ABOUT ONE PROBLEM OF PLANE ELASTICITY FOR A POLYGONAL DOMAIN WITH A CURVILINEAR HOLE

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Abstract

In the present paper we consider a plane problem of elasticity for a polygonal domain with a curvilinear hole, which is composed of the rectilinear segment (parallel to the abscissa axis) and arc of the circumference. The problem is solved by the methods of conformal mappings and boundary value problems of analytic functions. The sought complex potentials are constructed effectively (in the analytical form). Estimates of the obtained solutions are derived in the neighborhood of angular points.

Key words and phrases: Conformal mapping, Kolosov-Muskhelishvili formula, Riemann-Hilbert problem for circular ring.

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1 Introduction

The application of the methods of conformal mappings and boundary value problems of analytic functions has proved to be the most effective way of solving boundary value problems of elasticity and plate bending. However, if for a simply-connected domain these methods yield effective results (especially for domains mapped onto the circle by rational functions), they still remain poorly adapted to the use for multiply-connected domains. The difficulty consists in the effective construction of a conformally mapping function in general form. Nevertheless, for some practically important classes of doubly-connected domains bounded by polygons (including the polygonal domain with a curvilinear 2-gonal hole considered here) we may succeed in constructing effectively (in the analytical form) functions conformally mapping this domain onto the circular ring. In addition to this, the Kolosov-Muskhelishvili methods make it possible to decompose these problems into two Riemann-Hilbert problems for the circular ring and by solving the latter problems to construct the sought complex potentials in the analytical form. Estimates of the obtained solutions are derived in the neighborhood of angular points.

2 Statement of the Problem

Let the homogeneous and isotropic elastic plate on the plane z of a complex variable occupy a finite doubly-connected domain S, bounded by the convex polygon $(A) = A_1A_2\dots A_p$ and curvilinear hole (B). We will assume that (A) is the external boundary of the domain S, and we will denote by A_i $(i = 1,\dots,p)$ the vertices and by $L_0^{(k)}$ the sides of the polygon (A) (i.e. $L_0^{(k)} = A_kA_{k+1}$, $k = \overline{1,p}$, $A_{p+1} = A_1$). (B) is the internal boundary composed of the segment $L_1^{(1)} = B_1B_2$ and arc of the circumference $L_1^{(2)} = B_1\breve{}B_2$ (i.e. (B) represents the curvilinear 2-gonal hole). The values of the internal angles of the domain S at the vertices A_k and B_m will be denoted by $\pi\alpha_k^0$ and $\pi\beta_m^0$ $(m = 1,2)$ (we mean the angles between the segment B_1B_2 and the tangent arc $L_1^{(2)}$ in the points B_m), while the angles between the x axis and outward normals to the contours L_0 $\left(L_0 = \bigcup\limits_{k=1}^{p} L_0^{(k)}\right)$ and L_1 $\left(L_1 = L_1^{(1)} \bigcup L_1^{(2)}\right)$ will denoted by $\alpha(t)$ and $\beta(t)$ (here we mean that $\beta(t) = \frac{\pi}{2}, t \in L_1^{(1)}, \beta(t) = \pi + \arg t,\ t \in L_1^{(2)}$). The positive direction on $L = L_0 \bigcup L_1$ well be assumed to be that which keeps domain S to the left (see Figure. 1).

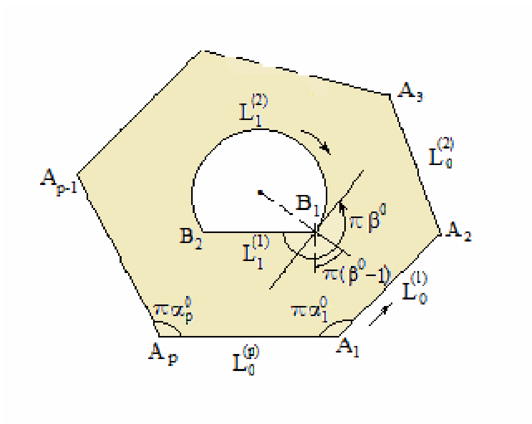

Fig. 1.

Assume that the sides L_k $(k = 1, \dots, p)$ are under the action of constant normal displacements $\nu_n(t)$ $\left(\nu_n(t) = \nu_n^k = const, t \in L_0^{(k)}, k = \overline{1,p}\right)$, the tangent stress equals zero, while the remaining part of the boundary $L = L_0 \bigcup L_1$ is free from external forces.

