DERIVATION OF SYSTEM OF THE EQUATIONS OF EQUILIBRIUM FOR SHALLOW SHELLS AND PLATES, HAVING DOUBLE POROSITY

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Abstract

We consider the three-dimensional system of the equations of elastic static equilibrium of bodies with double porosity. From this system of the equations, using a method of a reduction of I. Vekua, we receive the equilibrium equations for the shallow shells having double porosity. Further we consider a case of plates of constant thickness in more detail. Namely, the system of the equations corresponding to approximations $N = 1$ it is written down in a complex form and we express the general solution of these systems through analytic functions of complex variable and solutions of the Helmholtz equation. The received general representations of decisions give the opportunity to analytically solve boundary value problems about elastic equilibrium of plates with double porosity.

Key words and phrases: Double porosity, Shallow shells, General solution, Boundary value problems.

AMS subject classification: 74K25, 74B20, 74F10, 74G05.

1 Introduction

A model of elastic equilibrium of porous media with double porosity (Fig. 1) was constructed in [1-3]. The theory justified in these papers combines the previously proposed model of Barenblatt for media with double porosity [4] and that of Biot for media with ordinary porosity [5]. For a detailed account of the development of the theory of porous media and relevant references see [6].Various issues related to the elastic equilibrium of bodies with double porosities are treated in [7-15].

In the works mentioned above three-dimensional or two-dimensional problems of a double poroelasticity were considered. But we do not know of any works where problems of double porous elasticity would have been considered for shells or plates.For this reason, we considered the elastic equilibrium of porous shallow shells.

2 Basic three-dimensional relations

Let an elastic body with double porosity occupy the domain $\overline{\Omega} \subset R^3$. Denote by (x^1, x^2, x^3) a point of the domain $\overline{\Omega}$ in the arbitrary curvilinear system of coordinates. Let the domain $\overline{\Omega}$ be filled with an elastic isotropic homogenous medium having double porosity. The considered solid body is characterized by the displacement vector $\mathbf{u} = (u^1, u^2, u^3)$, and also by the fluid pressures $p_1(x^1, x^2, x^3)$ and $p_2(x^1, x^2, x^3)$ occurring respectively in the pores and fissures of the porous medium.

Then a homogeneous system of static equilibrium equations is written in the form [13]

$$\overset{\circ}{\nabla}_i \sigma^{ij} = 0, \tag{1}$$

where $\overset{\circ}{\nabla}_i$ are symbols of a spatial covariant derivative; σ^{ij} are contravariant components of stress tensor; the summation over the recurring index i is assumed to be made from 1 to 3.

Formulas that interrelate the stress components, the displacement vector components and the pressures p_1, p_2 have the form [13]

$$\sigma^{ij} = (\lambda di\nu \mathbf{u} - \beta_1 p_1 - \beta_2 p_2)\mathrm{g}^{ij} + 2\mu\varepsilon^{ij}, \tag{2}$$

where λ and μ are the Lamé parameters; β_1 and β_2 are the effective stress parameters; g^{ij} are the contravariant components of the spatial metric tensor; ε^{ij} are contravariant components of the deformation tensor

$$\varepsilon^{ij} = 0.5(\overset{\circ}{\nabla}{}^i u^j + \overset{\circ}{\nabla}{}^j u^i);$$

$\overset{\circ}{\nabla}_i = \mathrm{g}^{ij}\overset{\circ}{\nabla}_j$; the contravariant and covariant components of a vector of displacement are connected by a relation $u^i = \mathrm{g}^{ij} u_j$.

In the stationary case, the values p_1 and p_2 satisfy the following system of equations [13]

$$\begin{cases} (k_1 \overset{\circ}{\nabla}{}^2 - \gamma)p_1 + (k_{12}\overset{\circ}{\nabla}{}^2 + \gamma)p_2 = 0, \\ (k_{21}\overset{\circ}{\nabla}{}^2 + \gamma)p_1 + (k_2 \overset{\circ}{\nabla}{}^2 - \gamma)p_2 = 0 \end{cases} \quad \text{in} \quad \Omega, \tag{3}$$

where $k_1 = \frac{\kappa_1}{\mu'}, k_2 = \frac{\kappa_2}{\mu'}, k_{12} = \frac{\kappa_{12}}{\mu'}, k_{21} = \frac{\kappa_{21}}{\mu'}$; μ' is fluid viscosity; κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity; k_{12} and k_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases; $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures; is the $\overset{\circ}{\nabla}{}^2 = \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}{}^i$ three-dimensional Laplace operator.

It is easy to show that if $\gamma > 0,\ k_1k_2 - k_{12}k_{21} > 0$, then the system of equations (3) is equivalent to two independent equations: to the Laplace equation [16]

$$\nabla^2 \widetilde{p}_1 = 0 \quad in \quad \Omega \tag{4}$$

and to the Helmholtz equation

$$\nabla^2 \widetilde{p}_2 - \zeta^2 \widetilde{p}_2 = 0 \quad in \quad \Omega, \tag{5}$$

where

$$\widetilde{p}_1 := (k_1 + k_{21})p_1 + (k_2 + k_{12})p_2, \quad \widetilde{p}_2 := p_1 - p_2,$$

$$\zeta^2 := \frac{\gamma(k_1 + k_2 + k_{12} + k_{21})}{k_1k_2 - k_{12}k_{21}} > 0.$$

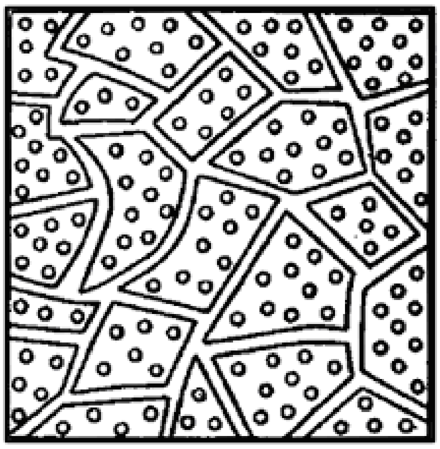

Fig. 1. Material with double porosity

3 Reduction of three-dimensional relations (1)-(5)

Let Ω represent a shell with constant thickness $2h$, symmetric concerning the middle surface ω. ω is smooth bilateral surface. We will denote the set of side surfaces of a shell through Γ. Surfaces of ω and Γ in each point are crossed at the right angle. We assume that thickness $2h$ is much smaller in comparison with other sizes of a shell.

We will consider the coordinate system which is normal connected with a middle of surface. In this system the radius vector $\mathbf{R}$ of any point M of domain Ω is expressed by means of a formula (see fig. 2)

$$\mathbf{R} = \mathbf{r}(x^1, x^2) + x^3 \mathbf{n}(x^1, x^2),$$

where x^1, x^2 are Gaussian parameters of the surface ω; $\mathbf{r}$ and $\mathbf{n}$ are radius vector and normal of the point $(x^1, x^2) \in \omega$. x^3 is the relative length from the point M to the surface ω.

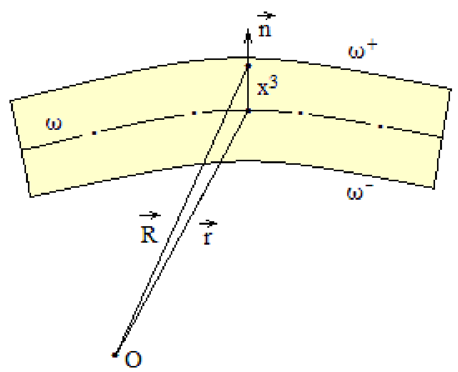

Fig. 2. The considered shell of constant thickness

We will start at first a reduction of equations (4) and (5). For this purpose we will write down these equations in a form

$$\frac{1}{\sqrt{\mathrm{g}}}\partial_\alpha(\sqrt{\mathrm{g}}\mathrm{g}^{\alpha\beta}\partial_\beta\widetilde{\mathrm{p}}_1) + \frac{1}{\sqrt{\mathrm{g}}}\partial_3(\sqrt{\mathrm{g}}\partial_3\widetilde{\mathrm{p}}_1) = 0, \tag{4'}$$

$$\frac{1}{\sqrt{\mathrm{g}}}\partial_\alpha(\sqrt{\mathrm{g}}\mathrm{g}^{\alpha\beta}\partial_\beta\widetilde{\mathrm{p}}_2) + \frac{1}{\sqrt{\mathrm{g}}}\partial_3(\sqrt{\mathrm{g}}\partial_3\widetilde{\mathrm{p}}_2) - \zeta^2\widetilde{\mathrm{p}}_2 = 0, \tag{5'}$$

where g are discriminant of the appropriate metric square form; $\partial_j :\equiv \frac{\partial}{\partial x^j}$, $j = 1, 2, 3$; the summation over the recurring Greek index is assumed to be made 1 to 2.

