DIFFERENCE SCHEME FOR ONE SYSTEM OF NONLINEAR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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(Received: 14.02.2016; accepted: 05.06.2016)

Abstract

Nonlinear parabolic integro-differential model which is based on Maxwell system is considered. Large time behavior of solutions of the initial-boundary value problem with mixed boundary condition is given. Finite difference scheme is investigated. Wider class of nonlinearity is studied than one has been investigated before.

Key words and phrases: System of nonlinear integro-differential equations, asymptotic behavior, finite difference scheme.

AMS subject classification: 45K05, 65M06.

1 Introduction

Let us consider the following system of nonlinear integro-differential equations

$$\frac{\partial W}{\partial t} + \mathbf{rot}\left[a\left(\int_0^t |\mathbf{rot}\, W|^2\, d\tau\right)\mathbf{rot}\, W\right] = 0. \tag{1.1}$$

The model (1.1) can be obtained by the reduction of system of Maxwell equations [22] to the integro-differential model. At first that reduction was made in [6].

If the magnetic field has the form $W = (0, U, V)$, where $U = U(x,t),\ \ V = V(x,t)$, then we have

$$\mathbf{rot}\, W = \left(0,\ -\frac{\partial V}{\partial x},\ \frac{\partial U}{\partial x}\right)$$

and from (1.1) we obtain the following system of nonlinear integro-differential equations:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left[a\left(\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial U}{\partial x}\right], \\
\frac{\partial V}{\partial t} &= \frac{\partial}{\partial x}\left[a\left(\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial V}{\partial x}\right].
\end{aligned} \tag{1.2}$$

One must note that the systems of type (1.1) and (1.2), as we already mentioned, at first appeared in [6].

Study of the models of type (1.1) has begun in the works [6] and [7]. In those works, in particular, the theorems of existence of solution of the initial-boundary value problem with first kind boundary conditions for scalar and one-dimensional space case when $a(S) = 1 + S$ and uniqueness for more general cases are proven. One-dimensional scalar variant for the case $a(S) = (1 + S)^p$, $0 < p \leq 1$ is studied in [4]. Investigations for multi-dimensional space cases at first are carried out in [3]. Multidimensional space cases are also discussed in the following works [2], [5], [23], [24].

Asymptotic behavior as $t \to \infty$ of solutions of initial-boundary value problems for (1.1) type models are studied in [1], [10], [11], [16] - [18] and in a number of other works as well. In those works main attention is paid to one-dimensional analogs. Two-dimensional case for the (1.2) type so called averaged integro-differential system is considered in [21].

Note that integro-differential parabolic models of (1.1) type are complex and still yields to the investigation only for special cases (see, for example, [1] - [6], [9], [23] - [25], [27] and references therein).

Interest to above-mentioned differential and integro-differential models is more and more increasing and initial-boundary value problems with different kinds of boundary and initial conditions are considered. Particular attention should be paid to construction of numerical solutions and to their importance for integro-differential models. Finite element analogues and Galerkin method algorithm as well as settling of semi-discrete and finite difference schemes for (1.1) type one-dimensional integro-differential models are studied in [9], [12] - [16], [18] - [20], [27] and in other works as well.

The literature on the questions of existence, uniqueness, regularity, asymptotic behavior of the solutions and numerical resolution of the initial-boundary value problems to (1.1) type models and models like it is very rich (see, for example, [18] and references therein).

Investigation of semi-discrete scheme for (1.1) type system for one-dimensional and two component magnetic field is given in [12].

Our aim in this note is to study the fully-discrete finite difference schemes for numerical solution of initial-boundary value problem with mixed boundary conditions for the special case of (1.1) system which is given in (1.2). Attention is paid to the investigation more wide cases of nonlinearity than already were studied. In particular, the following case of the diffusion coefficient is studied $a(S) = (1+S)^p$, $0 < p \le 1$.

The paper is organized as follows. In the second section the statement of the problem unique solvability and large time behavior of solution of corresponding initial-boundary value problem are given. In the third section the finite difference scheme is constructed and its stability and convergence are proved.

2 Unique Solvability and Long-time Behavior of Solution with Mixed Boundary Conditions

In the cylinder $(0,1)\times(0,\infty)$ let us consider the following initial-boundary value problem for system (1.2) for the case $a(S) = (1+S)^p$, $0 < p \le 1$:

$$\begin{aligned}
&\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[\left(1 + \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)^p \frac{\partial U}{\partial x}\right] = 0,\\
&\frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\left[\left(1 + \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)^p \frac{\partial V}{\partial x}\right] = 0,
\end{aligned} \tag{2.1}$$

$$U(0,t) = V(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = \left.\frac{\partial V(x,t)}{\partial x}\right|_{x=1} = 0, \tag{2.2}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \tag{2.3}$$

where $0 < p \le 1$, U_0 and V_0 are given functions.

