SOME STATICALLY DEFINABLE PROBLEMS FOR CYLINDRICAL SHELLS

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Abstract

In this paper we consider some statically definable problems for a cylindrical shell with constant thickness. The expand of middle surface of the shell on the plane is a rectangle. The shell is so thin that Hooke's law does not apply. It means that, the body of the transverse stress field is assigned beforehand and for the tangential stress components the system of equations is obtained. This system of equations is based on the physical boundary conditions and the problems are solved analytically.

Key words and phrases: Cylindrical shell, Boundary value problems.
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1 Introduction

I. Vekua in his famous monograph ([1], chapter III) considered the theory of shells so called statically definable problems. In this chapter, the author shows that, on applying Hooke's law to a homogeneous isotropic elastic body it may be assumed that the medium has similar resistivity in any direction. As a rule, this property is not globally fulfilled in the case of thin shells. The degree of resistance to the deformation of the thin shells, is evidently weaker in the transverse direction than in longitudinal directions. I. Vekua concluded that in computing the elastic shell the application of the generalized Hooke's law is not always admissible. The static definability of the problem is achieved when the the transverse stress field $\boldsymbol{P}^3$ is known in advance. This allows us to obtain, for the definition of the tangential stress field, the system of first-order partial differential equations for two unknown functions. On adding some physical boundary conditions to this system, to be formulated below, we obtain the problem wich enables the determination of the tangential stress field. The deformation of a shell is not determined in this case and the stress-strain relations are not employed.

I. Vekua takes the basic correlations in this case and cites the formulation of boundary value problems. Later, Vekua considers convex shells. The major relation he writes the isometric-conjugate coordinate system and explores the boundary value problems using the apparatus of a generalized analytic function theory.

Our aim is to consider statically definable problems for cylindrical shells. Some boundary value problems solution turned out to be very simple and the obvious solution is constructed without using the method of complex variable function theory.

2 Main system equation of statically definable problems for cylindrical shells Let be a cylindrical shells of constant thickens

Let Ω be a cylindrical shells of constant thickness $2h$. Denote by S the middle surface of the domain Ω. The position vector $\boldsymbol{R}$ in the normally related coordinate system with the middle surface may be expressed by the formula

$$\boldsymbol{R} = \boldsymbol{r}(x^1, x^2) + x^3\boldsymbol{n}(x^1, x^2), \tag{1}$$

where x^1, x^2 are Gaussian parameters of the surface S, $\boldsymbol{r} = \boldsymbol{r}(x^1, x^2)$ and $\boldsymbol{n} = \boldsymbol{n}(x^1, x^2)$ are, respectively, position vector and unit vector of the normal to S at the $(x^1, x^2) \in S$.

When considering thin shells we may assume that insie the shell the vector $\boldsymbol{P}^3$ is contiuned according to the linear (with respect to x^3) law

$$\boldsymbol{P}^3 = \frac{1}{2}\left(-\overset{(+)}{\boldsymbol{P}} + \overset{(-)}{\boldsymbol{P}}\right) - \frac{x^3}{2h}\left(\overset{(+)}{\boldsymbol{P}} + \overset{(-)}{\boldsymbol{P}}\right) \quad \text{in } \Omega, \tag{2}$$

where $\boldsymbol{P}^3|_{x^3=h} = -\overset{(+)}{\boldsymbol{P}}$, $\boldsymbol{P}^3\,|_{x^3=-h} = \overset{(-)}{\boldsymbol{P}}$ are the given functions.

In this case for tangential stress field $\boldsymbol{T}^\alpha = P^{\alpha\beta}\boldsymbol{R}_\beta$ on every surface $\hat{S} : x^3 = const$ I. Vekua obtains the following equations [1]

$$\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{\mathrm{g}}\,\boldsymbol{T}^\alpha) + \boldsymbol{X} = 0 \ \text{ on } \hat{S} : x^3 = const, \tag{3}$$

where

$$\boldsymbol{X} = \frac{1}{\sqrt{\mathrm{g}}}\partial_\alpha(\sqrt{\mathrm{g}}\,\boldsymbol{T}^\alpha) + \frac{1}{\sqrt{\mathrm{g}}}\partial_\alpha(\sqrt{\mathrm{g}}\,\boldsymbol{T}^\alpha) + \boldsymbol{\Phi}, \tag{4}$$

$\boldsymbol{\Phi}$ is a vector of volume force.

