SOLUTION OF BOUNDARY VALUE PROBLEMS OF SPHERICAL SHELLS BY THE VEKUA METHOD FOR APPROXIMATION $N = 2$

B. Gulua

Faculty of Exact and Natural Sciences and
I. Vekua Institute of Applied Mathematics of
I. Javakhishvili Tbilisi State University
2 University Str., Tbilisi 0186, Georgia
Sokhumi State University
9 Anna Politkovskaia Str., Tbilisi 0186,Georgia

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Abstract

In the present paper we consider the geometrically nonlinear and non-shallow spherical shells, when components of the deformation tensor have nonlinear terms. By means of I. Vekua's method the system of equilibrium equations in two variables is obtained. Using complex variable functions and the method of the small parameter approximate solutions are constructed for $N = 2$ in the hierarchy by I. Vekua. Concrete problem has been solved.

Key words and phrases: Non-shallow shells, geometrically nonlinear theory, small parameter, spherical shells.

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1 Equations of Equilibrium of an Elastic Medium

There are many different method of passage (reduction) from three- dimensional problems of elasticity to tow-dimensional problems of the theory of shells. I.Vekua had obtained the equations of shallow shells [1],[2]. It means that the interior geometry of the shell does not vary in thickness. This method for non-shallow shells in case of geometrical and physical non-linear theory was generalized by T.Meunargia [3],[4].

A complete system of equations of the three-dimensional nonlinear theory of elasticity can be written as:

$$\partial_i \sqrt{g}\boldsymbol{\sigma}^i + \sqrt{g}\boldsymbol{\Phi} = 0, \quad \Big(\partial_i = \frac{\partial}{\partial x^i}\Big), \tag{1}$$

$$\boldsymbol{\sigma}^i = \lambda\Big(\boldsymbol{R}^j\partial_j\boldsymbol{U} + \frac{1}{2}\partial^j\boldsymbol{U}\partial_j\boldsymbol{U}\Big)\Big(\boldsymbol{R}^i + \partial^i\boldsymbol{U}\Big)$$

$$+\mu\Big(\boldsymbol{R}^i\partial^j\boldsymbol{U}+\boldsymbol{R}^j\partial^i\boldsymbol{U}+\partial^i\boldsymbol{U}\partial^j\boldsymbol{U}\Big)\Big(\boldsymbol{R}_j+\partial_j\boldsymbol{U}\Big),$$

where x^1, x^2, x^3 are curvilinear coordinates, g is the discriminant of the metric tensor of the space, $\boldsymbol{\Phi}$ is an external force, $\boldsymbol{\sigma}^i$ are contravariant stress vectors, λ and μ are Lame's constants, $\boldsymbol{R}_i$ and $\boldsymbol{R}^i$ are covariant and contravariant base vectors of the space and $\boldsymbol{U}$ is the displacement vector.

2 I. Vekua's reduction method

In the present paper we use I. Vekua's reduction method for the nonlinear theory of non-shallow shells (I. Vekua used the method for linear theory of shallow shells) the essence of which consists, without going into detals, in the following: since the system of Legendre polynomials $P_m(\frac{x_3}{h})$ is complete in the interval $[-h,h]$, for equation (1) the equivalent infinite system of 2-D equations is obtained [2-3]

$$\nabla_\alpha \overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha}-\frac{2m+1}{h}\left(\overset{(m-1)}{\boldsymbol{\sigma}}{}^{3}+\overset{(m-3)}{\boldsymbol{\sigma}}{}^{3}+...\right)+\overset{(m)}{\boldsymbol{F}}=0, \tag{2}$$

where

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^{i},\overset{(m)}{\boldsymbol{\Phi}}\right)=\frac{2m+1}{2h}\int\limits_{-h}^{h}\big(\vartheta\boldsymbol{\sigma}^i,\vartheta\boldsymbol{\Phi}\big)P_m\left(\tfrac{x_3}{h}\right)dx_3,$$

$$\overset{(m)}{\boldsymbol{F}}=\overset{(m)}{\boldsymbol{\Phi}}+\frac{2m+1}{2h}\left(\overset{(+)}{\vartheta}\,\overset{(+)}{\boldsymbol{\sigma}}{}_3-(-1)^m\,\overset{(-)}{\vartheta}\,\overset{(-)}{\boldsymbol{\sigma}}{}_3\right).$$

Thus we have obtained the infinite system of 2-D equations (2), for which the boundary conditions of the face surfaces ($x_3=\pm h$) are satisfied, i.e. $\overset{(\pm)}{\boldsymbol{\sigma}}{}^3=\boldsymbol{\sigma}^3(x^1,x^2,\pm h)$ is the preassigned vector field and is entered in the equilibrium equations.