We apply I. Vekua's method [17] to a reduction of equations (4′) and (5′). We accept the following assumptions of geometrical character

$$1 - k_1 x^3 \cong 1,\ 1 - k_2 x^3 \cong 1,\ x^3\partial_\alpha k_1 \cong 0,\ x^3\partial_\alpha k_2 \cong 0,\ -h \le x^3 \le h. \tag{6}$$

These requirements mean that main curvature k_1 and k_2 of a surface are small (shallow shell), or thickness of shell is small (thin shell). From assumptions (6) it follows that the spatial covariant $(\mathbf{R}_a)$ and contravariant

$(\mathbf{R}^a)$ basis vectors are approximately equal to the corresponding basis vectors of a middle surface $(\mathbf{r}_a)$, $(\mathbf{r}^a)$. Therefore, corresponding covariant and contravariant components and discriminants of metric tensors of space and a middle surface are also approximately equal

$$\begin{aligned}&\mathbf{R}_a \cong \mathbf{r}_\alpha,\ \mathbf{R}^a \cong \mathbf{r}^a,\ \mathbf{R}^3 = \mathbf{r}_3 = \mathbf{n},\ \mathrm{g}_{\alpha\beta} \cong a_{\alpha\beta},\\ &\mathrm{g}^{\alpha\beta} \cong a^{\alpha\beta},\ \mathrm{g} \cong a,\ \partial_3\sqrt{\mathrm{g}} \cong -2H\sqrt{a},\end{aligned} \tag{7}$$

where $a_{\alpha\beta} = \mathbf{r}_\alpha \mathbf{r}_\beta$, $a^{\alpha\beta} = \mathbf{r}^\alpha \mathbf{r}^\beta$; a is the discriminant of quadratic form of the surface ω; $H = \frac{1}{2}(k_1+k_2) = \frac{1}{2}b_\alpha^\alpha$ are middle curvatures of the midsurface ω; $b_{\alpha\beta}, b_\alpha^\beta$, are covariant and mixed components of the tensor of curvature of the midsurface ω.

Taking into account the formula (6) and (7), equation (4') and (5') take the form

$$\nabla^2\widetilde{p}_\alpha + \partial_{33}\widetilde{p}_\alpha - 2H\partial_3\widetilde{p}_\alpha - \delta_{\alpha 2}\zeta^2\widetilde{p}_\alpha = 0, \quad \alpha = 1, 2, \tag{8}$$

where $\nabla^2 = \frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}\nabla^\alpha) \equiv \nabla_\alpha\nabla^\alpha$; ∇_α and ∇^α are symbols of a covariant and contravariant derivatives on the midsurface ω; $\delta_{\alpha 2}$ is the Kronecker delta.

On the faces of the shell we set one of the following boundary conditions $(a = 1, 2)$

$$\begin{aligned}&1)\quad \widetilde{p}_\alpha(x^1,x^2,h) = f_\alpha^+(x^1,x^2), \quad \widetilde{p}_\alpha(x^1,x^2,-h) = f_\alpha^-(x^1,x^2);\\ &2)\quad \partial_3\widetilde{p}_\alpha|_{x^3=h} = -q_\alpha^+(x^1,x^2), \quad \partial_3\widetilde{p}_\alpha|_{x^3=-h} = q_\alpha^-(x^1,x^2);\\ &3)\quad (\partial_3\widetilde{p}_\alpha + \kappa_\alpha^+\widetilde{p}_\alpha)|_{x^3=h} = l_\alpha^+(x^1,x^2),\\ &\qquad (\partial_3\widetilde{p}_\alpha - \kappa_\alpha^-\widetilde{p}_\alpha)|_{x^3=-h} = -l_\alpha^-(x^1,x^2)\\ &\qquad \text{(no summation)},\end{aligned} \tag{9}$$

where $f_\alpha^\pm, q_\alpha^\pm, l_\alpha^\pm$ are known functions; $\kappa_\alpha^\pm$ are known constants.

It is assumed that $\widetilde{p}_1$ and $\widetilde{p}_2$ are rather smooth functions $\widetilde{p}_\alpha \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ and they are presented as a uniformly convergent series at Legendre's polynomials

$$\widetilde{p}_\alpha(x^1,x^2,x^3) = \sum_{k=0}^{\infty} \overset{(k)}{\widetilde{p}}_\alpha(x^1,x^2) P_k\left(\frac{x^3}{h}\right), \tag{10}$$

where $P_k\left(\frac{x^3}{h}\right)$ are the Legendre polynomials of order k;

$$\overset{(k)}{\widetilde{p}}_\alpha(x^1,x^2) = \left(k+\frac{1}{2}\right)\frac{1}{h}\int_{-h}^{h} \widetilde{p}_\alpha(x^1,x^2,x^3) P_k\left(\frac{x^3}{h}\right) dx^3.$$

From (10)

$$\partial_3 \widetilde{p}_\alpha = \sum_{k=0}^{\infty} \frac{2k+1}{h} \left(\overset{(k+1)}{\widetilde{p}_\alpha} + \overset{(k+3)}{\widetilde{p}_\alpha} + \cdots \right) P_k \left(\frac{x^3}{h} \right). \tag{11}$$

If we differentiate both parts of equality (11), we will obtain

$$\partial_{33} \widetilde{p}_\alpha = \sum_{k=0}^{\infty} \frac{2k+1}{h^2} \sum_{m=1}^{\infty} m(2k+2m+1) \overset{(k+2m)}{\widetilde{p}_\alpha} P_k \left(\frac{x^3}{h} \right). \tag{12}$$

If we substitute relation (10)-(12) into equation (8) and take into account that the system of Legendre polynomials is full on the segment $[-1;1]$ we obtain an infinite system of equations

$$\begin{aligned} \nabla^2 \overset{(k)}{\widetilde{p}_\alpha} + \frac{2k+1}{h^2} \sum_{m=1}^{\infty} \left[m(2k+2m+1) \overset{(k+2m)}{\widetilde{p}_\alpha} - 2hH \overset{(k+2m-1)}{\widetilde{p}_\alpha} \right] \\ -\delta_{\alpha 2} \zeta^2 \overset{(k)}{\widetilde{p}_\alpha} = 0, \qquad k = 0, 1, 2, \dots \end{aligned} \tag{13}$$

We will rewrite equations (13) as follows

$$\begin{aligned} a)\ \nabla^2 \overset{(k)}{\widetilde{p}_\alpha} + \frac{2k+1}{h^2} \Bigg[(2k+3) \left(\overset{(k+2)}{\widetilde{p}_\alpha} + \overset{(k+4)}{\widetilde{p}_\alpha} + \cdots \right) + (2k+7) \\ \times \left(\overset{(k+4)}{\widetilde{p}_\alpha} + \overset{(k+6)}{\widetilde{p}_\alpha} + \cdots \right) + \cdots - 2hH \left(\overset{(k+1)}{\widetilde{p}_\alpha} + \overset{(k+3)}{\widetilde{p}_\alpha} + \cdots \right) \Bigg] \\ -\delta_{\alpha 2} \zeta^2 \overset{(k)}{\widetilde{p}_\alpha} = 0, \qquad k = 0, 1, 2, \dots \end{aligned} \tag{14}$$

$$b)\ \nabla^2 \overset{(k)}{\widetilde{p}_\alpha} + \frac{2k+1}{h^2}\Bigg[\frac{2k+3}{(k+2)(k+3)}\Big((k+2)(k+3)\overset{(k+2)}{\widetilde{p}_\alpha}$$

$$+(k+4)(k+5)\overset{(k+4)}{\widetilde{p}_\alpha}+\cdots\Big)+\left(\frac{2(2k+5)}{(k+4)(k+5)}-\frac{2k+3}{(k+2)(k+3)}\right)$$

$$\times\Big((k+4)(k+5)\overset{(k+4)}{\widetilde{p}_\alpha}+(k+6)(k+7)\overset{(k+6)}{\widetilde{p}_\alpha}+\cdots\Big)$$

$$+\cdots-2hH\left(\frac{1}{(k+1)(k+2)}\right. \tag{15}$$

$$\times\left((k+1)(k+2)\overset{(k+1)}{\widetilde{p}_\alpha}+(k+3)(k+4)\overset{(k+3)}{\widetilde{p}_\alpha}+\cdots\right)$$

$$+\left(\frac{1}{(k+3)(k+4)}-\frac{1}{(k+1)(k+2)}\right)\Big((k+3)(k+4)\overset{(k+3)}{\widetilde{p}_\alpha}$$

$$\left.+(k+5)(k+6)\overset{(k+5)}{\widetilde{p}_\alpha}+\cdots\Big)+\cdots\right)\Bigg]-\delta_{\alpha 2}\zeta^2\overset{(k)}{\widetilde{p}_\alpha}=0,\quad k=0,1,2,\ldots$$

$$c)\ \nabla^2 \overset{(k)}{\widetilde{p}_\alpha} + \frac{2k+1}{h^2}\Bigg[(k+2)(k+3)\overset{(k+2)}{\widetilde{p}_\alpha}+(k+4)(k+5)\overset{(k+4)}{\widetilde{p}_\alpha}+\cdots$$

$$-k(k+1)\left(\overset{(k+2)}{\widetilde{p}_\alpha}+\overset{(k+4)}{\widetilde{p}_\alpha}+\cdots\right)-4hH\left(\overset{(k+1)}{\widetilde{p}_\alpha}+\overset{(k+3)}{\widetilde{p}_\alpha}+\cdots\right)\Bigg] \tag{16}$$

$$-\delta_{\alpha 2}\zeta^2\overset{(k)}{\widetilde{p}_\alpha}=0,\quad k=0,1,2,\ldots$$