The following statement of existence and uniqueness of the solution takes place.

Theorem 2.1. If $0 < p \le 1$ *and* U_0, $V_0 \in H_0^2(0,1)$, *then where exists unique solution* (U, V) *of problem (2.1) - (2.3) such that:* $U, V \in L_2\left(0,\infty; H^2(0,1)\right)$, $U_{xt}, V_{xt} \in L_2\left(0,\infty; L_2(0,1)\right)$.

We use usual $L_2(0,1)$ and Sobolev spaces $H^k(0,1)$, $H_0^k(0,1)$.

For proving existence part in theorem above the Galerkin modified method and compactness arguments as in [26], [28] for nonlinear parabolic equations is used. Applying this technique the existence theorems for one-component analogs of (1.1) type integro-differential models are studied in [3] - [7].

As to uniqueness of solution we assume that there exist two different (U_1, V_1) and (U_2, V_2) solutions of problem (2.1) - (2.3) and introduce the differences $Z = U_2 - U_1$ and $R = V_2 - V_1$. To show that $Z = R \equiv 0$ the following identity, analogue of Hadamard formula, is mainly used:

$$\left\{\left(1+\int_0^t \left[\left(\frac{\partial U_2}{\partial x}\right)^2+\left(\frac{\partial V_2}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial U_2}{\partial x}\right.$$

$$\left.-\left(1+\int_0^t \left[\left(\frac{\partial U_1}{\partial x}\right)^2+\left(\frac{\partial V_1}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial U_1}{\partial x}\right\}\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)$$

$$+\left\{\left(1+\int_0^t \left[\left(\frac{\partial U_2}{\partial x}\right)^2+\left(\frac{\partial V_2}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial V_2}{\partial x}\right.$$

$$\left.-\left(1+\int_0^t \left[\left(\frac{\partial U_1}{\partial x}\right)^2+\left(\frac{\partial V_1}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial V_1}{\partial x}\right\}\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\right)$$

$$=\int_0^1 \frac{d}{d\mu}\left(1+\int_0^t\left\{\left[\frac{\partial U_1}{\partial x}+\mu\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)\right]^2\right.\right.$$

$$\left.\left.+\left[\frac{\partial V_1}{\partial x}+\mu\left(\frac{\partial V_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)\right]^2\right\}d\tau\right)^p$$

$$\times\left[\frac{\partial U_1}{\partial x}+\mu\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)\right]d\mu\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)$$

$$+\int_0^1 \frac{d}{d\mu}\left(1+\int_0^t\left\{\left[\frac{\partial U_1}{\partial x}+\mu\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)\right]^2\right.\right.$$

$$\left.\left.+\left[\frac{\partial V_1}{\partial x}+\mu\left(\frac{\partial V_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)\right]^2\right\}d\tau\right)^p$$

$$\times\left[\frac{\partial V_1}{\partial x}+\mu\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\right)\right]d\mu\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\right).$$

The following theorem shows that asymptotic behavior of solution of problem (2.1) - (2.3) has an exponential character. The validity of theorem below can be proven by using methodology analogous as in [1], [10], [11], [16] - [18].

Theorem 2.2. *If* $0 < p \le 1$ *and* $U_0 \in H^3(0,1)$, $U_0(0) = V_0(0) = \frac{dU_0(x)}{dx}\Big|_{x=1} = \frac{dV_0(x)}{dx}\Big|_{x=1} = 0$, *then for the solution of problem (2.1) - (2.3) the following estimates hold as* $t \to \infty$*:*

$$\left|\frac{\partial U(x,t)}{\partial x}\right| + \left|\frac{\partial U(x,t)}{\partial t}\right| \le C\exp\left(-\frac{t}{2}\right),$$

$$\left|\frac{\partial V(x,t)}{\partial x}\right| + \left|\frac{\partial V(x,t)}{\partial t}\right| \le C\exp\left(-\frac{t}{2}\right),$$

uniformly in x *on* $[0,1]$.

Here C denotes positive constant independent of t.

3 Difference Scheme

In the finite rectangle $[0,1]\times[0,T]$, where T is a positive constant let us study difference scheme for initial-boundary value problem (2.1) - (2.3).