The equations of the state (2) may be write as:

$$
\begin{aligned}
&\overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha} = \lambda\Big(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m)}{\boldsymbol{u}}\Big)\boldsymbol{r}^{\alpha} + \mu\Big(\boldsymbol{r}^{\alpha}\partial_{\beta}\overset{(m)}{\boldsymbol{u}}\Big)\boldsymbol{r}^{\beta} + \mu\partial^{\alpha}\overset{(m)}{\boldsymbol{u}} \\
&+\sum_{m_1=0}^{\infty}\Bigg\{B^{m}_{m_1}\left[\lambda\overset{(m_1)}{u}{}'^{3}\boldsymbol{r}^{\alpha} + \mu\overset{(m_1)}{u}{}'^{\alpha}\boldsymbol{n}\right] \\
&+\sum_{m_2=0}^{\infty}\Bigg\{A^{m}_{m_1m_2}\left[\frac{\lambda}{2}\Big(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}}\Big)\boldsymbol{r}^{\alpha} + \lambda\Big(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m_1)}{\boldsymbol{u}}\Big)\partial^{\alpha}\overset{(m_2)}{\boldsymbol{u}}\right. \\
&\left.+\mu\Big(\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\partial^{\beta}\overset{(m_2)}{\boldsymbol{u}}\Big)\boldsymbol{r}_{\beta} + \mu\Big(\boldsymbol{r}^{\alpha}\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}} + \boldsymbol{r}^{\beta}\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\Big)\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}}\right] \\
&+B^{m}_{m_1m_2}\left[\frac{\lambda}{2}\Big(\overset{(m_1)}{\boldsymbol{u}}{}'\overset{(m_2)}{\boldsymbol{u}}{}'\Big)\boldsymbol{r}^{\alpha} + \mu\overset{(m_1)}{u}{}'^{\alpha}\overset{(m_2)}{\boldsymbol{u}}{}'\right] \qquad\qquad (3) \\
&+C^{m}_{m_1m_2}\left[\lambda\overset{(m_1)}{u}{}'^{3}\partial^{\alpha}\overset{(m_2)}{\boldsymbol{u}} + \mu\Big(\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\overset{(m_2)}{\boldsymbol{u}}{}'\Big)\boldsymbol{n} + \mu\Big(\boldsymbol{n}\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\Big)\overset{(m_2)}{\boldsymbol{u}}{}'\right] \\
&+\sum_{m_3=0}^{\infty}\Bigg\{\bar{A}^{mm_3}_{m_1m_2}\left[\frac{\lambda}{2}\Big(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}} + \overset{(m_1)}{\boldsymbol{u}}{}'\overset{(m_2)}{\boldsymbol{u}}{}'\Big)\partial^{\alpha}\overset{(m_3)}{\boldsymbol{u}}\right. \\
&\left.+\mu\Big(\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\partial^{\alpha}\overset{(m_2)}{\boldsymbol{u}}\Big)\partial_{\beta}\overset{(m_3)}{\boldsymbol{u}}\right] + \mu C^{mm_3}_{m_1m_2}\big(\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\overset{(m_2)}{\boldsymbol{u}}{}'\big)\overset{(m_3)}{\boldsymbol{u}}{}'\Bigg\}\Bigg\}\Bigg\}
\end{aligned}
$$

where

$$
\overset{(m)}{\boldsymbol{u}} = \frac{2m+1}{2h}\int_{-h}^{h}\boldsymbol{u}P_m\Big(\frac{x_3}{h}\Big)dx_3, \quad \overset{(m)}{\boldsymbol{u}}{}' = \frac{2m+1}{h}\Big(\overset{(m+1)}{\boldsymbol{u}} + \overset{(m+3)}{\boldsymbol{u}} + \dots\Big),
$$

$$
\begin{aligned}
B^{m}_{m_1} &= \frac{2m+1}{2h}\int_{-h}^{h}\Big(1+\frac{x_3}{R}\Big)P_{m_1}\Big(\frac{x_3}{h}\Big)P_m\Big(\frac{x_3}{h}\Big)dx_3 \\
&= \delta^{m}_{m_1} + \frac{h}{R}\left[\frac{m+1}{2m+3}\delta^{m+1}_{m_1} + \frac{m}{2m-1}\delta^{m-1}_{m_1}\right], \\
B^{m}_{m_1m_2} &= \frac{2m+1}{2h}\int_{-h}^{h}\Big(1+\frac{x_3}{R}\Big)P_{m_1}\Big(\frac{x_3}{h}\Big)P_{m_2}\Big(\frac{x_3}{h}\Big)P_m\Big(\frac{x_3}{h}\Big)dx_3 \\
&= \sum_{s=0}^{\min(m_1,m_2)}\alpha_{m_1m_2s}B^{m}_{m_1+m_2-2s}, \\
A^{m}_{m_1m_2} &= \frac{2m+1}{2h}\int_{-h}^{h}\frac{P_{m_1}\big(\frac{x_3}{h}\big)P_{m_2}\big(\frac{x_3}{h}\big)P_m\big(\frac{x_3}{h}\big)}{1+\frac{x_3}{R}}dx_3
\end{aligned}
$$

$$
= \begin{cases} -\dfrac{R(2m+1)}{h}\dfrac{\partial}{\partial y}\displaystyle\sum_{r=0}^{\min(m_1,m_2)} \alpha_{m_1m_2r}P_{m_1+m_2-2r}\left(-\dfrac{R}{h}\right) \\ \times Q_m\left(-\dfrac{R}{h}\right), \qquad m_1+m_2-2r\le m \\ -\dfrac{R(2m+1)}{h}\dfrac{\partial}{\partial y}\displaystyle\sum_{r=0}^{\min(m_1,m_2)} \alpha_{m_1m_2r}P_m\left(-\dfrac{R}{h}\right) \\ \times Q_{m_1+m_2-2r}\left(-\dfrac{R}{h}\right), \qquad m_1+m_2-2r> m \end{cases}
$$

$$
\bar{A}^{mm_3}_{m_1m_2} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\frac{P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_{m_3}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)}{\left(1+\frac{x_3}{R}\right)^2}dx_3
$$

