality in (4.15), we have

$$\Delta J(\upsilon) = \int_0^T \left[\frac{d\psi_1}{dt}\frac{d\delta\upsilon}{dt} + \psi_1\delta\upsilon\right] dt + R. \tag{4.17}$$

It is clear that the expression

$$\left\langle J'(\upsilon), \delta\upsilon \right\rangle = \int_0^T \left[\psi_1\delta\upsilon + \frac{d\psi_1}{dt}\frac{d\delta\upsilon}{dt}\right] dt \tag{4.18}$$

determines the linear bounded functional at $\delta\upsilon$ on $W_2^1[0,T]$. Linearity of functional (4.18) is obvious. Using the Cauchy-Bunyakovsky inequality and estimations (4.8), (3.3), (4.4) we get the boundedness of functional (4.18) with by to $\delta \upsilon$ on $W_2^1[0,T]$.

Now we give the estimate the reminder term R which is determined by equality (4.16).

Again, using the Cauchy-Bunyakovsky inequality, the boundedness of the embedding $W_2^1[0,T] \to C[0,T]$ and estimation (4.4), (4.12), we get

$$R \leq c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)} + \|g\|_{L_2(0,T)}\right] \|\delta\upsilon\|^2_{W_2^1[0,T]}.$$

Taking into account this estimation from (4.17) we get that functional (3.1) is Frechet differentiable on V and formula (4.9) is valid for its gradient.

Now show that mapping $\upsilon \to J'(\upsilon)$ continuously acts from V to $W_2^1[0,T]$. Let

$$\delta\psi(x,t) = \psi(x,t;\upsilon+\delta\upsilon) - \psi(x,t;\upsilon), \delta\psi_1(t) = \psi_1(t;\upsilon+\delta\upsilon) - \psi_1(t;\upsilon).$$

From (4.5), (4.6) it follows that $\delta\psi_1(t)$ is the generalized solution from $W_2^1[0,T]$ of the boundary value problem

$$-\frac{d^2\delta\psi_1}{dt^2} + \delta\psi_1 = \int_\Omega \left[\frac{\partial\delta u}{\partial t}\psi + \frac{\partial u}{\partial t}\delta\psi + \frac{\partial\delta u}{\partial t}\delta\psi\right]dx,\ 0<t<T,$$

$$\frac{d\delta\psi_1}{dt}|_{t=0} = \frac{d\delta\psi_1}{dt}|_{t=T} = 0.$$

For solution of this boundary value problem as in (4.8), we get the estimation

$$\|\delta\psi_1\|_{W_2^1[0,T]} \leq c\left[\left\|\frac{\partial\delta u}{\partial t}\right\|_{L_2(Q)} \cdot \|\psi\|_{L_2(Q)} + \left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q)} \|\delta\psi\|_{L_2(Q)} + \left\|\frac{\partial\delta u}{\partial t}\right\|_{L_2(Q)} \|\delta\psi\|_{L_2(Q)}\right]. \tag{4.19}$$

Furthermore, for the function $\delta\psi(x,t)$, as in (4.12), we can get the estimation

$$\|\delta\psi\|_{W_2^1(Q)} \leq c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)} + \|g\|_{L_2[0,T]}\right] \|\delta\upsilon\|_{W_2^1[0,T]}. \tag{4.20}$$

Then taking into account estimations (3.3), (4.4), (4.12), (4.20) from (4.19) we get $\|\delta\psi_1\|_{W_2^1[0,T]} \to 0$ as $\|\delta\upsilon\|_{W_2^1[0,T]} \to 0$. Hence and from (4.9) it follows that the mapping $\upsilon \to J'(\upsilon)$ is continuously acting from V to $W_2^1[0,T]$. Theorem 4.1 is proved.

5 Necessary optimality condition

Theorem 5.1 *Let the condition of theorem 4.1 be fulfilled. Then for optimality of the control* $\upsilon_* = \upsilon_*(t) \in V$ *in problem* (2.1)-(2.3), (2.5), (3.1) *it is necessary that the inequality*

$$\int_0^T \left[\psi_{1*}(t)\left(\upsilon(t) - \upsilon_*(t)\right) + \frac{d\psi_{1*}(t)}{dt}\left(\frac{d\upsilon(t)}{dt} - \frac{d\upsilon_*(t)}{dt}\right)\right] dt \geq 0 \qquad (5.1)$$

be fulfilled for any $\upsilon = \upsilon(t) \in V$, *where* $\psi_{1*}(t) = \psi_1(t; \upsilon_*)$ *is the solution of problem* (4.5), (4.6) *for* $\upsilon = \upsilon_*(t)$.

Proof. The set determined by relation (2.5) is convex in $W_2^1[0,T]$. Furthmore, by Theorem 4.1 the functional $J(\upsilon)$ is continuously by differentiable Frechet on V and its gradient at the point $\upsilon \in V$ is defined by equality (4.9). Then by theorem 5 from [13, p. 28] fulfilled of the inequality $\langle J'(\upsilon_*), \upsilon - \upsilon_* \rangle \geq 0$ on the element $\upsilon_* \in V_*$ is necessary for all $\upsilon \in V$. Hence and from (4.9) it follows the validity of inequality (5.1) for all $\upsilon \in V$. Theorem 5.1 is proved.

Remark 5.1 All obtained results are also valid in the case when in equation (2.1) instead of the Laplace operator we take a self-adjoint second order elliptic operator with smooth coefficients and the addend term $\sum_{i=1}^{n} \upsilon_i(t) \frac{\partial u}{\partial x_i}$ is present, so that $\upsilon_i(t) \in V$, $i = 1, ..., n$.

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