SOME BOUNDARY VALUE PROBLEMS FOR PLANE THEORY OF ELASTICITY FOR DOUBLY-CONNECTED DOMAIN BOUNDED BY POLYGONS

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(Received: 06.07.2016; accepted: 22.10.2016)

Abstract

We consider a plane problem of elasticity for double-connected domain bounded by polygons. The problem is solved by the methods of conformal mappings and boundary value problems of analytic functions. The sought complex potentials are constructed effectively (in the analytical form). Estimates of the obtained solutions are derived in the neighborhood of angular points.

Key words and phrases: Conformal mapping, Kolosov-Muskhelishvili formula, Riemann-Hilbert problem for circular ring.

AMS subject classification: 74B05.

1 Introduction

As is known (see [1]), the boundary value problems of plane theory of elasticity for complex potentials of Muskhelishvili is reduced to boundary value problems of analytic functions. The more effective ways of solving these problems are based on the construction of a conformally mapping function of the given domain onto canonical domains (a circle, a circular ring). Therefore the above-mentioned methods are little-suited for the effective solution of problems in multi-connected domains. Nevertheless, for some practically important classes of multi-connected domain one manages to construct effectively (analytically) the conformally mapping function of that domain onto a circular ring. These classes involve doubly-connected domains bounded by polygons and their modifications (polygonal domains with a curvilinear hole and so on). The Kolosov-Muskhelishvili methods make it possible to decompose these problems into two Riemann-Hilbert problems for the circular ring and by solving the latter problems to construct the sought complex potentials in the analytical form.

2 Statement of the problem

Suppose S is a doubly-connected domain in the z plane of the complex variable, bounded by convex polygons (A) and (B). We will assume that (A) is the external boundary of the domain S, and B is the internal boundary, and we will denote by A_k $(k = \overline{1,p})$ and B_m $m = \overline{1,q}$ the vertices (and their affixes), and $L_0^{(k)} = A_k A_{k+1}$ $(k = \overline{1,p}, A_{p+1} = A_1)$ and $L_1^{(m)} = B_m B_{m+1}$ $(B_{q+1} = B_1,\ m = \overline{1,q})$ the sides of the polygons A and B. The values of the internal angles of the domain S at the vertices A_k and B_m will be denoted by $\pi\alpha_k^0$ and $\pi\beta_m^0$, while the angles between the x axis and outward normals to the contours L_0 $\left(L_0 = \bigcup\limits_{k=1}^{p} L_0^{(k)}\right)$ and L_1 $\left(L_1 = \bigcup\limits_{m=1}^{q} L_1^{(m)}\right)$ will be denoted by $\alpha(t)$ and $\beta(t)$. The positive direction on $L = L_0 \bigcup L_1$ well be assumed to be that which keeps domain S to the left.

Assume that the sides of the boundary L_0 and L_1 of S are under the action of normal stresses with the principal vectors P_k^0 and P_m^1, while the tangential displacements $v_\tau(t) = 0$ $t \in L_0 \cup L_1$ and the normal displacements $v_n(t) = const,\ t \in L_0 \cup L_1$.

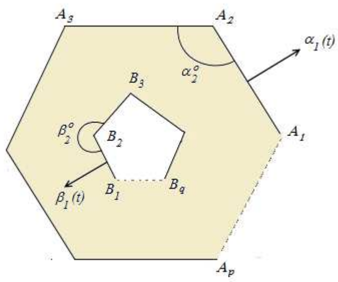

Fig. 1.

The problem consists in defining the elastic equilibrium of the plate and establishing the situation in which the concentration of stresses occurs near the angular points.

Similar problems of the plane theory of elasticity and plate bending have been are considered in [2, 4].

3 Solution of the problem

Let us recall some results concerning the conformal mapping of a doubly-connected domain S onto the circular ring $D_0\{1 < |\zeta| < R_0\}$ (see [4]).