$$
= \begin{cases} -\dfrac{R^2(2m+1)}{h^2}\dfrac{\partial}{\partial y}\displaystyle\sum_{r=0}^{\min(m_1,m_2)} \alpha_{m_1m_2r}\sum_{s=0}^{\min(m_3,m)}\alpha_{m_3ms}P_{m_1+m_2-2r}\left(-\dfrac{R}{h}\right) \\ \times Q_{m_3+m-2s}\left(-\dfrac{R}{h}\right), \qquad m_1+m_2-2r\le m_3+m-2s \\ -\dfrac{R^2(2m+1)}{h^2}\dfrac{\partial}{\partial y}\displaystyle\sum_{r=0}^{\min(m_1,m_2)} \alpha_{m_1m_2r}\sum_{s=0}^{\min(m_3,m)}\alpha_{m_3ms}P_{m_3+m-2s}\left(-\dfrac{R}{h}\right) \\ \times Q_{m_1+m_2-2r}\left(-\dfrac{R}{h}\right), \qquad m_1+m_2-2r> m_3+m-2s \end{cases}
$$

$$
\begin{aligned}
C^{m}_{m_1m_2} &= \frac{2m+1}{2h}\int\limits_{-h}^{h}P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3 \\
&= \sum_{r=0}^{\min(m_1,m_2)}\alpha_{m_1m_2r}\delta^{r}_{m_1+m_2-2r}, \\
C^{mm_3}_{m_1m_2} &= \frac{2m+1}{2h}\int\limits_{-h}^{h}P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_{m_3}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3 \\
&= \sum_{r=0}^{\min(m_1,m_2)}\alpha_{m_1m_2r}\sum_{s=0}^{\min(m_3,m)}\alpha_{m_3ms}\delta^{m_3+m-2s}_{m_1+m_2-2r},
\end{aligned}
$$

The equations of the state (2) may be write as:

$$
\begin{aligned}
\overset{(m)}{\boldsymbol{\sigma}}{}^{3} = \sum_{m_1=0}^{\infty} \Bigg\{ B^{m}_{m_1} \Big[\lambda(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m_1)}{\boldsymbol{u}})\boldsymbol{n} + \mu(\boldsymbol{n}\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}})\boldsymbol{r}_{\beta} \Big] \\
+\bar{B}^{m}_{m_1} \Big[(\lambda+\mu)\overset{(m_1)}{u}{}'_{3}\boldsymbol{n} + \mu\overset{(m_1)}{\boldsymbol{u}}{}' \Big] + \sum_{m_2=0}^{\infty} \Bigg\{ B^{m}_{m_1 m_2} \Big[\lambda(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m_1)}{\boldsymbol{u}})\overset{(m_2)}{\boldsymbol{u}}{}' \\
+\mu(\boldsymbol{r}^{\beta}\overset{(m_1)}{\boldsymbol{u}}{}')\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}} + \mu(\overset{(m_1)}{\boldsymbol{u}}{}'\partial^{\beta}\overset{(m_2)}{\boldsymbol{u}})\boldsymbol{r}_{\beta} \Big] \\
+\bar{B}^{m}_{m_1 m_2} \Big[(\lambda+\mu)(\overset{(m_1)}{\boldsymbol{u}}{}'\overset{(m_2)}{\boldsymbol{u}}{}')\boldsymbol{n} + (\lambda+2\mu)(\boldsymbol{n}\overset{(m_1)}{\boldsymbol{u}}{}')\overset{(m_2)}{\boldsymbol{u}}{}' \Big] \qquad (4) \\
+C^{m}_{m_1 m_2} \Big[\frac{\lambda}{2}(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}})\boldsymbol{n} + \mu(\boldsymbol{n}\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}})\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}} \Big] \\
+\sum_{m_3=0}^{\infty} \Bigg\{ \left(\frac{\lambda}{2}+\mu\right) \bar{B}^{mm_3}_{m_1 m_2}(\overset{(m_1)}{\boldsymbol{u}}{}'\overset{(m_2)}{\boldsymbol{u}}{}')\overset{(m_3)}{\boldsymbol{u}}{}' \\
+C^{mm_3}_{m_1 m_2} \Big[\frac{\lambda}{2}(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}})\overset{(m_3)}{\boldsymbol{u}}{}' + \mu(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\overset{(m_2)}{\boldsymbol{u}}{}')\partial_{\beta}\overset{(m_3)}{\boldsymbol{u}} \Big] \Bigg\} \Bigg\} \Bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{B}^{m}_{m_1} = \frac{2m+1}{2h}\int_{-h}^{h}\left(1+\frac{x_3}{R}\right)^2 P_{m_1}\Big(\frac{x_3}{h}\Big)P_{m}\Big(\frac{x_3}{h}\Big)dx_3 = \delta^{m}_{m_1} \\
+ \frac{2h}{R}\left[\frac{m+1}{2m+3}\delta^{m+1}_{m_1} + \frac{m}{2m-1}\delta^{m-1}_{m_1}\right] + \frac{h^2}{R^2}\left[\frac{m(m+1)\delta^{m-2}_{m_1}}{(2m-1)(2m-3)}\right. \\
+ \left.\frac{2m^2+2m-1}{(2m-1)(2m+3)}\delta^{m}_{m_1} + \frac{(m+1)(m+2)}{(2m+3)(2m+5)}\delta^{m+2}_{m_1}\right],
\end{aligned}
$$