The problem consists in defining the elastic equilibrium of the plate and establishing the situation in which the concentration of stresses occurs near the angular points and which in turn depends on the behavior of Kolosov-Muskhelishvili potentials at these points.

3 Some Additional Propositions

Dirichlet's problem for a circular ring. Suppose $D_{00}(1 < |z| < r)$ is a circular ring bounded by circles $\gamma_{00}(|z| = r)$ and $\gamma_{11}(|z| = 1)$. We will consider the following problem: it is required to obtain a function $\Phi(z) = u+iv$, holomorphic in the ring D_{00}, with respect to the boundary condition

$$\mathrm{Re}[\Phi(t)] = f_j(t),\ t \in \gamma_{00} \cup \gamma_{11},\ j = 0, 1. \tag{1}$$

The necessary and sufficient condition for problem (1) to be solvable has the form

$$\int\limits_{\gamma_{00}} \frac{f_0(t)}{t} dt = \int\limits_{\gamma_{11}} \frac{f_1(t)}{t} dt$$

while the solution itself is given by the formula

$$\Phi(z) = \frac{1}{\pi i} \sum_{j=-\infty}^{\infty} \left[\int\limits_{\gamma_{00}} \frac{f_0(t)}{t - r^{2j} z} dt + \int\limits_{\gamma_{11}} \frac{f_1(t)}{t - r^{2j} z} dt \right] + ic_{11} + c_{22},$$

where $c_{22} = \frac{1}{4\pi} \int\limits_0^{2\pi} f_1(t) d\vartheta$, c_{11} is an arbitrary real constant.

The conformal mapping of a doubly-connected domain, bounded by polygons, onto a circular ring. Suppose S^0 is a doubly-connected domain bounded by the convex polygons (A') and (B') with vertices A'_k $(k = \overline{1,p})$ and B'_m $(m = \overline{1,q})$ and internal (with respect to the domain $S^{(0)}$) vertex angles $\pi\alpha'_k$ and $\pi\beta'_k$. We will consider the following problem: it is required to find of the function $z = \omega_0(\varsigma)$ which conformally maps the circular ring $D_0(1 < |\varsigma| < R_0)$ onto the domain $S^{(0)}$.

The derivative of the function $\omega_0(\varsigma)$ is the solution of the Riemann-Hilbert problems for the circular ring

$$\begin{aligned}&\mathrm{Re}[ite^{-i\alpha_0(t)}\omega_0'(t)] = 0, \;\; t\in\gamma_{00}(|\varsigma| = R_0)\\&\mathrm{Re}[ite^{-i\beta_0(t)}\omega_0'(t)] = 0, \;\; t\in\gamma_{11}(|\varsigma| = 1)\\&(\alpha_0(t) = \alpha[\omega_0(t)], \;\; \beta_0(t) = \beta[\omega_0(t)])\,,\end{aligned} \tag{2}$$

and with the conditions

$$\prod_{k=1}^{p} R_0^2 a_k'^{\alpha_k'-1} \prod_{m=1}^{q} R_0^2 b_m'^{\beta_m'-1} = 1$$

(a_1^k and b_k' are the inverse images of the points A_k' and B_m') the solution is given by the formula

$$\omega_0'(\varsigma) = k^0 \prod_{j=-\infty}^{\infty}\prod_{k=1}^{p}(R_0^{2j}\varsigma - a_k')^{\alpha_k'-1}\prod_{m=1}^{q}(R_0^{2j}\varsigma - b_m')^{\beta_m'-1}R_0^{2\delta_j},$$

with k^0 as an arbitrary real constant, $\delta_j = \begin{cases} 0, & j\geq 0,\\ 1, & j\leq -1.\end{cases}$

4 Solution of the Problem

Let us now assume that the regular open polygon with sides ε_n are inscribed in the arc $L_1^{(2)}$ and denote the obtained doubly-connected domain $S^{(n)}$. Applying the results obtained above for the domain $S^{(n)}$ and treating the domain S as a limit case of the domain $S^{(n)}$as $n\to\infty$ (i.e. $\varepsilon_n\to 0$), the boundary conditions for $\omega'(\zeta)$ are written in the form

$$\begin{aligned}&\mathrm{Re}[i\sigma e^{-i\alpha_0(\sigma)}\omega'(\sigma)] = 0, \;\; \sigma\in l_0(|\sigma| = R)\\&\mathrm{Re}[i\sigma e^{-i\Delta_0(\sigma)}\omega'(\sigma)] = 0, \;\; \sigma\in l_1(|\sigma| = 1)\end{aligned} \tag{3}$$

where

$$\Delta_0(\sigma) = \begin{cases} \dfrac{\pi}{2}, & \sigma\in l_1^{(1)},\\ \arg\sigma, & \sigma\in l_1^{(2)},\end{cases}$$

$l_1^{(1)}$ and $l_1^{(2)}$ are the arcs of the circumference l_1 which correspond to the segments $L_1^{(1)}$ and $L_1^{(2)}$.