Choose the Gaussian parameters x^1, x^2 in the middle of the surface. Let x, y, z be a rectangular Cartesian coordinate system unit $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the direction of the x-axis y-axis and z-axis respectively. Then

$$x^1 = x, \quad x^2 = R\varphi, ; \ 0 \leq \varphi < 2\pi, \tag{5}$$

where R is a radius of transverse of middle surface; The angle φ measured from the z positive direction to clockwise direction, if we look at the positive direction of the axis x (Fig. 1).

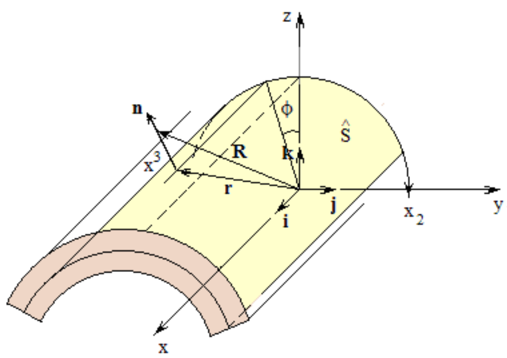

Fig. 1.

In this case the radius vector $\boldsymbol{r}$ has the form

$$\boldsymbol{r} = x^1\boldsymbol{i} + R\sin\varphi\boldsymbol{j} + R\cos\varphi\boldsymbol{k} = x^1\boldsymbol{i} + R\sin\frac{x^2}{R}\boldsymbol{j} + R\cos\frac{x^2}{R}\boldsymbol{k}, \tag{6}$$

and $\boldsymbol{n}$ is expressed by the formula

$$\boldsymbol{n} = \sin\frac{x^2}{R}\boldsymbol{j} + \cos\frac{x^2}{R}\boldsymbol{k}. \tag{7}$$

according of formulas (1), (6), (7) the radius-vector $\boldsymbol{R}$ has the form

$$\boldsymbol{R} = x^1\boldsymbol{i} + (R + x^3)\sin\frac{x^2}{R}\boldsymbol{j} + (R + x^3)\cos\frac{x^2}{R}\boldsymbol{k}. \tag{8}$$

from formula (8) we obtain the basic spatial vector

$$\boldsymbol{R}_1 = \partial_1\boldsymbol{R} = \boldsymbol{i}, \tag{9}$$

$$\boldsymbol{R}_2 = \partial_2\boldsymbol{R} = \left(1 + \frac{x^3}{R}\right)\cos\frac{x^2}{R}\boldsymbol{j} - \left(1 + \frac{x^3}{R}\right)\sin\frac{x^2}{R}\boldsymbol{k}, \tag{10}$$

$$\boldsymbol{R}_3 = \partial_3\boldsymbol{R} = \boldsymbol{n} = \sin\frac{x^2}{R}\boldsymbol{j} + \cos\frac{x^2}{R}\boldsymbol{k}. \tag{11}$$

From (9)-(11) for covariant and contravariant components of first basic quadratic form we get the formulas

$$\begin{aligned} &g_{11} = \boldsymbol{R}_1\boldsymbol{R}_1 = 1,\ g_{22} = \boldsymbol{R}_2\boldsymbol{R}_2 = \left(1+\frac{x^3}{R}\right)^2, \\ &g_{33} = \boldsymbol{n}\boldsymbol{n} = 1,\ g_{12} = g_{21} = g_{31} = g_{32} = 0; \end{aligned} \tag{12}$$

$$g^{11} = 1,\ g^{22} = \left(1+\frac{x^3}{R}\right)^{-2},\ \ g^{33} = 1,\ \ g^{12} = g^{21} = g^{31} = g^{32} = 0. \tag{13}$$

By (12) the discriminant of the metric quadratic form has the form

$$g = \begin{vmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(1+\frac{x^3}{R}\right)^2. \tag{14}$$

For the contravariant spatial basic vector we get the following formulas

$$\boldsymbol{R}^1 = \boldsymbol{R}_1 = \boldsymbol{i}, \tag{15}$$

$$\boldsymbol{R}^2 = g^{22}\boldsymbol{R}_2 = \left(1+\frac{x^3}{R}\right)^{-2}\boldsymbol{R}_2, \tag{16}$$

$$\boldsymbol{R}^3 = \boldsymbol{R}_3 = \boldsymbol{n}. \tag{17}$$

Consider also on the surface $\hat{S} : x^3 = const$ the contravariants coefficient of the second major quadratic form

$$\hat{b}_{11} = 0,\ \ \hat{b}_{22} = \boldsymbol{n}\boldsymbol{R}_{33} = -\frac{1}{R}\left(1+\frac{\hat{x}^3}{R}\right),\ \ \hat{b}_{12} = \hat{b}_{21} = 0. \tag{18}$$