$$
\begin{aligned}
\bar{B}^{m}_{m_1 m_2} &= \frac{2m+1}{2h}\int_{-h}^{h}\left(1+\frac{x_3}{R}\right)^2 P_{m_1}\Big(\frac{x_3}{h}\Big)P_{m_2}\Big(\frac{x_3}{h}\Big)P_{m}\Big(\frac{x_3}{h}\Big)dx_3 \\
&= \sum_{s=0}^{\min(m_1,m_2)} \alpha_{m_1 m_2 s}\bar{B}^{m}_{m_1+m_2-2s},
\end{aligned}
$$

$$
\begin{aligned}
\bar{B}^{m_3 m}_{m_1 m_2} &= \frac{2m+1}{2h}\int_{-h}^{h}\left(1+\frac{x_3}{R}\right)^2 P_{m_1}\Big(\frac{x_3}{h}\Big)P_{m_2}\Big(\frac{x_3}{h}\Big)P_{m_3}\Big(\frac{x_3}{h}\Big)P_{m}\Big(\frac{x_3}{h}\Big)dx_3 \\
&= \sum_{s=0}^{\min(m_1,m_2)} \alpha_{m_1 m_2 s} \sum_{r=0}^{\min(m_3,m)} \alpha_{m_3 m_r}\bar{B}^{m_3+m-2r}_{m_1+m_2-2s}.
\end{aligned}
$$

Here we have used the formulas of F. Neumann and J. Adams:

$$\int_{-1}^{1} \frac{P_m(t)dt}{x-t} = 2Q_m(x), \quad (|x| > 1),$$

$$P_m(x)P_n(x) = \sum_{r=0}^{\min(m,n)} \alpha_{mnr} P_{m+n-2r}(x).$$

respectively, where $Q_m(x)$ is the Legender function of second order, and

$$\alpha_{mnr} = \frac{A_{m-r}A_rA_{n-r}}{A_{m+n-r}} \frac{2m+2n-4r+1}{2m+2n-2r+1}, \quad A_m = \frac{1\cdot 3\cdots(2m-1)}{m!}.$$

3 Approximation of Order $N = 2$

The displacement vector $\boldsymbol{U}(x^1, x^2, x^3)$ are expressed by the following formula [1, 2] (approximation $N = 1$)

$$\boldsymbol{U}(x^1, x^2, x^3) = \mathbf{u}(x^1, x^2) + \frac{x^3}{h}\mathbf{v}(x^1, x^2) - \frac{1}{2}\left(\frac{3(x^3)^2}{h^2} - 1\right)\mathbf{w}(x^1, x^2).$$

Here $\mathbf{u}(x^1, x^2)$, $\mathbf{v}(x^1, x^2)$ and $\mathbf{w}(x^1, x^2)$ are the vector fields on the middle surface $x^3 = 0$, $2h$ is the thickness of the shell, x^3 is a thickness coordinate $(-h \le x^3 \le h)$, x^1 and x^2 are isometric coordinates on the spherical surface

$$x^1 = \tan\frac{\theta}{2}\cos\varphi, \quad x^2 = \tan\frac{\theta}{2}\sin\varphi,$$

where θ and φ are the geographical coordinates.

Let us construct the solutions of the form [2, 5]

$$u_i = \sum_{k=1}^{\infty} \overset{k}{u}_i \varepsilon^k, \qquad v_i = \sum_{k=1}^{\infty} \overset{k}{v}_i \varepsilon^k, \qquad w_i = \sum_{k=1}^{\infty} \overset{k}{w}_i \varepsilon^k, \quad (i = 1, 2, 3),$$

where u_i, v_i and w_i are the components of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively, $\varepsilon = \frac{h}{R_0}$ is a small parameter, R_0 is the radius of the midsurface of the sphere.

Using I. Vekua's method and complex variable functions the system of equilibrium equations can be represented in the form

$$
\begin{aligned}
&4\mu\frac{\partial}{\partial\bar z}\left(\frac{1}{\Lambda}\frac{\partial \overset{k}{u}_+}{\partial\bar z}\right)+2(\lambda+\mu)\frac{\partial\overset{k}{\theta}}{\partial\bar z}+2\lambda\frac{\partial \overset{k}{v}_3}{\partial\bar z}=\overset{k}{X}_+,\\
&4\mu\frac{\partial}{\partial\bar z}\left(\frac{1}{\Lambda}\frac{\partial \overset{k}{w}_+}{\partial\bar z}\right)+2(\lambda+\mu)\frac{\partial\overset{k}{\Theta}}{\partial\bar z}-5\mu\left[2\frac{\partial \overset{k}{v}_3}{\partial\bar z}+\frac{3}{h}\overset{k}{w}_+\right]=\overset{k}{Z}_+,\\
&\mu\left(\nabla^2\overset{k}{v}_3+3\overset{k}{\Theta}\right)-3\left[\lambda\overset{k}{\theta}+(\lambda+2\mu)\overset{k}{v}_3\right]=\overset{k}{Y}_3,
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&4\mu\frac{\partial}{\partial\bar z}\left(\frac{1}{\Lambda}\frac{\partial \overset{k}{v}_+}{\partial\bar z}\right)+2(\lambda+\mu)\frac{\partial\overset{k}{\vartheta}}{\partial\bar z}+6\lambda\frac{\partial \overset{k}{w}_3}{\partial\bar z}-3\mu\left(2\frac{\partial \overset{k}{u}_3}{\partial\bar z}+\overset{k}{v}_+\right)=\overset{k}{Y}_+,\\
&\mu\left(\nabla^2\overset{k}{u}_3+\overset{k}{\vartheta}\right)=\overset{k}{Y}_3,\\
&\mu\nabla^2\overset{k}{w}_3-5\left[\lambda\overset{k}{\vartheta}+3(\lambda+2\mu)\overset{k}{w}_3\right]=\overset{k}{Z}_3,\quad (k=1,2,...),
\end{aligned}
\tag{6}
$$