On $[0,1]\times[0,T]$ let us introduce a net with mesh points denoted by $(x_i,t_j) = (ih, j\tau)$, where $i = 0,1,...,M$; $j = 0,1,...,N$ with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i,t_j) is designed by $(u_i^j,\, v_i^j)$ and the exact solution to the problem (2.1) - (2.3) by $(U_i^j,\, V_i^j)$. We will use the following known notations [29]:

$$r_{t,i}^j = \frac{r_i^{j+1} - r_i^j}{\tau}, \quad r_{x,i}^j = \frac{r_{i+1}^j - r_i^j}{h}, \quad r_{\bar{x},i}^j = \frac{r_i^j - r_{i-1}^j}{h}.$$

Introduce inner products and norms:

$$(r^j, g^j) = h\sum_{i=1}^{M-1} r_i^j g_i^j, \quad (r^j, g^j] = h\sum_{i=1}^{M} r_i^j g_i^j,$$

$$\|r^j\| = (r^j, r^j)^{1/2}, \quad \|r^j]| = (r^j, r^j]^{1/2}.$$

For the problem (2.1) - (2.3) let us consider the following finite difference scheme:

$$\begin{aligned}
&\frac{u_i^{j+1} - u_i^j}{\tau} - \left\{\left(1 + \tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2\right]\right)^p u_{\bar{x},i}^{j+1}\right\}_x = f_{1,i}^j,\\
&\frac{v_i^{j+1} - v_i^j}{\tau} - \left\{\left(1 + \tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2\right]\right)^p v_{\bar{x},i}^{j+1}\right\}_x = f_{2,i}^j,
\end{aligned} \tag{3.1}$$

$$i = 1,2,...,M-1; \;\; j = 0,1,...,N-1,$$

$$u_0^j = v_0^j = u_{\bar{x},M}^j = v_{\bar{x},M}^j = 0, \quad j = 0, 1, ..., N, \tag{3.2}$$

$$u_i^0 = U_{0,i}, \ , v_i^0 = V_{0,i}, \quad i = 0, 1, ..., M. \tag{3.3}$$

It is not difficult to get the inequalities:

$$\|u^n\|^2 + \sum_{j=1}^{n} \|u_{\bar{x}}^j]|^2 \tau < C, \quad \|v^n\|^2 + \sum_{j=1}^{n} \|v_{\bar{x}}^j]|^2 \tau < C, \\ n = 1, 2, ..., N, \tag{3.4}$$

where here and below C is a positive constant independent from τ and h.

The a priori estimates (3.4) guarantee the stability of the scheme (3.1) - (3.3). Note, that using the analogous technique as proving Theorem 3.1 below, it is easy to prove the uniqueness of the solution of the scheme (3.1) - (3.3) too.

The principal aim of the present section is the proof of the following statement.

Theorem 3.1. *If problem (2.1) - (2.3) has a sufficiently smooth solution $(U(x,t), V(x,t))$, then the solution $u^j = (u_1^j, u_2^j, \dots, u_M^j)$, $v^j = (v_1^j, v_2^j, \dots, v_M^j)$, $j = 1, 2, \dots, N$ of the difference scheme (3.1) - (3.3) tends to the solution of continuous problem (2.1) - (2.3) $U^j = (U_1^j, U_2^j, \dots, U_M^j)$, $V^j = (V_1^j, V_2^j, \dots, V_M^j)$, $j = 1, 2, \dots, N$ as $\tau \to 0$, $h \to 0$ and the following estimates are true:*

$$\|u^j - U^j\| \le C(\tau + h), \quad \|v^j - V^j\| \le C(\tau + h). \tag{3.5}$$

Proof. For $U = U(x,t)$ and $V = V(x,t)$ we have:

$$U_{t,i}^j - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p U_{\bar{x},i}^{j+1} \right\}_x = f_{1,i}^j + \psi_{1,i}^j,$$

$$V_{t,i}^j - \left\{ \left(1 + \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p V_{\bar{x},i}^{j+1} \right\}_x = f_{2,i}^j + \psi_{2,i}^j, \tag{3.6}$$

$$i = 1, 2, \dots, M-1,$$

$$U_0(t) = V_0(t) = U_{\bar{x},M}(t) = V_{\bar{x},M}(t) = 0, \tag{3.7}$$

$$U_i(0) = U_{0,i}, \quad V_i(0) = V_{0,i}, \quad i = 0, 1, \dots, M, \tag{3.8}$$