By solving problem (3) we obtain the formula

$$\omega'(\varsigma) = K^0 e^{\gamma(\varsigma)} A(\varsigma), \tag{4}$$

where

$$\gamma(\varsigma)=\frac{1}{2\pi i}\int\limits_{l_1}\frac{\ln[\sigma^{-2}e^{2i\Delta_0(\sigma)}]}{\sigma-\varsigma}d\sigma+\frac{1}{2\pi i}\sum_{j=-\infty}^{\infty}\int\limits_{l_1}\frac{\ln[\sigma^{-2}e^{2i\Delta_0(\sigma)}]}{\sigma-R^{2j}\varsigma}d\sigma,$$

K^0 is an arbitrary real constant, $\sum\limits_{j=-\infty}^{\infty}$ indicates that $j=0$ is omitted, $A(\varsigma)=\prod\limits_{j=-\infty}^{\infty}\prod\limits_{k=1}^{p}(R^{2j}\varsigma-a_k)^{a_k^0-1}$.

Based on the results given in [4, §78], we conclude that the function $e^{\gamma(\varsigma)}$ near the points $b_k(k=1,2)$ can be written in the form

$$e^{\gamma(\varsigma)}=(\varsigma-b_1)^{\beta^0-1}(\varsigma-b_2)^{\beta^0-1}\Omega^0(\varsigma),$$

where $\beta^0=\beta_1^0=\beta_2^0$, Ω^0 is the function holomorphic near the point b_k and tending to definite nonzero limits as $\varsigma\to b_k$.

Thus, for a conformally mapping function bounded at the points b_1 and b_2 (i.e. of the class $h(b_1,b_2)$ (see [4])), from (4) we obtain the formula

$$\omega'(\varsigma)=K^0(\varsigma-b_1)^{\beta^0-1}(\varsigma-b_2)^{\beta^0-1}\Omega^0(\varsigma)A(\varsigma). \tag{5}$$

When solving mixed problems of the plane theory of elasticity by the method of Kolosov-Muskhelishvili [5], in the case of doubly-connected domains bounded by broken lines it becomes possible to reduce these problems to two Riemann-Hilbert problems for a circular ring with respect to the complex potentials $\varphi(z)$ and $\psi(z)$. By virtue of the well known formulas (see [1], §41), for finding the complex potentials $\varphi(z)$ and $\psi(z)$ we obtain the boundary conditions

$$\mathrm{Re}[e^{-i\nu(t)}\varphi(t)]=f_{j1}(t),\ t\in L_j\ (j=0,1), \tag{6}$$

$$\mathrm{Re}[e^{-i\nu(t)}(\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}]=f_{j2}(t),\ t\in L_j\ (j=0,1), \tag{7}$$

where

$$\nu(t)=\begin{cases}\alpha(t), & t\in L_0,\\ \beta(t), & t\in L_1;\end{cases}$$

$$f_{01}=\frac{1}{\kappa+1}[c(t)+2\mu\nu_n(t)+\mathrm{Re}e^{-i\alpha(t)}(c_0^1+ic_0^2)],\ t\in L_0;$$

$$f_{11}=\frac{1}{\kappa+1}\mathrm{Re}e^{-i\beta(t)}(c_1^1+ic_1^2),\ t\in L_1;$$

$$f_{02}=f_{01}(t)-\frac{2\mu\nu_n(t)}{\kappa+1},\ t\in L_0;$$

$$f_{12}(t)=f_{11}(t),\ t\in L_1(t);$$

$$c(t)=\sum_{r=1}^{k}\int\limits_{L_0^{(r)}}N(\tau)\sin(\alpha_k-\alpha_r)ds=c_k,\ \ t\in L_0^{(k)},\ \ (\overline{k=1,p})$$

$\kappa = \frac{\lambda+3\mu}{\lambda+\mu}$, λ and μ are Lamé's constants, $N(\tau)$ is a normal stress, $c_k(k = \overline{1,p})$ and $c_j^m(j = 0,1,\ m = 1,2)$ are arbitrary real constants. During the solution of the problem these constants must be defined in such away that functions $\varphi(z)$ and $\bar{z}\varphi'(z)+\psi(z)$ were continuously extendable to the closed domain $S+L$.