where $z=x^1+ix^2, \Lambda=\dfrac{4R_0^2}{(1+z\bar z)^2}, \nabla^2=\dfrac{4}{\Lambda}\dfrac{\partial^2}{\partial z\partial\bar z}$ and

$$\overset{k}{u}_+=\overset{k}{u}_1+i\overset{k}{u}_2,\quad \overset{k}{v}_+=\overset{k}{v}_1+i\overset{k}{v}_2,\quad \overset{k}{w}_+=\overset{k}{w}_1+i\overset{k}{w}_2,$$

$$\overset{k}{\theta}=\frac{1}{\Lambda}\left(\frac{\partial\overset{k}{u}_+}{\partial z}+\frac{\partial\overline{\overset{k}{u}}_+}{\partial\bar z}\right),\quad \overset{k}{\vartheta}=\frac{1}{\Lambda}\left(\frac{\partial\overset{k}{v}_+}{\partial z}+\frac{\partial\overline{\overset{k}{v}}_+}{\partial\bar z}\right),$$

$$\overset{k}{\Theta}=\frac{1}{\Lambda}\left(\frac{\partial\overset{k}{w}_+}{\partial z}+\frac{\partial\overline{\overset{k}{w}}_+}{\partial\bar z}\right).$$

Introducing the well-known differential operators

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x^1}-i\frac{\partial}{\partial x^2}\right),\quad \frac{\partial}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x^1}+i\frac{\partial}{\partial x^2}\right).$$

$\overset{k}{X}_+, \overset{k}{Y}_+, \overset{k}{Z}_+, \overset{k}{X}_3, \overset{k}{Y}_3, \overset{k}{Z}_3$ are the components of external force and well-known quantity, defined by functions $\overset{0}{u}_i, ..., \overset{k-1}{u}_i, \overset{0}{v}_j, ..., \overset{k-1}{v}_j$, $\overset{0}{w}_i, ..., \overset{k-1}{w}_i$ $(i,j=1,2,3)$.

The complex representation of a general solutions of systems (5) end (6) are written in the following form

$$\overset{k}{u}_{+}=-\frac{5\lambda+6\mu}{3\lambda+2\mu}\frac{1}{\pi}\iint\limits_{D}\frac{\Lambda(\zeta,\bar{\zeta})\varphi'(\zeta)d\xi d\eta}{\bar{\zeta}-\bar{z}}+\left(\frac{1}{\pi}\iint\limits_{D}\frac{\Lambda(\zeta,\bar{\zeta})d\xi d\eta}{\bar{\zeta}-\bar{z}}\right)\overline{\varphi'(z)}$$
$$-\overline{\psi(z)}-\frac{2\lambda}{\lambda+2\mu}\left(\frac{1}{\gamma_1}\frac{\partial\chi_1(z,\bar{z})}{\partial\bar{z}}+\frac{1}{\gamma_2}\frac{\partial\chi_2(z,\bar{z})}{\partial\bar{z}}\right),$$
$$\overset{k}{w}_{+}=\frac{2}{3}\left(\frac{\gamma_2}{\gamma_1}\frac{\partial\chi_1(z,\bar{z})}{\partial\bar{z}}+\frac{\gamma_1}{\gamma_2}\frac{\partial\chi_2(z,\bar{z})}{\partial\bar{z}}+i\frac{\partial\chi_3(z,\bar{z})}{\partial\bar{z}}+\frac{2\lambda}{3\lambda+2\mu}\overline{\varphi''(z)}\right),$$
$$\overset{k}{v}_{3}=\chi_1(z,\bar{z})+\chi_2(z,\bar{z})-\frac{2\lambda}{3\lambda+2\mu}\left(\varphi'(z)+\overline{\varphi'(z)}\right),$$

$$\overset{k}{v}_{+}=i\frac{\partial\chi_4(z,\bar{z})}{\partial\bar{z}}-\frac{\lambda}{10(\lambda+\mu)}\frac{\partial\chi_5(z,\bar{z})}{\partial\bar{z}}+\frac{16(\lambda+\mu)}{3(\lambda+2\mu)}\overline{f''(z)}$$
$$-\frac{1}{\pi}\iint\limits_{D}\frac{\Lambda(\zeta,\bar{\zeta})f'(\zeta)d\xi d\eta}{\bar{\zeta}-\bar{z}}-\left(\frac{1}{\pi}\iint\limits_{D}\frac{\Lambda(\zeta,\bar{\zeta})d\xi d\eta}{\bar{\zeta}-\bar{z}}\right)\overline{f'(z)}-2\overline{g'(z)},$$
$$\overset{k}{u}_{3}=\frac{\lambda}{20(\lambda+\mu)}\chi_5(z,\bar{z})+g(z)+\overline{g(z)}$$
$$-\frac{1}{\pi}\iint\limits_{D}\Lambda(\zeta,\bar{\zeta})\left[f'(z)+\overline{f'(z)}\right]\ln|\zeta-z|d\xi d\eta,$$
$$\overset{k}{w}_{3}=\chi_5(z,\bar{z})-\frac{2\lambda}{3(\lambda+2\mu)}\left(f'(z)+\overline{f'(z)}\right),$$