From the equations

$$t - A_k = i|t - A_k|e^{i\alpha_k(t)},\ t \in L_0^{(k)};$$

$$t - B_m = i|t - B_m|e^{i\beta_m(t)},\ t \in L_1^{(m)},$$

(where $\alpha_k(t)$ and $\beta_m(t)$ are the angles between the x axis and outward normals to the contours $L_0^{(k)}$ and $L_1^{(m)}$) we get

$$\begin{aligned} \operatorname{Re}\left[ie^{-i\alpha(t)} \cdot t\right] &= \operatorname{Re}\left[ie^{-i\alpha(t)} \cdot A(t)\right],\ \ t \in L_0; \\ \operatorname{Re}\left[ie^{-i\beta(t)} \cdot t\right] &= \operatorname{Re}\left[ie^{-i\alpha(t)} \cdot B(t)\right],\ \ t \in L_1, \end{aligned} \tag{1}$$

where $A(t),\ B(t),\ \alpha(t)$ and $\beta(t)$ are the piecewise constant functions:

$$A(t) = A_k,\ \ \alpha(t) = \alpha_k(t),\ \ t \in L_0^{(k)};$$

$$B(t) = B_k,\ \ \beta(t) = \beta_k(t),\ \ t \in L_1^{(m)}.$$

From conditions (1) regarding the function $z = \omega(\zeta)$ ($z = \omega(\zeta)$ is the function conformally mapping the circular ring D onto the domain S), after differentiation with respect to the abscissa s, we obtain for the circular ring D the following Riemann-Hilbert boundary value problem

$$\begin{aligned} &\operatorname{Re}[i\sigma e^{-i\alpha_0(\sigma)}\omega'(\sigma)] = 0,\ \ \sigma \in l_0,\ (|\zeta| = R); \\ &\operatorname{Re}[i\sigma e^{-i\Delta_0(\sigma)}\omega'(\sigma)] = 0,\ \ \sigma \in l_1,\ (|\zeta| = 1), \end{aligned} \tag{2}$$

where $\alpha_0(\sigma) = \alpha[\omega(\sigma)]$, $\beta_0(\sigma) = \beta[\omega(\sigma)]$, L_1 is mapped into l_1 and L_0 is mapped into l_0 by $z = \omega(\zeta)$ conformally map.

The boundary value problem (2) with respect to the function $i \ln \omega'(\zeta)$ is reduced in to the Dirichlet problem

$$\operatorname{Re}[i \ln \omega'(\zeta)] = f(\sigma),\ \ \sigma \in l = l_0 \cup l_1, \tag{3}$$

where

$$f(\sigma) = \frac{i}{2} \ln\left[R^2\sigma^{-2}e^{2i\alpha_0(\sigma)}\right],\ \sigma \in l_0;$$

$$f(\sigma) = \frac{i}{2} \ln\left[\sigma^{-2}e^{2i\beta_0(\sigma)}\right],\ \sigma \in l_1,$$

whose condition of solvability in the class $h(b_1, ..., b_q)$ (for this class, see [6]) has the form

$$\prod_{k=1}^{p}\left(\frac{a_k}{R}\right)^{\alpha_k^0-1}\prod_{m=1}^{q}(b_m)^{\beta_m^0-1}=1, \tag{4}$$

(a_k and b_m are the inverse images of the points A_k and B_m), and the solution itself of the given class is given by the formula

$$\omega_0'(\zeta)=K^0\prod_{j=-\infty}^{\infty}G\left(R^{2j}\zeta\right)g\left(R^{2j}\zeta\right)R^{2\delta_j}, \tag{5}$$

where

$$G(\zeta)=\prod_{k=1}^{p}(\zeta-a_k)^{\alpha_k^0-1},\quad g(\zeta)=\prod_{m=1}^{q}(\zeta-b_m)^{\beta_m^0-1},$$

$$\delta_j=\begin{cases}0, & j\geq 0,\\ 1, & j\leq -1,\end{cases}$$

with K^0 as an arbitrary real constant.

From the given results we obtain the factorization of the coefficient of problem (2) in the following form

$$\begin{aligned} e^{2i\alpha_0(\sigma)}R^2\sigma^{-2}&=\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}},\quad \sigma\in l_0,\\ e^{2i\beta_0(\sigma)}\sigma^{-2}&=\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}},\quad \sigma\in l_1,\end{aligned} \tag{6}$$

where $\omega'(\zeta)$ is defined by formula (5).