where

$$\psi_{k,i} = O(\tau + h), \quad k = 1, 2.$$

Let $z_i^j = u_i^j - U_i^j$ and $w_i^j(t) = v_i^j - V_i^j$. From (2.1) - (2.3) and (3.6) - (3.8) we have:

$$
\begin{aligned}
& z_{t,i}^j - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p u_{\bar{x},i}^{j+1} \right. \\
& \left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p U_{\bar{x},i}^{j+1} \right\}_x = -\psi_{1,i}^j, \\
& w_{t,i}^j - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p v_{\bar{x},i}^{j+1} \right. \\
& \left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p V_{\bar{x},i}^{j+1} \right\}_x = -\psi_{2,i}^j, \\
& z_0^j = w_0^j = z_{\bar{x},M}^j = w_{\bar{x},M}^j = 0, \\
& z_i^0 = w_i^0 = 0.
\end{aligned}
\tag{3.9}
$$

Multiplying the first equation of system (3.9) scalarly by $\tau z^{j+1} = \tau(z_1^{j+1}, z_2^{j+1}, \ldots, z_{M-1}^{j+1})$, using the discrete analogue of the formula of integration by parts we get

$$
\begin{aligned}
& \|z^{j+1}\|^2 - (z^{j+1}, z^j) + \tau h \sum_{i=1}^{M} \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p u_{\bar{x},i}^{j+1} \right. \\
& \left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p U_{\bar{x},i}^{j+1} \right\} z_{\bar{x},i}^{j+1} = -\tau(\psi_1^j, z^{j+1}),
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
& \|w^{j+1}\|^2 - (w^{j+1}, w^j) + \tau h \sum_{i=1}^{M} \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p v_{\bar{x},i}^{j+1} \right. \\
& \left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p V_{\bar{x},i}^{j+1} \right\} w_{\bar{x},i}^{j+1} = -\tau(\psi_2^j, w^{j+1}).
\end{aligned}
$$

Adding these two equalities we have

$$
\begin{gathered}
\|z^{j+1}\|^2-(z^{j+1},z^j)+\|w^{j+1}\|^2-(w^{j+1},w^j)\\
+\tau h\sum_{i=1}^{M}\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)^p u_{\bar{x},i}^{j+1}\right.\\
\left.-\left(1+\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)^p U_{\bar{x},i}^{j+1}\right\}z_{\bar{x},i}^{j+1}\\
+\tau h\sum_{i=1}^{M}\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)^p v_{\bar{x},i}^{j+1}\right.\\
\left.-\left(1+\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)^p V_{\bar{x},i}^{j+1}\right\}w_{\bar{x},i}^{j+1}\\
=-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).
\end{gathered}
\tag{3.10}
$$

Note that,

$$
\begin{gathered}
\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)^p u_{\bar{x},i}^{j+1}-\right.\\
\left.\left(1+\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)^p U_{\bar{x},i}^{j+1}\right\}\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)\\
+\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)^p v_{\bar{x},i}^{j+1}-\right.\\
\left.\left(1+\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)^p V_{\bar{x},i}^{j+1}\right\}\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)\\
=\int_0^1\frac{d}{d\mu}\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]^2\right.\right.\\
\left.\left.+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]^2\right\}\right)^p\\
\times\left[U_{\bar{x},i}^{j+1}+\mu(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1})\right]d\mu\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)
\end{gathered}
$$

$$+\int_0^1 \frac{d}{d\mu}\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$\left.\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)^p$$

$$\times\left[V^{j+1}_{\bar{x},i}+\mu(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i})\right]d\mu\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)$$

$$=2p\int_0^1\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$\left.\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)^{p-1}$$

$$\times\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]\left(u^k_{\bar{x},i}-U^k_{\bar{x},i}\right)+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]\right.$$

$$\left.\times\left(v^k_{\bar{x},i}-V^k_{\bar{x},i}\right)\right\}\left[U^{j+1}_{\bar{x},i}+\mu(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i})\right]d\mu\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)$$

$$+\int_0^1\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$\left.\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)^p$$

$$\times\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)d\mu\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)$$

$$+2p\int_0^1\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$\left.\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)^{p-1}$$

$$\times\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]\left(u^k_{\bar{x},i}-U^k_{\bar{x},i}\right)\right.$$

$$\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]\left(v^k_{\bar{x},i}-V^k_{\bar{x},i}\right)\right\}$$

$$\times\left[V^{j+1}_{\bar{x},i}+\mu(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i})\right]d\mu\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)$$