where $\zeta=\xi+i\eta$, $\varphi(z)$,$\psi(z)$,$f(z)$ and $g(z)$ are any analytic functions of z, $\chi_1(z,\bar{z})$, $\chi_2(z,\bar{z})$, $\chi_3(z,\bar{z})$, $\chi_4(z,\bar{z})$ and $\chi_5(z,\bar{z})$, are the general solutions of the following Helmholtz's equations, respectively:

$$\Delta\chi_\alpha(z,\bar{z})-\gamma_\alpha^2\chi_\alpha(z,\bar{z})=0,\quad \alpha=1,2,$$
$$\gamma_\alpha=\frac{6(\lambda+\mu)}{\lambda+2\mu}\left[1\pm\sqrt{\frac{\lambda-4\mu}{\lambda+\mu}}\right],$$

$$\Delta\chi_3(z,\bar{z})-15\chi_3(z,\bar{z})=0,\quad \Delta\chi_4(z,\bar{z})-3\chi_4(z,\bar{z})=0,$$
$$\Delta\chi_5(z,\bar{z})-\gamma^2\chi_5(z,\bar{z})=0,\quad \gamma^2=\frac{60(\lambda+\mu)}{\lambda+2\mu}.$$

D is the domain of the plane Ox^1x^2 onto which the midsurface S of the shell is mapped topologically.

Here we present a general scheme of solution of boundary problems when the domain D is a circle of radius R [6, 7, 8, 9].

The second boundary problem (in displacements) for any k takes the form

$$\begin{gathered}\overset{k}{u}_{+}=-\frac{5\lambda+6\mu}{3\lambda+2\mu}\frac{1}{\pi}\iint\limits_{D}\frac{\Lambda(\zeta,\bar{\zeta})\varphi'(\zeta)d\xi d\eta}{\bar{\zeta}-\bar{z}}\\ +\left(\frac{1}{\pi}\iint\limits_{D}\frac{\Lambda(\zeta,\bar{\zeta})d\xi d\eta}{\bar{\zeta}-\bar{z}}\right)\overline{\varphi'(z)}-\overline{\psi(z)}\\ -\frac{2\lambda}{\lambda+2\mu}\left(\frac{1}{\gamma_1}\frac{\partial\chi_1(z,\bar{z})}{\partial\bar{z}}+\frac{1}{\gamma_2}\frac{\partial\chi_2(z,\bar{z})}{\partial\bar{z}}\right)=P_{+},\ \ |z|=R,\end{gathered}\tag{7}$$

$$\begin{gathered}\overset{k}{w}_{+}=\frac{2}{3}\left(\frac{\gamma_2}{\gamma_1}\frac{\partial\chi_1(z,\bar{z})}{\partial\bar{z}}+\frac{\gamma_1}{\gamma_2}\frac{\partial\chi_2(z,\bar{z})}{\partial\bar{z}}\right.\\ \left.+i\frac{\partial\chi_3(z,\bar{z})}{\partial\bar{z}}+\frac{2\lambda}{3\lambda+2\mu}\overline{\varphi''(z)}\right)=S_{+},\ \ |z|=R,\end{gathered}\tag{8}$$

$$\overset{k}{v}_{3}=\chi_1(z,\bar{z})+\chi_2(z,\bar{z})-\frac{2\lambda}{3\lambda+2\mu}\left(\varphi'(z)+\overline{\varphi'(z)}\right)=Q_3,\ \ |z|=R,\tag{9}$$

$$\begin{gathered}\overset{k}{v}_{+}=i\frac{\partial\chi_4(z,\bar{z})}{\partial\bar{z}}-\frac{\lambda}{10(\lambda+\mu)}\frac{\partial\chi_5(z,\bar{z})}{\partial\bar{z}}\\ +\frac{16(\lambda+\mu)}{3(\lambda+2\mu)}\overline{f''(z)}-\frac{1}{\pi}\iint\limits_{D}\frac{\Lambda(\zeta,\bar{\zeta})f'(\zeta)d\xi d\eta}{\bar{\zeta}-\bar{z}}\\ -\left(\frac{1}{\pi}\iint\limits_{D}\frac{\Lambda(\zeta,\bar{\zeta})d\xi d\eta}{\bar{\zeta}-\bar{z}}\right)\overline{f'(z)}-2\overline{g'(z)}=Q_{+},\ \ |z|=R,\end{gathered}\tag{10}$$

$$\begin{gathered}\overset{k}{u}_{3}=\frac{\lambda}{20(\lambda+\mu)}\chi_5(z,\bar{z})+g(z)+\overline{g(z)}\\ -\frac{1}{\pi}\iint\limits_{D}\Lambda(\zeta,\bar{\zeta})\left[f'(z)+\overline{f'(z)}\right]\ln|\zeta-z|d\xi d\eta=P_3,\ \ |z|=R,\end{gathered}\tag{11}$$

$$\overset{k}{w}_{3}=\chi_5(z,\bar{z})-\tfrac{2\lambda}{3(\lambda+2\mu)}\left(f'(z)+\overline{f'(z)}\right)=S_3,\ \ |z|=R,\tag{12}$$

where P_{+}, Q_{+}, S_{+}, P_3, Q_3 and S_3 are the known values.