Let us now return to the considered problem. On the basis of the well-known Kolosov-Muskhelishvili's formulas (see [1], §41) for finding complex potential $\varphi(z)$ and $\psi(z)$ in this case we have the boundary conditions

$$\mathrm{Re}\left[e^{-i\nu(t)}\varphi(t)\right]=f(t),\quad t\in L; \tag{7}$$

$$\mathrm{Re}\left[e^{-i\nu(t)}(\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)})\right]=f_1(t),\quad t\in L, \tag{8}$$

where

$$f(t)=(\varkappa+1)^{-1}\{\mathrm{Re}[e^{-i\nu(t)}\left(i\int_0^s(N(t_0)+iT(t_0))\right)e^{i\nu(t_0)}ds_0+$$

$$+c_1^0+ic_2^0]+2\mu v_n(t)\};$$

$$f_1(t)=\mathrm{Re}[e^{-i\nu(t)}\left(i\int_0^s(N(t_0)+iT(t_0))\right)e^{i\nu(t_0)}ds_0+$$

$$+c_1^1 + ic_2^1], \ \ t \in L,$$

$\nu(t) = \alpha(t), \ t \in L_0$; $\nu(t) = \beta(t), \ t \in L_1$; c_1^k and c_2^k $(k = 0, 1)$ are arbitrary real constants, $\varkappa = \dfrac{\lambda + 3\mu}{\lambda + \mu}$ is the Muskhelishvili's constant, λ and μ are the Lame constants.

After the conformal mapping of the domain S onto the circular ring D, the boundary conditions (7) and (8) are written in the form

$$\text{Re}\left[e^{-i\nu_0(\sigma)}\varphi_0(\sigma)\right] = f_0(\sigma), \ \ \sigma \in l; \tag{9}$$

$$\text{Re}\left[e^{-i\nu_0(\sigma)}(\varphi_0(\sigma) + \sigma\overline{\varphi_0'(\sigma)} + \overline{\psi_0(\sigma)})\right] = f_1^0(\sigma), \ \ \sigma \in l, \tag{10}$$

where $\nu_0(\sigma) = \nu[\omega(\sigma)]$; $\varphi_0(\sigma) = \varphi[\omega(\sigma)]$; $\psi_0(\sigma) = \psi[\omega(\sigma)]$; $f_0(\sigma) = f[\omega(\sigma)]$; $f_1^0(\sigma) = f_1[\omega(\sigma)]$.

Let us consider problem (9). From the boundary condition (6), this problem for the function

$$\Omega(\zeta) = [\zeta\omega'(\zeta)]^{-1}\varphi_0(\zeta)$$

reduces to the Dirichlet problem for the a circular ring

$$\text{Re}[\Omega(\sigma)] = [\sigma\omega'(\sigma)]e^{i\nu_0(\sigma)}f_0(\sigma), \ \ \sigma \in l. \tag{11}$$

A solvability condition of problem (11) has the form (see [5])

$$\int_l \frac{f_0(\sigma)e^{i\nu_0(\sigma)}}{\sigma\omega'(\sigma)}\frac{d\sigma}{\sigma} = 0, \tag{12}$$

and its solution is given by the formula

$$\Omega(\zeta) = M(\zeta), \tag{13}$$

where

$$M(\zeta) = \frac{1}{\pi i}\sum_{j=-\infty}^{\infty}\int_l \frac{f_0(\sigma)e^{i\nu_0(\sigma)}}{(\sigma - R^{2j}\zeta)\sigma\omega'(\sigma)}d\sigma + iE_1, \tag{14}$$

where E_1 is an arbitrary real constant.

Thus, for the function $\varphi_0(\zeta)$ we obtain the formula

$$\varphi_0(\zeta) = \zeta\omega'(\zeta)M(\zeta), \tag{15}$$

where $M(\zeta)$ is defined by formula (14).

Taking into account the form of the function $\omega'(\zeta)$ in the neighborhood of the point a_k $(k = \overline{1,p})$, we conclude that for the continuous extension of the function $\varphi_0(\zeta)$ in the domain $D \cup l$ it is necessary that the conditions

$$M(a_k) = 0, \quad k = \overline{1,p} \tag{16}$$

be fulfilled.