Given the properties of Legendre polynomials

$$\begin{aligned}&\overset{(2m)}{\widetilde{p}_\alpha}+\overset{(k+2m+2)}{\widetilde{p}_\alpha}+\cdots=\frac{1}{2}(f_\alpha^+ + (-1)^k f_\alpha^-)\\&-\left(\overset{(k+2m-2)}{\widetilde{p}_\alpha}+\overset{(k+2m-4)}{\widetilde{p}_\alpha}+\cdots\right),\quad m=1,2,\ldots;\end{aligned} \tag{17}$$

$$\overset{(k+1)}{\widetilde{p}_\alpha}+\overset{(k+3)}{\widetilde{p}_\alpha}+\cdots=\frac{1}{2}(f_\alpha^+ - (-1)^k f_\alpha^-)-\left(\overset{(k-1)}{\widetilde{p}_\alpha}+\overset{(k-3)}{\widetilde{p}_\alpha}+\cdots\right); \tag{18}$$

$$
\begin{aligned}
(\partial_3 \widetilde{p}_\alpha)|_{x^3=h} &= \frac{1}{2h}\sum_{k=1}^{\infty} k(k+1)\overset{(k)}{\widetilde{p}}_\alpha,\\
(\partial_3 \widetilde{p}_\alpha)|_{x^3=-h} &= \frac{1}{2h}\sum_{k=1}^{\infty} (-1)^{k+1}k(k+1)\overset{(k)}{\widetilde{p}}_\alpha .
\end{aligned}
\tag{19}
$$

From relations (19) we obtain

$$
\begin{aligned}
&(k+2m)(k+2m+1)\overset{(k+2m)}{\widetilde{p}_\alpha} +(k+2m+2)(k+2m+3)\overset{(k+2m+2)}{\widetilde{p}_\alpha}\\
&+\cdots = -h(q^+_\alpha + (-1)^k q^-_\alpha) - [(k+2m-2)(k+2m-1)\overset{(k+2m-2)}{\widetilde{p}_\alpha}\\
&+(k+2m-4)(k+2m-3)\overset{(k+2m-4)}{\widetilde{p}_\alpha} +\cdots];
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&(k+2m-1)(k+2m)\overset{(k+2m-1)}{\widetilde{p}_\alpha} +(k+2m+1)(k+2m+2)\\
&\times \overset{(k+2m+1)}{\widetilde{p}_\alpha} +\cdots = -h(q^+_\alpha - (-1)^k q^-_\alpha)\\
&-[(k+2m-3)(k+2m-2)\overset{(k+2m-3)}{\widetilde{p}_\alpha}\\
&+(k+2m-5)(k+2m-4)\overset{(k+2m-5)}{\widetilde{p}_\alpha} +\cdots];
\end{aligned}
\tag{21}
$$

By substituting formulas (17) and (18) in equations (14), we obtain

$$
\begin{aligned}
&a)\nabla^2 \overset{(k)}{\widetilde{p}}_\alpha + \frac{2k+1}{h^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big[(2k+4m+3)\\
&\times\left(\frac{1}{2}(f^+_\alpha + (-1)^k f^-_\alpha) - \overset{(k+2m-2n)}{\widetilde{p}_\alpha}\right)\\
&-2hH\frac{1}{2}(f^+_\alpha - (-1)^k f^-_\alpha) - \overset{(k-2n-1)}{\widetilde{p}_\alpha}\Bigg)\Bigg]\\
&-\delta_{\alpha 2}\zeta^2 \overset{(k)}{\widetilde{p}}_\alpha = 0, \quad k=0,1,2,\cdots;
\end{aligned}
\tag{22}
$$

By substituting formulas (20) and (21) in equations (15), we obtain

$$
\begin{aligned}
& b)\ \nabla^2 \overset{(k)}{\widetilde{p}_\alpha} + \frac{2k+1}{h^2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \\
& \times \left[\left(\frac{m(2k+2m+1)}{(k+2m)(k+2m+1)} - \frac{(m-1)(2k+2m-1)}{(k+2m-2+\delta_{k0})(k+2m-1)} \right) \right. \\
& \times \left(-h(q^{+}(-1)^k q^{-})(k+2m-2n)(k+2m-2n+1) \overset{(k+2m-2n)}{\widetilde{p}_\alpha} \right) \qquad (23) \\
& -2hH \left(\frac{1}{(k+2m-1)(k+2m)} - \frac{\delta^*_{m1}}{(k+2m-3)(k+2m-2)} \right) \\
& \times \left. \left(-h(q^{+}-(-1)^k q^{-})(k+2m-2n-1)(k+2m-2n) \overset{(k+2m-2n-1)}{\widetilde{p}_\alpha} \right) \right] \\
& -\delta_{\alpha 2}\zeta^2 \overset{(k)}{\widetilde{p}_\alpha} = 0, \quad k = 0,1,2,\cdots,
\end{aligned}
$$

where

$$
\delta^*_{k0} = \begin{cases} 0, & k \neq 0, \\ 1, & k = 0, \end{cases} \qquad \delta^*_{m1} = \begin{cases} 0, & m = 1, \\ 1, & m \neq 1. \end{cases}
$$

If in equations (16) we substitute formulas (20) and (18) and take into account boundary conditions 3), we obtain the following equations

$$
\begin{aligned}
& c)\ \nabla^2 \overset{(k)}{\widetilde{p}_\alpha} - \tfrac{2k+1}{h^2} \left[h(k^{+}_\alpha \widetilde{p}^{+}_\alpha + (-1)^k \kappa^{-}_\alpha \widetilde{p}^{-}_\alpha) + (k-2)(k-1) \right. \\
& \times \overset{(k-2)}{\widetilde{p}_\alpha} + (k-4)(k-2) \overset{(k-4)}{\widetilde{p}_\alpha} + \cdots k(k+1) \left(\tfrac{1}{2}(\widetilde{p}^{+}_\alpha + (-1)^k \widetilde{p}^{-}_\alpha) \right. \\
& - \left. \left(\overset{(k-2)}{\widetilde{p}_\alpha} + \overset{(k-4)}{\widetilde{p}_\alpha} + \cdots \right) \right) + 4hH \left(\tfrac{1}{2}(\widetilde{p}^{+}_\alpha - (-1)^k \widetilde{p}^{-}_\alpha) \right. \qquad (24) \\
& - \left. \left. \left(\overset{(k-1)}{\widetilde{p}_\alpha} + \overset{(k-3)}{\widetilde{p}_\alpha} + \cdots \right) \right) \right] - \delta_{a2}\zeta^2 \overset{(k)}{\widetilde{p}_\alpha} \\
& = -\tfrac{2k+1}{h^2} (\kappa^{+}_\alpha l^{+}_\alpha + (-1)^k \kappa^{-}_\alpha l^{-}_\alpha),
\end{aligned}
$$

where

$$
\widetilde{p}^{+}_\alpha = \widetilde{p}_\alpha(x^1, x^2, h), \quad \widetilde{p}^{-}_\alpha = \widetilde{p}_\alpha(x^1, x^2, -h).
$$

Equations (24) include unknown values $\widetilde{p}^{+}_\alpha$ and $\widetilde{p}^{+}_\alpha$ whom we will exclude when we pass to finite approximations.