Next $\varphi'(z)$, $\psi(z)$, P_{+}, S_{+} and Q_3 are expanded by the series

$$\begin{gathered}\varphi'(z)=\sum_{0}^{\infty}a_nz^n,\quad \psi(z)=\sum_{0}^{\infty}b_nz^n,\quad \chi_1(z,\bar{z})=\sum_{-\infty}^{\infty}\alpha_nI_n(\gamma_1r)e^{in\vartheta},\\ \chi_2(z,\bar{z})=\sum_{-\infty}^{\infty}\beta_nI_n(\gamma_2r)e^{in\vartheta},\quad \chi_3(z,\bar{z})=\sum_{-\infty}^{\infty}\delta_nI_n(\sqrt{15}r)e^{in\vartheta},\\ P_{+}=\sum_{-\infty}^{\infty}A_ne^{in\vartheta},\quad S_{+}=\sum_{-\infty}^{\infty}B_ne^{in\vartheta},\quad Q_3=\sum_{-\infty}^{\infty}C_ne^{in\vartheta},\end{gathered}\tag{13}$$

where $I_n(\cdot)$ is modified Bessel function of n-th order, and are substituted in the boundary conditions (7-9) we have

$$\begin{aligned}
&\frac{5\lambda+6\mu}{3\lambda+2\mu}\sum_{0}^{\infty}\frac{a_n\varepsilon_n}{R^{n+1}}e^{i(n+1)\vartheta}-\varepsilon_0\sum_{0}^{\infty}R^{n-1}\bar{a}_n e^{i(1-n)\vartheta}\\
&-\sum_{0}^{\infty}R^n\bar{b}_n e^{-in\vartheta}-\frac{\lambda}{\lambda+2\mu}\sum_{-\infty}^{\infty}(\alpha_n I_{n+1}(\gamma_1 R)\\
&+\beta_n I_{n+1}(\gamma_2 R))\,e^{i(n+1)\vartheta}=\sum_{-\infty}^{\infty}A_n e^{in\vartheta},
\end{aligned}\tag{14}$$

$$\begin{aligned}
&\sum_{-\infty}^{\infty}(\alpha_n I_n(\gamma_1 R)+\beta_n I_n(\gamma_2 R))\,e^{in\vartheta}\\
&-\frac{2\lambda}{3\lambda+2\mu}\sum_{0}^{\infty}(a_n e^{in\vartheta}+\bar{a}_n e^{-in\vartheta})R^n=\sum_{-\infty}^{\infty}B_n e^{in\vartheta},
\end{aligned}\tag{15}$$

$$\begin{aligned}
&\sum_{-\infty}^{\infty}\left(\frac{\gamma_2}{3}\alpha_n I_{n+1}(\gamma_1 R)+\frac{\gamma_1}{3}\beta_n I_{n+1}(\gamma_2 R)+\frac{i\sqrt{15}}{3}\delta_n I_{n+1}(\sqrt{15}R)\right)\\
&\times e^{i(n+1)\vartheta}+\frac{4\lambda}{3(3\lambda+2\mu)}\sum_{0}^{\infty}nR^{n-1}a_n e^{i(n-1)\vartheta}=\sum_{-\infty}^{\infty}C_n e^{in\vartheta},
\end{aligned}\tag{16}$$

where $\varepsilon_n=2\int\limits_0^R\Lambda(\rho)\rho^{2n+1}d\rho$.

Compare the coefficients at identical degrees. We obtain the following system of equations

$$\begin{aligned}
&\frac{5\lambda+6\mu}{3\lambda+2\mu}\frac{\varepsilon_0}{R}a_0-\frac{\varepsilon_0}{R}\bar{a}_0-\frac{\lambda}{\lambda+2\mu}\left(I_1(\gamma_1 R)\alpha_0+I_1(\gamma_2 R)\beta_0\right)=A_1,\\
&I_0(\gamma_1 R)\alpha_0+I_0(\gamma_2 R)\beta_0-\frac{2\lambda}{3\lambda+2\mu}(a_0+\bar{a}_0)=B_0,
\end{aligned}\tag{17}$$

$$\begin{aligned}
&\frac{5\lambda+6\mu}{3\lambda+2\mu}\frac{\varepsilon_n}{R^{n+1}}a_n-\frac{\lambda}{\lambda+2\mu}\left(I_{n+1}(\gamma_1 R)\alpha_n+I_{n+1}(\gamma_2 R)\beta_n\right)=A_{n+1},\\
&-\varepsilon_0 R^n\bar{a}_{n+1}-R^n\bar{b}_n\\
&-\frac{\lambda}{\lambda+2\mu}\left(I_n(\gamma_1 R)\alpha_{-n-1}+I_n(\gamma_2 R)\beta_{-n-1}\right)=A_{-n},\quad n\geq 1\\
&I_n(\gamma_1 R)\alpha_n+I_n(\gamma_2 R)\beta_n-\frac{2\lambda}{3\lambda+2\mu}R^n a_n=B_n,\quad n\geq 1
\end{aligned}\tag{18}$$

$$
\begin{aligned}
&\frac{\gamma_2}{3} I_{n+1}(\gamma_1 R)\alpha_n + \frac{\gamma_1}{3} I_{n+1}(\gamma_2 R)\beta_n + \frac{i\sqrt{15}}{3} I_{n+1}(\sqrt{15}R)\delta_n \\
&+\frac{4\lambda}{3(3\lambda+2\mu)}(n+2)R^{n+1}a_{n+2} = C_{n+1}, \quad n \geq -1 \\
&\frac{\gamma_2}{3} I_{n-1}(\gamma_1 R)\bar{\alpha}_n + \frac{\gamma_1}{3} I_{n-1}(\gamma_2 R)\bar{\beta}_n \\
&+\frac{i\sqrt{15}}{3} I_{n-1}(\sqrt{15}R)\bar{\delta}_n = \overline{C}_{-n+1}, \quad n \geq 2.
\end{aligned} \tag{19}
$$