Since $\varphi'(z) = \varphi_0'(\zeta)[\omega'(\zeta)]^{-1}$, from (15) we have

$$\varphi'(z) = M(\zeta) + \zeta\frac{\omega''(\zeta)}{\omega'(\zeta)}M(\zeta) + \zeta M'(\zeta). \tag{17}$$

Based on the results obtained in (see [6], §26) as to the behavior of a Cauchy type integral near the density discontinuity points, we conclude that near the points b_k $(k = \overline{1,q})$ the function $M(\zeta)$ has the form

$$M(\zeta) = \frac{K_1^{(k)}}{(\zeta - b_k)^{\beta_k^0 - 1}} + M_k^0(\zeta), \quad k = \overline{1,q},$$

where $M_k^0(\zeta)$ is the function that near the point b_k admits the following estimate

$$\left|M_k^0(\zeta)\right| < \frac{C^{(k)}}{|\zeta - b_k|^{\delta_k}}, \quad \delta_k < \beta_k^0 - 1, \tag{18}$$

where $K_1^{(k)}$ and $C^{(k)}$ are the well-defined constant.

Taking into account the behavior of the conformally mapping function near the angular points (see [7], §37), we obtain

$$\omega(\zeta) = B_k + (\zeta - b_k)^{\beta_k^0}\Omega_k(\zeta),$$

$$\zeta\frac{\omega''(\zeta)}{\omega'(\zeta)} = \frac{b_k(\beta_k^0 - 1)}{\zeta - b_k} + \Omega_k^*(\zeta), \quad k = \overline{1,q},$$

where $\Omega_k(b_k) \neq 0$, $\Omega_k^*(\zeta)$ is the regular part of the Loran decomposition of the function $\zeta\dfrac{\omega''(\zeta)}{\omega'(\zeta)}$ near the point b_k.

By the above reasoning, from (17) we obtain the estimate

$$\varphi'(z) = \frac{K^{(k)}}{(\zeta - b_k)^{\beta_k^0 - 1}} + \Omega_k^0(\zeta), \quad k = \overline{1,q}, \tag{19}$$

and thus near a point B which is one of the points B_k $(k = \overline{1,q})$ we have the estimates

$$\left|\varphi'(z)\right| < M_1\left|z - B\right|^{\frac{1}{\beta^0} - 1}, \quad M_1 = const. \tag{20}$$

By a similar reasoning to the above, it is proved that $\varphi'(z)$ is almost bounded (i.e. has singularities of logarithmic type) near the points A_k ($k = \overline{1,q}$).

After finding the function $\varphi(z)$, the definition of the function $\psi(z)$ by (8) reduces to the following problem which is analogous to problem (7)

$$\operatorname{Re}\left[e^{i\nu(t)}R(t)\right] = N(t), \tag{21}$$

where

$$R(z) = \psi(z) + P(z)\varphi'(z),$$
$$N(t) = f_1(t) - \operatorname{Re}\left[e^{i\nu(t)}\left(\overline{\varphi(t)} + (\bar{t} - P(t))\varphi'(t)\right)\right], \quad t \in L,$$

$P(z)$ is an interpolation polynomial satisfying the condition $P(B_k) = \overline{B}_k$ ($k = \overline{1,q}$).

After the conformal mapping of the domain S onto the circular ring D from (10) the problem (21) is written in the form

$$\operatorname{Re}\left[e^{i\nu_0(\sigma)}R_0(\sigma)\right] = N_0(\sigma), \quad \sigma \in l, \tag{22}$$

where

$$R_0(\sigma) = \psi_0(\sigma) + P_0(\sigma)\varphi_0'(\sigma); \quad P_0(\sigma) = P[\omega(\sigma)];$$
$$N_0(\sigma) = f_1^0(\sigma) - \operatorname{Re}[e^{i\nu_0(\sigma)}(\overline{\varphi_0(\sigma)} + (\overline{\omega(\sigma)} - P_0(\sigma))\varphi_0'(\sigma))], \ \sigma \in l.$$

The use of the polynomial $P(z)$ makes bounded the right-hand part of the boundary condition (21). The solution of the problem (22) can be constructed in an analogous manner as above (see problem (9)), while the solvability condition (with the assumption that the function $R_0(\zeta)$ is continuous up to the boundary) will be analogous to conditions (12) and (16). All these conditions will be represented as a non-homogeneous system with real coefficients with respect to unknown real constants. It is proved that the obtained system is uniquely solvable and therefore the problem posed has a unique solution.

Acknowledgment

The designated project has been fulfilled by financial support of the Shota Rustaveli National Science Foundation (Grant SRNSF/FR /358/5-109/14).

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