Thus, we obtained three infinite systems of equations (22), (23) and (24) which correspond to the boundary conditions (9, 1) (9, 2) and (9, 3) (set on surfaces $x^3 = \pm h$). In all these formulas it is meant that

$$\overset{(k)}{\widetilde{p}_\alpha} = 0, \text{ when } k > 0.$$

If we take now

$$\widetilde{p}_\alpha(x^1, x^2, x^3) \approx \sum_{k=0}^{\infty} \overset{(k)}{\widetilde{p}_\alpha}(x^1, x^2) P_k\left(\frac{x^3}{h}\right),$$

where N is some non-negative integer. For $\overset{(0)}{\widetilde{p}_\alpha}, \overset{(1)}{\widetilde{p}_\alpha}, \cdots, \overset{(k)}{\widetilde{p}_\alpha}$ on the boundary of the domain ω one of the following boundary conditions can be set

$$1)\ \overset{(k)}{\widetilde{p}_\alpha} = \overset{(k)}{\widetilde{f}_\alpha}(M), \quad k = 0, 1, \cdots\ M \in \partial\omega;$$

$$2)\ \partial_\nu \overset{(k)}{\widetilde{p}_\alpha} = \overset{(k)}{\mathrm{g}_\alpha}(M), \quad k = 0, 1, \cdots\ M \in \partial\omega;$$

$$3)\ \partial_\nu \overset{(k)}{\widetilde{p}_\alpha} + k \overset{(k)}{\widetilde{p}_\alpha} = \overset{(k)}{h_\alpha}(M), \quad k = 0, 1, \cdots\ M \in \partial\omega;$$

where ν is an external normal of a contour.

Further we carry out a reduction of system of the equations (1), (2) using the method of Vekua. At the same time we almost repeat verbatim reasonings in the monograph [16]. Therefore we will present the reduced two-dimensional equations at once without details of derivation

$$\begin{cases} \nabla_\alpha \overset{(k)}{\sigma^{\alpha\beta}} - b_\alpha^\beta \overset{(k)}{\sigma^{\alpha 3}} - \dfrac{2k+1}{h}\left(\overset{(k-1)}{\sigma^{3\beta}} + \overset{(k-3)}{\sigma^{3\beta}} + \cdots\right) \\ \qquad\qquad -2H\overset{(k)}{\sigma^{3\beta}} = \overset{(k)}{F^{(\beta)}}, \quad \beta = 1, 2, \\ \nabla_\alpha \overset{(k)}{\sigma^{\alpha 3}} + b_{\alpha\beta}^\beta \overset{(k)}{\sigma^{\alpha\beta}} - \dfrac{2k+1}{h}\left(\overset{(k-1)}{\sigma^{33}} + \overset{(k-3)}{\sigma^{33}} + \cdots\right) \\ \qquad\qquad -2H\overset{(k)}{\sigma^{33}} = \overset{(k)}{F^{(3)}}, \end{cases} \tag{25}$$

where $k = 0, 1, \ldots$; $\overset{(k)}{\sigma^{ij}} = \left(k + \frac{1}{2}\right) \frac{1}{h} \int\limits_{-h}^{h} \sigma^{ij}(x^1, x^2, x^3) P_k\left(\frac{x^3}{h}\right) dx^3$; $\overset{(-k)}{\sigma^{ij}} = 0$, when $k > 0$;

$$\overset{(k)}{F_j} = \frac{2k+1}{2h}(\sigma_{3j}(x^1, x^2, h) - (-1)^k \sigma_{3j}(x^1, x^2, -h));$$

$$\overset{(k)}{\sigma}_{\alpha\beta}=\left(\lambda\overset{(k)}{\theta}-\beta_1^*\overset{(k)}{p_1}-\beta_2^*\overset{(k)}{p_2}\right)a_{\alpha\beta}+2\mu\overset{(k)}{e}_{\alpha\beta},$$

$$\overset{(k)}{\sigma}_{\alpha 3}=2\mu\overset{(k)}{e}_{\alpha 3}, \tag{26}$$

$$\overset{(k)}{\sigma}_{33}=\lambda\overset{(k)}{\theta}-\beta_1^*\overset{(k)}{p_1}-\beta_2^*\overset{(k)}{p_2}+2\mu\overset{(k)}{e}_{33},$$

where $\overset{(k)}{\sigma}_{ij}=a_{ik}a_{jl}\overset{(k)}{\sigma}{}^{kl}$; $\overset{(k)}{e}_{ij}=\left(k+\frac{1}{2}\right)\frac{1}{h}\int\limits_{-h}^{h}\varepsilon_{ij}(x^1,x^2,x^3)P_k\left(\frac{x^3}{h}\right)dx^3$;

$$\beta_1^*=\frac{\beta_1+\beta_2}{k_0},\quad \beta_2^*=\frac{\beta_1(k_2+k_{12})-\beta_2(k_1+k_{21})}{k_0},\quad k_0=k_1+k_2+k_{12}+k_{21}.$$

$$\overset{(k)}{e}_{\alpha\beta}=\frac{1}{2}\left(\nabla_\alpha\overset{(k)}{u_\beta}+\nabla_\beta\overset{(k)}{u_\alpha}-2b_{\alpha\beta}\overset{(k)}{u_3}\right),$$

$$\overset{(k)}{e}_{\alpha 3}=\frac{1}{2}\left(\nabla_\alpha\overset{(k)}{u_3}+b_{\alpha\beta}\overset{(k)}{u}{}^{\beta}+\frac{1}{h}\overset{(k)}{u'_\alpha}\right), \tag{27}$$

$$\overset{(k)}{e}_{33}=\frac{1}{h}\overset{(k)}{u'_3},$$

$$\overset{(k)}{\theta}=\overset{(k)}{e}{}^{i}_{i}=\nabla_\alpha\overset{(k)}{u}{}^{\alpha}-2H\overset{(k)}{u'_3},$$

where

$$\overset{(k)}{u_i}=\left(k+\frac{1}{2}\right)\frac{1}{h}\int\limits_{-h}^{h}u_i(x^1,x^2,x^3)P_k\left(\frac{x^3}{h}\right)dx^3;$$

$$\overset{(k)}{u'_j}=(2k+1)\left(\overset{(k+1)}{u_j}+\overset{(k+3)}{u_j}+\cdots\right).$$

Substituting relations (27) in formulas (26) and entering the obtained expressions into system (25), we obtain the system of poroelastic equilibrium of shallow shells with respect to the components of the displacement vector

$$\left\{\begin{aligned}&\mu\nabla_\alpha\left(\nabla^\alpha\overset{(k)}{u}{}^{\beta}\right)+\mu\nabla_\alpha\left(\nabla^\beta\overset{(k)}{u}{}^{\alpha}\right)+\lambda\nabla^\beta\left(\nabla_\alpha\overset{(k)}{u}{}^{\alpha}\right)+\overset{(k)}{M}{}^{\beta}\\&\qquad-\nabla_\beta\left(\beta_1^*\overset{(k)}{\widetilde{p}_1}+\beta_2^*\overset{(k)}{\widetilde{p}_2}\right)=\overset{(k)}{F}{}^{\beta},\quad\beta=1,2,\\&\mu\nabla_\alpha\left(\nabla^\alpha\overset{(k)}{u_3}\right)+\overset{(k)}{M_3}+\frac{2k+1}{h}\left(\beta_1^*\left(\overset{(k-1)}{\widetilde{p}_1}+\overset{(k-3)}{\widetilde{p}_1}+\cdots\right)\right.\\&\qquad\left.+\beta_2^*\left(\overset{(k-1)}{\widetilde{p}_2}+\overset{(k-3)}{\widetilde{p}_2}+\cdots\right)\right)=\overset{(k)}{F}{}^{3},\quad k=0,1,\ldots,\end{aligned}\right. \tag{28}$$

where $\overset{(k)}{M^j}$ are homogeneous linear differential expressions that contain functions $\overset{(k)}{U_j}$ and first order derivatives with the variable x^1, x^2.

In order to obtain the finite system of equations we accept the assumption

$$\mathbf{u}(x^1, x^2, x^3) = \sum_{k=0}^{N} \overset{(k)}{\mathbf{u}}(x^1, x^2) P_k\left(\frac{x^3}{h}\right).$$

Thus, in all expressions(25)-(28) received above we will assume that $\overset{(k)}{\mathbf{u}} = \mathbf{0}$, when $k > N$, we obtain the system of second-order partial differential equations. It contains $3N + 3$ unknowns $\overset{(k)}{u^i}$ $(k = 0, 1, ..., N;\ i = 1, 2, 3)$, and its order is equal to $6N + 6$.