Let us introduce the functions $f'(z)$, $g(z)$, $\chi_4(z,\bar{z})$ and $\chi_5(z,\bar{z})$, Q_+, P_3, S_3 by the series

$$
\begin{aligned}
&f'(z) = \sum_0^\infty c_n z^n, \quad g(z) = \sum_0^\infty d_n z^n, \\
&\chi_4(z,\bar{z}) = \sum_{-\infty}^\infty \alpha'_n I_n(\sqrt{3}r)e^{in\vartheta}, \quad \chi_5(z,\bar{z}) = \sum_{-\infty}^\infty \beta'_n I_n(\gamma r)e^{in\vartheta}, \\
&Q_+ = \sum_{-\infty}^\infty M_n e^{in\vartheta}, \quad P_3 = \sum_{-\infty}^\infty N_n e^{in\vartheta}, \quad S_3 = \sum_{-\infty}^\infty D_n e^{in\vartheta}.
\end{aligned} \tag{20}
$$

By substituting (20) into (10-12) we obtain

$$
\begin{aligned}
&\sum_{-\infty}^\infty \left(\frac{i\sqrt{3}}{2} I_{n+1}(\sqrt{3}R)\alpha'_n - \frac{\lambda\gamma}{20(\lambda+\mu)} I_{n+1}(\gamma R)\beta'_n\right) e^{i(n+1)\vartheta} \\
&-2\sum_1^\infty nR^{n-1}\bar{d}_n e^{i(1-n)\vartheta} + \sum_o^\infty \frac{\varepsilon_n c_n}{R^{n+1}} e^{i(n+1)\vartheta} \\
&+\varepsilon_0 \sum_0^\infty R^{n-1}\bar{c}_n e^{i(1-n)\vartheta} + \frac{16(\lambda+\mu)}{3(\lambda+2\mu)} \sum_1^\infty nR^{n-1}\bar{c}_n e^{i(1-n)\vartheta} \\
&= \sum_{-\infty}^\infty M_n e^{in\vartheta},
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&\frac{\lambda}{20(\lambda+\mu)} \sum_{-\infty}^\infty I_n(\gamma R)\beta'_n e^{in\vartheta} + \sum_0^\infty \left(R^n d_n e^{in\vartheta} + R^n \bar{d}_n e^{-in\vartheta}\right) \\
&-\varepsilon_0(c_0 + c_0) - \sum_0^\infty \left(\frac{\varepsilon_n c_n}{nR^n} e^{in\vartheta} + \frac{\varepsilon_n \bar{c}_n}{nR^n} e^{-in\vartheta}\right) = \sum_{-\infty}^\infty N_n e^{in\vartheta},
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&\sum_{-\infty}^\infty I_n(\gamma R)\beta'_n e^{in\vartheta} - \frac{2\lambda}{3(\lambda+2\mu)} \sum_0^\infty \left(R^n c_n e^{in\vartheta} + R^n \bar{c}_n e^{-in\vartheta}\right) \\
&= \sum_{-\infty}^\infty D_n e^{in\vartheta},
\end{aligned} \tag{23}
$$

We now find the coefficients c_n, d_n, α'_n and β'_n from following system of algebraic equations:

$$
\begin{aligned}
&\frac{i\sqrt{3}}{2}I_1(\sqrt{3}R)\alpha'_0-\frac{\lambda\gamma}{20(\lambda+\mu)}I_1(\gamma R)\beta'_0+\frac{\varepsilon_0}{R}(c_0+\bar{c}_0)=M_1,\\
&\frac{i\sqrt{3}}{2}I_n(\sqrt{3}R)\alpha'_{n-1}-\frac{\lambda\gamma}{20(\lambda+\mu)}I_n(\gamma R)\beta'_{n-1}+\frac{\varepsilon_{n-1}}{R^n}c_n=M_n,\\
&\frac{i\sqrt{3}}{2}I_n(\sqrt{3}R)\alpha'_{n+1}-\frac{\lambda\gamma}{20(\lambda+\mu)}I_n(\gamma R)\beta'_{n+1}-2(n+1)R^n d_{n+1}\\
&+\varepsilon_0 R^n c_{n+1}+\frac{16(\lambda+\mu)}{3(\lambda+2\mu)}(n+1)R^n c_{n+1}=\overline{M}_{-n},
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&\frac{\lambda}{20(\lambda+\mu)}I_0(\gamma R)\beta'_0+d_0+\bar{d}_0-\frac{\varepsilon_0}{R}(c_0+\bar{c}_0)=N_0,\\
&I_0(\gamma R)\beta'_0-\frac{2\lambda}{3(\lambda+2\mu)}(c_0+\bar{c}_0)=D_0,
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\frac{\lambda}{20(\lambda+\mu)}I_n(\gamma R)\beta'_n+R^n d_n+\bar{d}_0-\frac{\varepsilon_n}{nR^n}c_n=N_n,\\
&I_n(\gamma R)\beta'_n-\frac{2\lambda}{3(\lambda+2\mu)}R^n c_n=D_n.
\end{aligned}
\tag{26}
$$

It is easy to prove the absolute and uniform convergence of the series obtained in the circle (including the contours) when the functions set on the boundaries have sufficient smoothness.

Similarly the problem can be solved when on the boundary of the considered domain the values of stresses are given.

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