Let us consider problem (6). After the conformal mapping of the domain S onto the circular ring D , this problem for the function

$$\varphi^*(\varsigma) = \varsigma^{-1}\varphi[\omega(\varsigma)] \equiv \varsigma^{-1}\varphi_0(\varsigma) \tag{8}$$

reduces to the Riemann-Hilbert problem for a circular ring

$$\begin{aligned} &\mathrm{Re}[\sigma e^{-i\alpha_0(\sigma)}\varphi^*(\sigma)] = F_0(\sigma),\ \ \sigma \in l_0; \\ &\mathrm{Re}[\sigma e^{-i\Delta_0(\sigma)}\varphi^*(\sigma)] = F_1(\sigma),\ \ \sigma \in l_1; \end{aligned} \tag{9}$$

where $F_0(\sigma) = f_{01}[\omega(\sigma)],\ \ \sigma \in l_0;\ F_1(\sigma) = f_{11}[\omega(\sigma)],\ \ \sigma \in l_1.$

We easily observe that from the boundary conditions (3) we obtain the factorization of the coefficient of problem (6) in the following form

$$e^{2i\alpha_0(\sigma)}R^2\sigma^{-2} = \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}},\ \ \sigma \in l_0,$$

$$e^{2i\Delta_0(\sigma)}\sigma^{-2} = \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}},\ \ \sigma \in l_1,$$

where $\omega'(\varsigma)$ is defined by formula (5) and from the boundary conditions (8) for the function

$$\Omega(\varsigma) = \varphi_0(\varsigma)[\varsigma\omega'(\varsigma)]^{-1} \tag{10}$$

we obtain the Dirichlet problem for a circular ring

$$\begin{aligned} &\mathrm{Re}[\Omega(\sigma)] = F_0(\sigma)e^{i\alpha_0(\sigma)}[\sigma\omega'(\sigma)]^{-1},\ \ \sigma \in l_0, \\ &\mathrm{Re}[\Omega(\sigma)] = F_1(\sigma)e^{i\Delta_0(\sigma)}[\sigma\omega'(\sigma)]^{-1},\ \ \sigma \in l_1. \end{aligned} \tag{11}$$

A solvability condition of problem (11) has the form

$$\int\limits_{l_0} F_0(\sigma)e^{i\alpha_0(\sigma)}[\sigma\omega'(\sigma)]^{-1}d\sigma = \int\limits_{l_1} F_1(\sigma)e^{i\Delta_0(\sigma)}[\sigma\omega'(\sigma)]^{-1}d\sigma \tag{12}$$

and its solution is given by the formula

$$\Omega(\varsigma) = M(\varsigma), \tag{13}$$

where

$$M(\varsigma) = \frac{1}{\pi i}\sum_{j=-\infty}^{\infty}\left[\int_{l_0}\frac{F_0(\sigma)e^{i\alpha_0(\sigma)}[\sigma\omega'(\sigma)]^{-1}}{\sigma - R^{2j}\varsigma}d\sigma + \int_{l_1}\frac{F_1(\sigma)e^{i\Delta_0(\sigma)}[\sigma\omega'(\sigma)]^{-1}}{\sigma - R^{2j}\varsigma}d\sigma\right] + iE_1 \tag{14}$$

where E_1 is an arbitrary real constant.

Thus, using (10) and (13), for the function $\varphi_0(\varsigma)$ we obtain the formula

$$\varphi_0(\varsigma) = \varsigma\omega'(\varsigma)M(\varsigma). \tag{15}$$

Taking into account the form of the function $\omega'(\varsigma)$ in the neighborhood of the point $a_k(k = \overline{1,p})$, we conclude that for the continuous extension of the function $\varphi_0(\varsigma)$ in the domain $D + l$ it is necessary that the conditions

$$M(a_k) = 0, \quad k = \overline{1,p}. \tag{16}$$

Since $\varphi'(z) = \varphi_0'(\varsigma)[\omega'(\varsigma)]^{-1}$, from (15) we have

$$\varphi'(z) = M(\varsigma) + \varsigma\frac{\omega''(\varsigma)}{\omega'(\varsigma)}M(\varsigma) + \varsigma M'(\varsigma). \tag{17}$$