4 Approximation $N = 1$

a) In equations (22), (23) and (24) we need $\overset{(0)}{\widetilde{p}_\alpha}$ and $\overset{(1)}{\widetilde{p}_\alpha}$. At the same time $\overset{(k)}{\widetilde{p}_\alpha} = 0$, when $k > 3$. The system of equations (22) in this case takes the form

$$\begin{cases} \nabla^2 \overset{(0)}{\widetilde{p}_\alpha} - \left(\dfrac{3}{h^2} + \sigma_{\alpha 2}\zeta^2\right) \overset{(0)}{\widetilde{p}_\alpha} = -\dfrac{3}{2h^2}(f_\alpha^+ + f_\alpha^-) + \dfrac{H}{h}(f_\alpha^+ + f_\alpha^-), \\ \nabla^2 \overset{(1)}{\widetilde{p}_\alpha} - \left(\dfrac{15}{h^2} + \sigma_{\alpha 2}\zeta^2\right) \overset{(1)}{\widetilde{p}_\alpha} + \dfrac{6H}{h} \overset{(0)}{\widetilde{p}_\alpha} \\ = -\dfrac{15}{2h^2}(f_\alpha^+ - f_\alpha^-) + \dfrac{3H}{h}(f_\alpha^+ + f_\alpha^-). \end{cases} \tag{29}$$

In this case we restore functions $\overset{(0)}{\widetilde{p}_\alpha}$ as follows

$$\widetilde{p}_\alpha = \sum_{k=0}^{3} \overset{(k)}{\widetilde{p}_\alpha}(x^1, x^2) P_k\left(\frac{x^3}{h}\right),$$

where

$$\overset{(2)}{\widetilde{p}_\alpha} = \frac{1}{2}(f_\alpha^+ + f_\alpha^-) - \overset{(0)}{\widetilde{p}_\alpha},\ \ \overset{(3)}{\widetilde{p}_\alpha} = \frac{1}{2}(f_\alpha^+ - f_\alpha^-) - \overset{(1)}{\widetilde{p}_\alpha}.$$

In case of a plate of constant thickness $H = 0$ and the system of the equations (29) breaks up to the independent equations for $\overset{(0)}{\widetilde{p}_\alpha}$ and $\overset{(1)}{\widetilde{p}_\alpha}$

$$\Delta \overset{(0)}{\widetilde{p}_\alpha} - \left(\frac{3}{h^2} + \delta_{\alpha 2}\zeta^2\right) \overset{(0)}{\widetilde{p}_\alpha} = -\frac{3}{2h^2}(f_\alpha^+ + f_\alpha^-), \tag{30}$$

$$\Delta \overset{(1)}{\widetilde{p}_\alpha} - \left(\frac{15}{h^2} + \delta_{\alpha 2}\zeta^2\right) \overset{(1)}{\widetilde{p}_\alpha} = -\frac{15}{2h^2}(f_\alpha^+ - f_\alpha^-), \tag{31}$$

b) In the case when on the face of the shell the boundary conditions (9, 2)) is given we obtain the following system of equations

$$\begin{cases} \nabla^2 \overset{(0)}{\widetilde{p}_\alpha} - \frac{5H}{3h} \overset{(1)}{\widetilde{p}_\alpha} - \delta_{\alpha 2}\zeta^2 \overset{(0)}{\widetilde{p}_\alpha} = \frac{1}{2h}(q_\alpha^+ + q_\alpha^-) - \frac{H}{6}(q_\alpha^+ - q_\alpha^-), \\ \nabla^2 \overset{(1)}{\widetilde{p}_\alpha} - \left(\frac{5}{2h^2} + \delta_{\alpha 2}\zeta^2\right) \overset{(1)}{\widetilde{p}_\alpha} = \frac{15}{4h}(q_\alpha^+ - q_\alpha^-) - H(q_\alpha^+ + q_\alpha^-), \end{cases} \tag{32}$$

For $\overset{(0)}{\widetilde{p}_\alpha}$ we have the formula (30), where this time

$$\overset{(2)}{\widetilde{p}_\alpha} = -\frac{h}{6}(q_\alpha^+ + q_\alpha^-), \;\; \overset{(3)}{\widetilde{p}_\alpha} = -\frac{h}{12}(q_\alpha^+ - q_\alpha^-) - \frac{1}{6} \overset{(1)}{\widetilde{p}_\alpha}.$$

In case of a plate we will have the independent equations

$$\nabla \overset{(0)}{\widetilde{p}_\alpha} - \delta_{\alpha 2}\zeta^2 \overset{(0)}{\widetilde{p}_\alpha} = \frac{1}{2h}(q_\alpha^+ + q_\alpha^-), \tag{33}$$

$$\nabla \overset{(1)}{\widetilde{p}_\alpha} - \left(\frac{5}{2h^2} + \delta_{\alpha 2}\zeta^2\right) \overset{(1)}{\widetilde{p}_\alpha} = \frac{15}{4h}(q_\alpha^+ - q_\alpha^-) \tag{34}$$

c) If in the system of equations (24)

$$\widetilde{p_\alpha^+} = \overset{(0)}{\widetilde{p}_\alpha} + \overset{(1)}{\widetilde{p}_\alpha}, \;\; \widetilde{p_\alpha^-} = \overset{(0)}{\widetilde{p}_\alpha} - \overset{(1)}{\widetilde{p}_\alpha},$$

we obtain the following system of equations

$$\begin{cases} \nabla^2 \overset{(0)}{\widetilde{p}_\alpha} - \left(\frac{1}{h}(\kappa_\alpha^+ + \kappa_\alpha^-) + \delta_{\alpha 2}\zeta^2\right) \overset{(0)}{\widetilde{p}_\alpha} - \frac{1}{h}(\kappa_\alpha^+ - \kappa_\alpha^- + 2H) \overset{(1)}{\widetilde{p}_\alpha} \\ = \frac{1}{h^2}(\kappa_\alpha^+ l_\alpha^+ + \kappa_\alpha^- l_\alpha^-), \\ \nabla^2 \overset{(1)}{\widetilde{p}_\alpha} - \left(\frac{3}{h}(\kappa_\alpha^+ + \kappa_\alpha^-) + \frac{3}{2h^2} + \delta_{\alpha 2}\zeta^2\right) \overset{(1)}{\widetilde{p}_\alpha} \\ -\frac{3}{h}(\kappa_\alpha^+ - \kappa_\alpha^- + 2H) \overset{(0)}{\widetilde{p}_\alpha} = \frac{3}{h^2}(\kappa_\alpha^+ l_\alpha^+ - \kappa_\alpha^- l_\alpha^-). \end{cases} \tag{35}$$

In this case we restore functions $\widetilde{p}_\alpha$ as follows

$$\widetilde{p}_\alpha = \overset{(0)}{\widetilde{p}_\alpha} + \frac{x^3}{h} \overset{(1)}{\widetilde{p}_\alpha} .$$

In case of a plate the system of equations (35) takes the form

$$\begin{cases} \nabla^2 \overset{(0)}{\widetilde{p}_\alpha} - \left(\frac{1}{h}(\kappa_\alpha^+ + \kappa_\alpha^-) + \delta_{\alpha 2}\zeta^2 \right) \overset{(0)}{\widetilde{p}_\alpha} - \frac{1}{h}(\kappa_\alpha^+ - \kappa_\alpha^-) \overset{(1)}{\widetilde{p}_\alpha} \\ = \frac{1}{h^2}(\kappa_\alpha^+ l_\alpha^+ + \kappa_\alpha^- l_\alpha^-), \\ \nabla^2 \overset{(1)}{\widetilde{p}_\alpha} - \left(\frac{3}{h}(\kappa_\alpha^+ + \kappa_\alpha^-) + \frac{3}{2h^2} + \delta_{\alpha 2}\zeta^2 \right) \overset{(1)}{\widetilde{p}_\alpha} - \frac{3}{h}(\kappa_\alpha^+ - \kappa_\alpha^-) \overset{(0)}{\widetilde{p}_\alpha} \\ = \frac{3}{h^2}(\kappa_\alpha^+ l_\alpha^+ - \kappa_\alpha^- l_\alpha^-). \end{cases} \tag{36}$$

If in the system of equations (36) $\kappa_\alpha^+ = \kappa_\alpha^-$ then this system breaks up into independent equations

$$\nabla^2 \overset{(0)}{\widetilde{p}_\alpha} - \left(\frac{2\kappa_\alpha^+}{h} + \delta_{\alpha 2}\zeta^2 \right) \overset{(0)}{\widetilde{p}_\alpha} = \frac{1}{h^2}(\kappa_\alpha^+ l_\alpha^+ + \kappa_\alpha^- l_\alpha^-), \tag{37}$$

$$\nabla^2 \overset{(1)}{\widetilde{p}_\alpha} - \left(\frac{6\kappa_\alpha^+}{h} + \frac{3}{2h^2} + \delta_{\alpha 2}\zeta^2 \right) \overset{(1)}{\widetilde{p}_\alpha} = \frac{3}{h^2}(\kappa_\alpha^+ l_\alpha^+ + \kappa_\alpha^- l_\alpha^-), \tag{38}$$

In case of approximation $N = 1$ the displacement vector $\mathbf{u}$ has the form

$$\mathbf{u} = \overset{(0)}{\mathbf{u}} + \frac{x^3}{h} \overset{(1)}{\mathbf{u}} .$$