The mixed coefficients of the second major quadratic form would be different from zero only $\hat{b}_2^2$

$$\hat{b}_2^2 = g^{22}\hat{b}_{22} = -\frac{1}{R+\hat{x}^3}. \tag{19}$$

It is clear that, the Gaussian curvature of the cylindrical surface $\hat{S}$ is equal to zero, and the average curvature is half of the quantity $\hat{b}_2^2$

$$\hat{K} = 0,\ \ \hat{H} = \hat{k}_1 + \hat{k}_2 = 0.5(\hat{b}_1^1 + \hat{b}_2^2) = -\frac{1}{2(R+\hat{x}^3)}.$$

where $\hat{k}_1 = 0$ and $\hat{k}_2$ are major curvatures of the surface $\hat{S}$.

The basis of the obtained relation above the system of equations (3) for cylindrical shells takes the form

$$\begin{cases} \hat{\nabla}_1 P^{11} + \hat{\nabla}_2 P^{21} + X^1 = 0, \\ \hat{\nabla}_1 P^{12} + \hat{\nabla}_2 P^{22} + X^2 = 0, \quad \hat{S} : x^3 = const, \\ \hat{b}_{22} P^{22} + X^3 = 0, \end{cases} \tag{20}$$

where, by formula (4) and Weingarten formulas

$$\begin{aligned} &\boldsymbol{X} = X^j \boldsymbol{R}_j = \partial_\alpha P^{\alpha 3} \boldsymbol{n} + P^{\alpha 3} \partial_\alpha \boldsymbol{n} + \partial_3 (P^{3j} \boldsymbol{R}_j) \\ &+ \frac{1}{R + x^3} P^{3j} \boldsymbol{R}_j + \Phi^{3j} \boldsymbol{R}_j \\ &= \partial_\alpha P^{\alpha 3} \boldsymbol{n} + \frac{1}{R + x^3} P^{23} \boldsymbol{R}_2 + \partial_3 (P^{3j}) \boldsymbol{R}_j \\ &+ P^{32} \frac{1}{R + x^3} \boldsymbol{R}_2 + \frac{1}{R + x^3} P^{3j} \boldsymbol{R}_j + \phi^{3j} \boldsymbol{R}_j \\ &= \left(\partial_3 P^{31} + \frac{P^{31}}{R + x^3} \right) \boldsymbol{R}_1 + \left(\partial_3 P^{32} + \frac{3}{R + x^3} P^{32} \right) \boldsymbol{R}_2 \\ &+ \left(\partial_\alpha P^{\alpha 1} + + \partial_3 P^{33} + \frac{1}{R + x^3} P^{33} \right) \boldsymbol{n} \end{aligned}$$

from which obtained

$$\begin{aligned} &X^1 = \partial_3 P^{31} + \frac{P^{31}}{R + x^3}, \quad X^2 = \partial_3 P^{32} + \frac{3}{R + x^3} P^{32}, \\ &X^3 = \partial_\alpha P^{\alpha 3} + \partial_3 P^{33} + \frac{1}{R + x^3} P^{33}. \end{aligned} \tag{21}$$

On the basis obtained formulas above in system (20) the covariant derivatives coincide with the partial derivatives. In addition, by the third equations of this system the quantity P^{22}

$$\begin{cases} \partial_1 P^{11} + \partial_2 P^{21} = -X^1, \\ \partial_1 P^{12} + \partial_2 P^{22} = -X^2, \end{cases} \tag{22}$$

$$P^{22} = -\frac{x^3}{\hat{b}_{22}} = \frac{R}{1 + \dfrac{\hat{x}^3}{R}} X^3 \tag{23}$$

is determined.

If substitute (23) into the second equation of system (22), we get

$$\partial_1 P^{12} = -X^2 - \frac{R}{1 + \dfrac{\hat{x}^3}{R}} \partial_2 X^3. \tag{24}$$

3 Boundary value problems

Let the surface $\hat{S}$ be the following rectangle $\hat{S} = \{0 < x_1 < a \; 0 < x_2 < b\}$. Set the following boundary value problems (Fig. 2):

Problem I:

$$
\begin{aligned}
&x_1 = 0, \;\; x_1 = a: \;\; P^{11} = 0, \\
&x_2 = 0, \;\; x_2 = b: \;\; P^{22} = 0.
\end{aligned} \tag{25}
$$

$$
P^{22} = \frac{R}{1 + \frac{\hat{x}^3}{R}} X^3 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{\pi m x_1}{a} \sin \frac{\pi n x_2}{b}, \tag{26}
$$