Based on the results obtained in [4, §26] as to the behavior of a Cauchy type integral near the density discontinuity points, we conclude that near the points b_k $(k = 1,2)$ the function $M(\varsigma)$ has the form

$$M(\varsigma) = \frac{K_1^{(k)}}{(\varsigma - b_k)^{\beta^0 - 1}} + M_k^0(\varsigma), \; k = 1,2$$

where $M_k^0(\varsigma)$ is the function that near the point b_k admits the following estimate

$$|M_k^0(\varsigma)| < \frac{C}{|\varsigma - b_k|^{\delta_0}}, \;\; C = const, \;\; \delta < \beta^0 - 1,$$

where $K_1^{(k)}$ is the well-defined constant.

Taking into account the behavior of the conformally mapping function near the angular points (see [6], §37), we obtain

$$\omega(\varsigma) = B_k + (\varsigma - b_k)^{\beta^0}\Omega_k(\varsigma)$$

$$\varsigma\frac{\omega''(\varsigma)}{\omega'(\varsigma)} = \frac{b_k(\beta^0 - 1)}{\varsigma - b_k} + \Omega_k^*(\varsigma), \;\; k = 1,2$$

where $\Omega_k(b_k) \neq 0$, $\Omega_k^*(\varsigma)$ is the regular part of the Loran decomposition of the function $\varsigma\frac{\omega''(\varsigma)}{\omega'(\varsigma)}$.

By the above reasoning, from (17) we obtain the estimate

$$\varphi'(z) = \frac{K_0^{(k)}}{(\varsigma - b_k)^{\beta^0 - 1}} + M_0^k(\varsigma), \quad k = 1, 2, \quad K_0^{(k)} = const,$$

and thus near a point B which is one of the points B_k we have the estimates

$$|\varphi'(z)| < M_1|z - B|^{\frac{1}{\beta^0} - 1}, \quad |\varphi''(z)| < M_2|z - B|^{\frac{1}{\beta^0} - 2}, \quad M_1, M_2 = const.$$

By a similar reasoning to the above, it is proved that $\varphi'(z)$ is almost bounded near the points $A_k(k = \overline{1,p})$ (see [4], §77).

After finding the function $\varphi(z)$, the definition of the function $\psi(z)$ by (7) reduces to the following problem which is analogous to problem (6)

$$\begin{aligned} \mathrm{Re}[e^{i\nu(t)}R(t)] &= N_0(t), t \in L_0 \\ \mathrm{Re}[e^{i\nu(t)}R(t)] &= N_1(t), t \in L_1 \end{aligned} \tag{18}$$

where

$$R(z) = \psi(z) + P(z)\varphi'(z),$$
$$N_0(t) = f_{01}(t) - \mathrm{Re}[e^{i\nu(t)}(\overline{\varphi(t)} + (\bar{t} - P(t))\varphi'(t)], \ t \in L_0$$
$$N_1(t) = f_{12}(t) - \mathrm{Re}[e^{i\nu(t)}(\overline{\varphi(t)} + (\bar{t} - P(t))\varphi'(t)], \ t \in L_1$$

and $P(z)$ is an interpolation polynomial satisfying the condition $P(B_k) = \bar{B}_k$ $(k = 1, 2)$, $\bar{B}_k$ is a number conjugate to B_k and having the form

$$P(z) = \frac{z - B_2}{B_1 - B_2}\bar{B}_1 + \frac{z - B_1}{B_2 - B_1}\bar{B}_2$$

Since the functions $N_j(t)(j = 0, 1)$ are bounded, the problem of finding the function $\psi(z)$ reduces to the problem investigated above. The solution of problem (18) can be constructed in the same way as before, while the conditions for this problem to be solvable (the requirement that the function $\psi(z)+P(z)\varphi'(z)$ should be continuously extendable) will have a form similar to conditions (12) and (16). All these conditions are represented as an inhomogeneous system with real coefficient with respect to $(p+4)$ constants c_k $(k = \overline{1,p})$ c_j^m $(j = 0, 1, m = 1, 2)$ (two of these constants such as c_1^1 and c_1^2, can be considered to be zero) and E_1, E_2 (E_2 is a real constant which occurs when solving problem (18)). For the definition of these constant we have $(p + 4)$ equations. It is proved that the obtained system is uniquely solvable and therefore the problem posed has a unique solution.

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