The system of equations (25) can be written as

$$\begin{cases} \nabla_\alpha \overset{(0)}{\sigma^{\alpha\beta}} - b_\alpha^\beta \overset{(0)}{\sigma^{\alpha 3}} - 2H \overset{(0)}{\sigma^{\alpha\beta}} = \overset{(0)}{F^\beta}, \quad \beta = 1, 2, \\ \nabla_\alpha \overset{(0)}{\sigma^{\alpha 3}} + b_{\alpha\beta} \overset{(0)}{\sigma^{\alpha\beta}} - 2H \overset{(0)}{\sigma^{33}} = \overset{(0)}{F^3}, \\ \nabla_\alpha \overset{(1)}{\sigma^{\alpha\beta}} - b_\alpha^\beta \overset{(1)}{\sigma^{\alpha 3}} - \frac{3}{h} \overset{(0)}{\sigma^{3\beta}} - 2H \overset{(1)}{\sigma^{3\beta}} = \overset{(1)}{F^\beta}, \quad \beta = 1, 2, \\ \nabla_\alpha \overset{(1)}{\sigma^{\alpha 3}} + b_{\alpha\beta} \overset{(1)}{\sigma^{\alpha\beta}} - \frac{3}{h} \overset{(0)}{\sigma^{33}} - 2H \overset{(1)}{\sigma^{33}} = \overset{(1)}{F^3}, \end{cases} \tag{39}$$

where

$$\overset{(0)}{\sigma}{}^{\alpha\beta} = \left(\lambda \overset{(0)}{\theta} - \beta_1^* \overset{(0)}{\widetilde{p}_1} - \beta_2^* \overset{(0)}{\widetilde{p}_2} \right) a^{\alpha\beta} + 2\mu \overset{(0)}{e}{}^{\alpha\beta},$$

$$\overset{(1)}{\sigma}{}^{\alpha\beta} = \left(\lambda \overset{(1)}{\theta} - \beta_1^* \overset{(1)}{\widetilde{p}_1} - \beta_2^* \overset{(1)}{\widetilde{p}_2}\right) a^{\alpha\beta} + 2\mu \overset{(1)}{e}{}^{\alpha\beta},$$

$$\overset{(0)}{\sigma}{}^{\alpha 3} = 2\mu \overset{(0)}{e}{}^{\alpha 3}, \quad \overset{(1)}{\sigma}{}^{\alpha 3} = 2\mu \overset{(1)}{e}{}^{\alpha\beta},$$

$$\overset{(0)}{\sigma}{}^{33} = \lambda \overset{(0)}{\theta} - \beta_1^* \overset{(0)}{\widetilde{p}_1} - \beta_2^* \overset{(0)}{\widetilde{p}_2} + 2\mu \overset{(0)}{e}{}^{33},$$

$$\overset{(1)}{\sigma}{}^{33} = \lambda \overset{(1)}{\theta} - \beta_1^* \overset{(1)}{\widetilde{p}_1} - \beta_2^* \overset{(1)}{\widetilde{p}_2};$$

$$\overset{(0)}{e}_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha \overset{(0)}{u}_\beta + \nabla_\beta \overset{(0)}{u}_\alpha - 2b_{\alpha\beta} \overset{(0)}{u}_3\right),$$

$$\overset{(1)}{e}_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha \overset{(1)}{u}_\beta + \nabla_\beta \overset{(1)}{u}_\alpha - 2b_{\alpha\beta} \overset{(1)}{u}_3\right),$$

$$\overset{(0)}{e}_{\alpha 3} = \frac{1}{2}\left(\nabla_\alpha \overset{(0)}{u}_3 + b_{\alpha\beta} \overset{(0)}{u}{}^\beta + \frac{1}{h} \overset{(1)}{u}_\alpha\right),$$

$$\overset{(1)}{e}_{\alpha 3} = \frac{1}{2}\left(\nabla_\alpha \overset{(1)}{u}_3 + b_{\alpha\beta} \overset{(1)}{u}{}^\beta\right), \quad \overset{(1)}{e}_{33} = \frac{1}{h} \overset{(1)}{u}_3;$$

$$\overset{(0)}{\theta} = \nabla_\gamma \overset{(0)}{u}{}^\gamma - 2H \overset{(0)}{u}_3 + \frac{1}{2} \overset{(1)}{u}_3, \quad \overset{(1)}{\theta} = \nabla_\gamma \overset{(1)}{u}{}^\gamma - 2H \overset{(1)}{u}_3.$$

In case of a plate of constant thickness $b_{\alpha\beta} = 0$, $H = 0$ the system of the equations (39) in the Cartesian system of coordinates will take a form

$$\begin{cases} \partial_\alpha \overset{(0)}{\sigma}_{\alpha\beta} = \overset{(0)}{F}_\beta, \quad \beta = 1,2, \\ \partial_\alpha \overset{(0)}{\sigma}_{\alpha 3} = \overset{(0)}{F}_3, \\ \partial_\alpha \overset{(1)}{\sigma}_{\alpha\beta} - \frac{3}{h} \overset{(0)}{\sigma}_{3\beta} = \overset{(0)}{F}_\beta, \quad \beta = 1,2, \\ \partial_\alpha \overset{(1)}{\sigma}_{\alpha 3} - \frac{3}{h} \overset{(0)}{\sigma}_{33} = \overset{(1)}{F}_3, \end{cases} \tag{40}$$

$$
\begin{aligned}
&\overset{(0)}{\sigma}_{\alpha\beta} = \left(\lambda\partial_\gamma \overset{(0)}{u}_\gamma + \frac{\lambda}{h}\overset{(1)}{u}_3 - \beta_1^* \overset{(0)}{\widetilde{p}_1} - \beta_2^* \overset{(0)}{\widetilde{p}_2}\right)\delta_{\alpha\beta} + \mu\left(\partial_\alpha \overset{(0)}{u}_\beta + \partial_\beta \overset{(0)}{u}_\alpha\right),\\
&\overset{(1)}{\sigma}_{\alpha\beta} = \left(\lambda\partial_\gamma \overset{(1)}{u}_\gamma - \beta_1^* \overset{(1)}{\widetilde{p}_1} - \beta_2^* \overset{(1)}{\widetilde{p}_2}\right)\delta_{\alpha\beta} + \mu\left(\partial_\alpha \overset{(1)}{u}_\beta + \partial_\beta \overset{(1)}{u}_\alpha\right),\\
&\overset{(0)}{\sigma}_{\alpha 3} = \mu\left(\partial_\alpha \overset{(0)}{u}_3 + \frac{1}{h}\overset{(1)}{u}_\alpha\right),\\
&\overset{(1)}{\sigma}_{\alpha 3} = \mu\partial_\alpha \overset{(1)}{u}_3,\\
&\overset{(0)}{\sigma}_{33} = \lambda\partial_\gamma \overset{(0)}{u}_\gamma - \beta_1^* \overset{(0)}{\widetilde{p}_1} - \beta_2^* \overset{(0)}{\widetilde{p}_2} + \frac{\lambda+2\mu}{h}\overset{(1)}{u}_3,\\
&\overset{(1)}{\sigma}_{33} = \lambda\partial_\gamma \overset{(1)}{u}_\gamma - \beta_1^* \overset{(1)}{\widetilde{p}_1} - \beta_2^* \overset{(1)}{\widetilde{p}_2}.
\end{aligned}
\tag{41}
$$

If we substitute relations (41) in (40) we obtain the system of equations of poroelastic equilibrium of porous plates with double porosity in displacement vector components.This system of equations is divided into two independent systems:equations of stretch-press for unknowns $\overset{(0)}{u}_1, \overset{(0)}{u}_2, \overset{(1)}{u}_3, \overset{(0)}{\widetilde{p}_1}, \overset{(0)}{\widetilde{p}_2}$ and equations of bending for unknowns $\overset{(1)}{u}_1, \overset{(1)}{u}_2, \overset{(0)}{u}_3, \overset{(1)}{\widetilde{p}_1}, \overset{(1)}{\widetilde{p}_2}$. The system of equations of stretch-press

$$
\begin{cases}
\mu\Delta \overset{(0)}{u}_\beta + (\lambda+\mu)\partial_\beta \overset{(0)}{\vartheta} + \frac{\lambda}{h}\partial_\beta \overset{(1)}{u}_3 - \partial_\beta\left(\beta_1^* \overset{(0)}{\widetilde{p}_1} + \beta_2^* \overset{(0)}{\widetilde{p}_2}\right) + \overset{(0)}{F}_\beta = 0,\\
\beta = 1,2,\\
\mu\Delta \overset{(1)}{u}_3 - \frac{3\lambda}{h}\overset{(0)}{\vartheta} - \frac{3(\lambda+2\mu)}{h^2}\overset{(1)}{u}_3 + \frac{3}{h}\left(\beta_1^* \overset{(0)}{\widetilde{p}_1} + \beta_2^* \overset{(0)}{\widetilde{p}_2}\right) + \overset{(1)}{F}_3 = 0,
\end{cases}
\tag{42}
$$

The system of equations of bending

$$
\begin{cases}
\mu\Delta \overset{(1)}{u}_\beta + (\lambda+\mu)\partial_\beta \overset{(1)}{\vartheta} - \frac{3\mu}{h}\partial_\beta \overset{(0)}{u}_3 - \frac{3\mu}{h^2}\overset{(1)}{u}_\beta\\
-\partial_\beta\left(\beta_1^* \overset{(0)}{\widetilde{p}_1} + \beta_2^* \overset{(0)}{\widetilde{p}_2}\right) + \overset{(1)}{F}_\beta = 0, \quad \beta = 1,2,\\
\mu\Delta \overset{(0)}{u}_3 + \frac{\mu}{h}\overset{(1)}{\vartheta} + \overset{(0)}{F}_3 = 0,
\end{cases}
\tag{43}
$$

where $\overset{(0)}{\vartheta} = \partial_\gamma \overset{(0)}{u}_\gamma$, $\overset{(1)}{\vartheta} = \partial_\gamma \overset{(1)}{u}_\gamma$; $\overset{(0)}{\widetilde{p}_1}, \overset{(0)}{\widetilde{p}_2}, \overset{(1)}{\widetilde{p}_1}, \overset{(1)}{\widetilde{p}_2}$ satisfy one of systems (29), (32) or (36), depending on what boundary conditions for pressures are set on the face planes of a plate.