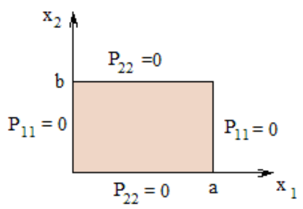

Fig. 1.

where C_{mn} is a known quantity. The quantities X_1 and X_2 are represented on every $x^3 = const$ by the following series

$$
\begin{aligned}
X^1 &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} X^1_{mn} \cos \frac{\pi m x_1}{a} \sin \frac{\pi n x_2}{b}, \\
X^2 &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} X^2_{mn} \sin \frac{\pi m x_1}{a} \cos \frac{\pi n x_2}{b},
\end{aligned} \tag{27}
$$

P^{11} and P^{12} are also represented by the series

$$
\begin{aligned}
P^{11} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{\pi m x_1}{a} \sin \frac{\pi n x_2}{b}, \\
P^{12} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} \cos \frac{\pi m x_1}{a} \cos \frac{\pi n x_2}{b}.
\end{aligned} \tag{28}
$$

(25) boundary conditions can be satisfied automatically. If the expansions P^{12} of (28) and P^{22} of (26). If we substitute into (22), and take into account (27), we get

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -\frac{\pi m}{a} B_{mn} \sin \frac{\pi m x_1}{a} \cos \frac{\pi n x_2}{b}
$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-X^2_{mn} - \frac{\pi n}{b}C_{mn}\right)\sin\frac{\pi m x_1}{a}\cos\frac{\pi n x_2}{b}.$$

from which

$$B_{mn} = \frac{a}{\pi m}X^2_{mn} + \frac{n}{m}\frac{a}{b}C_{mn}, \quad m,n = 1,2,\ldots.$$

By substituting expansions (7) (8) into (1) we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\pi m}{a}A_{mn}\cos\frac{\pi m x_1}{a}\cos\frac{\pi n x_2}{b}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-X^1_{mn} + \frac{\pi n}{b}B_{mn}\right)\cos\frac{\pi m x_1}{a}\sin\frac{\pi n x_2}{b}$$

from which

$$A_{mn} = \frac{a}{b}\frac{n}{m}B_{mn} - \frac{a}{mn}X^1_{mn}, \quad m,n = 1,2,\ldots.$$

Problem II (Fig. 3):

$$x_1 = 0, \ \ x_1 = a: \ \ P^{12} = 0,$$

$$x_2 = 0, \ \ x_2 = b: \ \ P^{21} = 0.$$

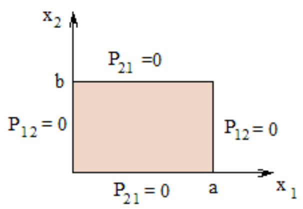

Fig. 1.

This time X_1 and X_2 are represented by the series

$$X^1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X^1_{mn}\sin\frac{\pi m x_1}{a}\cos\frac{\pi n x_2}{b},$$

$$X^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X^2_{mn}\cos\frac{\pi m x_1}{a}\sin\frac{\pi n x_2}{b}.$$

$$P^{22} = \frac{R}{1+\frac{\hat{x}^3}{R}}X^3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\cos\frac{\pi m x_1}{a}\cos\frac{\pi n x_2}{b},$$

where C_{mn} is a known quantity.

P^{12} and P^{11} are represented by the series

$$P^{12} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{\pi m x_1}{a} \sin \frac{\pi n x_2}{b},$$

$$P^{11} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos \frac{\pi m x_1}{a} \cos \frac{\pi n x_2}{b}.$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi m}{a} B_{mn} \cos \frac{\pi m x_1}{a} \sin \frac{\pi n x_2}{b}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(-X_{mn}^2 + \frac{\pi n}{b} C_{mn} \right) \cos \frac{\pi m x_1}{a} \sin \frac{\pi n x_2}{b},$$

from which

$$B_{mn} = -\frac{a}{\pi m} X_{mn}^2 + \frac{n}{m} \frac{a}{b} C_{mn}, \quad m, n = 1, 2, \dots,$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -\frac{\pi m}{a} A_{mn} \sin \frac{\pi m x_1}{a} \cos \frac{\pi n x_2}{b} =$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(-X_{mn}^1 - \frac{\pi n}{b} B_{mn} \right) \sin \frac{\pi m x_1}{a} \cos \frac{\pi n x_2}{b}.$$

we get

$$A_{mn} = \frac{a}{\pi m} X_{mn}^1 + \frac{a}{b} \frac{n}{m} B_{mn}, \quad m, n = 1, 2, \dots.$$

When these functions are sufficiently smooth, it is not difficult to prove that the series are absolutely and uniformly convergence.

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