5 The general solution of systems of equations (42) and (43)

On the plane Ox_1x_2 we introduce the complex variable $z = x_1 + ix_2$ ($i^2 = -1$) and the operators $\partial_z = 0.5(\partial_1 - i\partial_2), \partial_{\bar{z}} = 0.5(\partial_1 + i\partial_2)$ ($\bar{z} = x_1 - ix_2$). The two-dimensional Laplace operator is expressed as $\Delta = 4\partial_z\partial_{\bar{z}}$.

The homogenous system of equations (42) will be written in a complex form as follows

$$\begin{cases} \mu\Delta \overset{(0)}{u}_{+} + 2(\lambda+\mu)\partial_{\bar{z}} \overset{(0)}{\vartheta} - \dfrac{2\lambda}{h}\partial_{\bar{z}} \overset{(1)}{u}_{3} - 2\partial_{\bar{z}}\Big(\beta_1^* \overset{(0)}{\widetilde{p}}_1 + \beta_2^* \overset{(0)}{\widetilde{p}}_2\Big) = 0, \\ \mu\Delta \overset{(1)}{u}_{3} - \dfrac{3\lambda}{h} \overset{(0)}{\vartheta} - \dfrac{3(\lambda+2\mu)}{h^2} \overset{(1)}{u}_{3} + \dfrac{3}{h}\Big(\beta_1^* \overset{(0)}{\widetilde{p}}_1 + \beta_2^* \overset{(0)}{\widetilde{p}}_2\Big) + \overset{(1)}{F}_3 = 0, \end{cases} \tag{44}$$

where $\overset{(0)}{u}_{+} = \overset{(0)}{u}_{1} + i \overset{(0)}{u}_{2}$, $\overset{(0)}{\vartheta} = \partial_z + \partial_{\bar{z}} \overline{\overset{(0)}{u}_{+}}$; $\overset{(0)}{p}_1, \overset{(0)}{p}_2$ satisfy one of the equations (30), (33) or (37), depending on what boundary conditions for pressure are set on the surfaces $x_3 = \pm h$. For example, we take the case when on the face planes of the plate are given Dirichlet boundary conditions (9, 1)). When $\overset{(0)}{p}_1$ and $\overset{(0)}{p}_2$ satisfy the equations (30)

$$\Delta \overset{(0)}{p}_1 - \frac{3}{h^2} \overset{(0)}{\widetilde{p}}_1 = -\frac{3}{2h^2}(f_1^+ + f_1^-), \tag{45}$$

$$\Delta \overset{(0)}{p}_2 - \left(\frac{3}{h^2} + \zeta^2\right) \overset{(0)}{\widetilde{p}}_2 = -\frac{3}{2h^2}(f_2^+ + f_2^-), \tag{46}$$

We assume that $f_1^+ + f_1^-$ and $f_2^+ + f_2^-$ are constant values $f_\alpha^+ + f_\alpha^- = const$, $\alpha = 1, 2$.

We take the operator $2\partial_{\bar{z}}$ out of the brackets in the left-hand part of the first equation of system (44)

$$2\partial_{\bar{z}}\Big(2\mu\partial_z \overset{(0)}{u}_{+} + (\lambda+\mu) \overset{(0)}{\vartheta} + \frac{\lambda}{h} \overset{(1)}{u}_{3} - \beta_1^* \overset{(0)}{\widetilde{p}}_1 - \beta_2^* \overset{(0)}{\widetilde{p}}_2\Big) = 0. \tag{47}$$

Since (47) is a system of Cauchy-Riemann equations, we have

$$2\mu\partial_z \overset{(0)}{u}_{+} + (\lambda+\mu) \overset{(0)}{\vartheta} + \frac{\lambda}{h} \overset{(1)}{u}_{3} - \beta_1^* \overset{(0)}{\widetilde{p}}_1 - \beta_2^* \overset{(0)}{\widetilde{p}}_2 = a\varphi'(z), \tag{48}$$

where $\varphi'(z)$ are an arbitrary analytic function of z, a, are arbitrary nonzero constant. Summing equation (48) with the conjugate equation, we will obtain

$$(\lambda+2\mu) \overset{(0)}{\vartheta} + \frac{\lambda}{h} \overset{(1)}{u}_{3} - \beta_1^* \overset{(0)}{\widetilde{p}}_1 - \beta_2^* \overset{(0)}{\widetilde{p}}_2 = \frac{a}{2}(\varphi'(z) + \overline{\varphi'(z)}). \tag{49}$$

From the second equation of system (44) we will define $\overset{(0)}{\vartheta}$

$$\overset{(0)}{\vartheta}=\frac{\mu h}{3\lambda}\Delta\overset{(1)}{u_3}-\frac{\lambda+2\mu}{\lambda h}\overset{(1)}{u_3}+\frac{1}{\lambda}\left(\beta_1^*\overset{(0)}{\widetilde{p}_1}+\beta_2^*\overset{(0)}{\widetilde{p}_2}\right). \tag{50}$$

(50) we will substitute in (49), we obtain the following equation

$$\begin{aligned}&\Delta\left(2\mu\overset{(1)}{u_3}\right)-\frac{12(\lambda+\mu)}{(\lambda+2\mu)h^2}\left(2\mu\overset{(1)}{u_3}\right)=a\frac{3\lambda}{(\lambda+2\mu)h}(\varphi'(z)+\overline{\varphi'(z)})\\&-\frac{12\mu}{(\lambda+2\mu)h}\left(\beta_1^*\overset{(0)}{\widetilde{p}_1}+\beta_2^*\overset{(0)}{\widetilde{p}_2}\right)\end{aligned} \tag{51}$$

The general solution of equation (51) taking into account that $\overset{(0)}{\widetilde{p}_1}$ and $\overset{(0)}{\widetilde{p}_2}$ satisfy equations (45) and (46), will have the form

$$2\mu\overset{(1)}{u_3}=\chi(z,\bar{z})-\frac{\lambda ha}{4(\lambda+\mu)}(\varphi'(z)+\overline{\varphi'(z)})+a_1\overset{(0)}{\widetilde{p}_1}+a_2\overset{(0)}{\widetilde{p}_2}+b, \tag{52}$$

where $\chi(z,\bar{z})$ is the general solution of the following Helmholtz equation

$$\Delta\chi-\frac{12(\lambda+\mu)}{(\lambda+2\mu)h^2}\chi=0;$$

$$a_\alpha=\frac{12\mu h}{9\lambda+6-\delta_{\alpha 2}\zeta^2(\lambda+2\mu)h^2}\beta_\alpha^*,\quad \alpha=1,2;$$

$$b=-\frac{\lambda+2\mu}{8(\lambda+\mu)}[(f_1^++f_1^-)a_1+(f_2^++f_2^-)a_2].$$

Substituting equation (50) into (48) we obtain

$$2\mu\partial_z\overset{(0)}{u_+}+(\lambda+\mu)\frac{\mu h}{3\lambda}\Delta\overset{(1)}{u_3}-(3\lambda+2\mu)\frac{\mu}{\lambda h}\overset{(1)}{u_3}+\frac{\mu}{\lambda}\left(\beta_1^*\overset{(0)}{\widetilde{p}_1}+\beta_2^*\overset{(0)}{\widetilde{p}_2}\right)=a\varphi'(z).$$

Here the last formula are substituting the expression (52) for $\overset{(1)}{u_3}$

$$\begin{aligned}2\mu\partial_z\overset{(0)}{u_+}&=\frac{5\lambda+6\mu}{8(\lambda+\mu)}a\varphi'(z)-\frac{3\lambda+2\mu}{8(\lambda+\mu)}a\overline{\varphi'(z)}+\frac{\lambda h}{6(\lambda+\mu)}\partial_z\partial_{\bar{z}}\chi\\&+\partial_z\partial_{\bar{z}}\left(a_0\overset{(0)}{\widetilde{p}_1}+b_0\overset{(0)}{\widetilde{p}_2}\right)+c_0,\end{aligned} \tag{53}$$

where

$$a_0=\frac{4\mu h^2}{3(3\lambda+2\mu)}\beta_1^*,\quad b_0=\frac{4\mu h^2}{3+h^2\zeta^2}\frac{3-h^2\zeta^2}{9\lambda+6\mu-\zeta^2(\lambda+\mu)h^2}\beta_2^*,$$

$$c_0 = \frac{\mu}{2\lambda}(f_1^+ + f_1^-)\beta_1^* + \frac{3\mu}{2\lambda(3+h^2\zeta^2)}\frac{9\lambda+6\mu+\zeta^2(\lambda+\mu)h^2}{9\lambda+6\mu-\zeta^2(\lambda+\mu)h^2}(f_1^+ + f_1^-)\beta_2^*,$$

Let $a = \frac{8(\lambda+\mu)}{3\lambda+2\mu}$, then integrating on z the above formula (53), we obtain

$$2\mu \overset{(0)}{u}_+ = \kappa^*\varphi(z) - z\overline{\varphi' z} - \overline{\psi(z)} - \frac{\lambda h}{6(\lambda+\mu)}\partial_{\bar z}\chi(z,\bar z)$$

$$+ \int \Big(a_0 \overset{(0)}{\widetilde{p}_1} + b_0 \overset{(0)}{\widetilde{p}_2}\Big)dz + c_0 z, \tag{54}$$

where

$$\kappa^* = \frac{5\lambda+6\mu}{2\lambda+2\mu}.$$

$\psi(z)$ is an arbitrary analytic function of z. Thus, the general solution of (44) is represented by formulas (53) and (54). Substituting this expressions into corresponding formulas (41), for combinations of the stress tensor components we obtain the following formulas

$$\overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{11} = 2\left[\varphi'(z) + \overline{\varphi'(z)} + \frac{1}{h}\frac{\lambda}{\lambda+2\mu}\chi(z,\bar z) - a_0 \overset{(0)}{\widetilde{p}_1} - b_0 \overset{(0)}{\widetilde{p}_2} - c_0\right],$$

$$\overset{(0)}{\sigma}_{11} - \overset{(0)}{\sigma}_{22} + 2i \overset{(0)}{\sigma}_{12} = 2\Bigg[-z\overline{\varphi''(z)} - \overline{\psi'(z)} + \frac{h}{6}\frac{\lambda}{\lambda+\mu}\partial_{\bar z}\partial_{\bar z}\chi(z,\bar z)$$

$$+ \int \partial_{\bar z}\Big(a_0 \overset{(0)}{\widetilde{p}_1} + b_0 \overset{(0)}{\widetilde{p}_2}\Big)dz\Bigg], \tag{55}$$

The homogenous system of equations (43) will be written in the complex form as follows

$$\begin{cases} \mu\Delta \overset{(1)}{u}_+ + 2(\lambda+\mu)\partial_{\bar z} \overset{(1)}{\vartheta} \\ -\dfrac{6\mu}{h}\partial_{\bar z} \overset{(0)}{u}_3 - \dfrac{3\mu}{h^2} \overset{(1)}{u}_+ - 2\partial \bar z\Big(\beta_1^* \overset{(1)}{\widetilde{p}_1} + \beta_2^* \overset{(1)}{\widetilde{p}_2}\Big) = 0, \\ \mu\Delta \overset{(0)}{u}_3 + \dfrac{\mu}{h} \overset{(1)}{\vartheta} = 0. \end{cases} \tag{56}$$

When $\overset{(1)}{\widetilde{p}_1}$ and $\overset{(2)}{\widetilde{p}_2}$ satisfy equations (31)

$$\Delta \overset{(1)}{\widetilde{p}_1} - \frac{15}{h^2} \overset{(1)}{\widetilde{p}_1} = -\frac{15}{h^2}(f_1^+ - f_1^-), \tag{57}$$

$$\Delta \overset{(1)}{\widetilde{p}_2} - \Big(\frac{15}{h^2} + \zeta^2\Big) \overset{(1)}{\widetilde{p}_2} = -\frac{15}{h^2}(f_2^+ - f_2^-). \tag{58}$$

From the second equation of system (57)

$$2\partial_{\bar z}\overset{(0)}{u_3}+\frac{1}{h}\overset{(1)}{u_+}=\frac{i}{h}\partial_{\bar z}\omega, \tag{59}$$

where ω is yet an unknown real function.
From (60) we have

$$\overset{(1)}{u_+}=-2h\partial_{\bar z}\overset{(0)}{u_3}+i\partial_{\bar z}\omega. \tag{60}$$

Substituting (61) and $\vartheta=-h\Delta\overset{(0)}{u_3}$ in the first equation of system (57) and integrating on z we obtain the equation

$$-(\lambda+2\mu)h\Delta\overset{(0)}{u_3}+i\mu\left(\Delta\omega-\frac{3}{h^2}\omega\right)-2\Big(\beta_1^*\overset{(1)}{\widetilde{p}_1}+\beta_2^*\overset{(1)}{\widetilde{p}_2}\Big)=a\overline{f'(z)}, \tag{61}$$

where $f'(z)$ is an arbitrary analytic function of z; a is an arbitrary nonzero real constant. Summing equation (61) with the conjugate equation, we will obtain

$$-4(\lambda+2\mu)h\Delta\overset{(0)}{u_3}=a(f'(z)+\overline{f'(z)})+4\Big(\beta_1^*\overset{(1)}{\widetilde{p}_1}+\beta_2^*\overset{(1)}{\widetilde{p}_2}\Big).$$

From the last equation it follows that

$$\overset{(0)}{u_3}=-\frac{a}{16(\lambda+2\mu)}(\bar z f(z)+z\overline{f(z)})+\mathrm{g}(z)+\overline{\mathrm{g}(z)}$$

$$-\frac{1}{4(\lambda+2\mu)}\int\int\Big(\beta_1^*\overset{(1)}{\widetilde{p}_1}+\beta_2^*\overset{(1)}{\widetilde{p}_2}\Big)dzd\bar z, \tag{62}$$

where $\mathrm{g}(z)$ is an arbitrary analytic function of z.
Considering the imaginary part of the equation (62), we obtain the equation

$$2\mu\Big(\Delta\omega-\frac{3}{h^2}\omega\Big)=-i(f'(z)-\overline{f'(z)}),$$

whose general solution is represented as follows

$$\omega=\tau(z,\bar z)+i\frac{h^2}{6\mu}(f'(z)-\overline{f'(z)}), \tag{63}$$

where $\tau(z,\bar z)$ is the general solution of the following Helmholtz equation

$$\Delta\tau-\frac{3}{h^2}\tau=0,$$

Substituting formulas (61) and (62) in (59), we obtain ($a = 8(\lambda + 2\mu)$)

$$\overset{(1)}{u}_{+} = i\partial_{\bar{z}}\tau + \frac{4h^2(\lambda+2\mu)}{3\mu}\overline{f''(z)} + z\overline{f'(z)} + f'(z) + f(z) - 2h\overline{\mathrm{g}'(z)}$$

$$+\frac{1}{2(\lambda+2\mu)h}\int\left(\beta_1^* \overset{(1)}{\widetilde{p}_1} + \beta_2^* \overset{(1)}{\widetilde{p}_2}\right)dz. \tag{64}$$

Taking into account the value of a constant a we will rewrite a formula (61)

$$\overset{(0)}{u}_3 = -\frac{1}{2h}(\bar{z}f(z) + z\overline{f(z)}) + \mathrm{g}(z) + \overline{\mathrm{g}(z)}$$

$$-\frac{1}{4(\lambda+2\mu)}\int\int\left(\beta_1^* \overset{(1)}{\widetilde{p}_1} + \beta_2^* \overset{(1)}{\widetilde{p}_2}\right)dzd\bar{z}, \tag{65}$$

Substituting the expressions into corresponding formulas (41), we obtain expressions for combinations $\overset{(1)}{\sigma}_{11} + \overset{(1)}{\sigma}_{22}$, $\overset{(1)}{\sigma}_{11} - \overset{(1)}{\sigma}_{22} + 2i\overset{(1)}{\sigma}_{12}$ and $\overset{(0)}{\sigma}_{13} + i\overset{(0)}{\sigma}_{23}$.

The constructed general solution enables one to solve analytically a sufficiently wide class of boundary value problems of the elastic equilibrium of porous plates with double porosity.

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