$$+\int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$\left.\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)^p$$

$$\times\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)d\mu\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)$$

$$=2p\int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$\left.\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)^{p-1}$$

$$\times\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]\left(u^k_{\bar{x},i}-U^k_{\bar{x},i}\right)\right.$$

$$\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]\left(v^k_{\bar{x},i}-V^k_{\bar{x},i}\right)\right\}$$

$$\times\left\{\left[U^{j+1}_{\bar{x},i}+\mu(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i})\right]\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)\right.$$

$$\left.+\left[V^{j+1}_{\bar{x},i}+\mu(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i})\right]d\mu\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)\right\}d\mu$$

$$+\int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$\left.\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)^p$$

$$\times\left[\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)^2+\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)^2\right]d\mu$$

$$=2p\int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$\left.\left.+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)^{p-1}\xi^{j+1}_i(\mu)\xi^j_{t,\,i}(\mu)\,d\mu$$

$$+\int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2\right.\right.$$

$$+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\Big\}\Big)^p$$

$$\times\left[\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)^2+\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)^2\right]d\mu,$$

where

$$\xi^{j+1}_i(\mu)=\tau\sum_{k=1}^{j+1}\Big\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]\left(u^k_{\bar{x},i}-U^k_{\bar{x},i}\right)$$

$$+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]\left(v^k_{\bar{x},i}-V^k_{\bar{x},i}\right)\Big\},$$

$$\xi^0_i(\mu)=0$$

and therefore,

$$\xi^j_{t,\,i}(\mu)=\left[U^{j+1}_{\bar{x},i}+\mu(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i})\right]\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)$$

$$+\left[V^{j+1}_{\bar{x},i}+\mu(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i})\right]\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right).$$

Introducing the following notation

$$s^{j+1}_i(\mu)=\tau\sum_{k=1}^{j+1}\Big\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\Big\},$$

we have from the previous equality

$$\Big\{\Big(1+\tau\sum_{k=1}^{j+1}\left[(u^k_{\bar{x},i})^2+(v^k_{\bar{x},i})^2\right]\Big)^p u^{j+1}_{\bar{x},i}$$

$$-\Big(1+\tau\sum_{k=1}^{j+1}\left[(U^k_{\bar{x},i})^2+(V^k_{\bar{x},i})^2\right]\Big)^p U^{j+1}_{\bar{x},i}\Big\}\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)$$

$$+\Big\{\Big(1+\tau\sum_{k=1}^{j+1}\left[(u^k_{\bar{x},i})^2+(v^k_{\bar{x},i})^2\right]\Big)^p v^{j+1}_{\bar{x},i}$$

$$-\Big(1+\tau\sum_{k=1}^{j+1}\left[(U^k_{\bar{x},i})^2+(V^k_{\bar{x},i})^2\right]\Big)^p V^{j+1}_{\bar{x},i}\Big\}\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)$$

$$=2p\int_0^1\left(1+s^{j+1}_i(\mu)\right)^{p-1}\xi^{j+1}_i\xi^j_{t,\,i}\,d\mu$$

$$+\int_0^1 \left(1+s_i^{j+1}(\mu)\right)^p \left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right] d\mu.$$

After substituting this equality in (3.10) we get

$$\begin{gathered}
\|z^{j+1}\|^2-(z^{j+1},z^j)+\|w^{j+1}\|^2-(w^{j+1},w^j) \\
+2\tau hp\sum_{i=1}^{M}\int_0^1 \left(1+s_i^{j+1}(\mu)\right)^{p-1} \xi_i^{j+1}\xi_{t,\,i}^j\, d\mu \\
+\tau h\sum_{i=1}^{M}\int_0^1 \left(1+s_i^{j+1}(\mu)\right)^p \left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2\right. \\
\left.+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right] d\mu=-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).
\end{gathered} \tag{3.11}$$

Tacking into account restriction $p>0$ and relations $s_i^{j+1}(\mu)\geq 0$,

$$(r^{j+1},r^j)=\frac{1}{2}\|r^{j+1}\|^2+\frac{1}{2}\|r^j\|^2-\frac{1}{2}\|r^{j+1}-r^j\|^2,$$

$$\tau\,\xi_i^{j+1}\xi_{t,\,i}^j=\frac{1}{2}\left(\xi_i^{j+1}\right)^2-\frac{1}{2}\left(\xi_i^j\right)^2+\frac{\tau^2}{2}\left(\xi_{t,\,i}^j\right)^2,$$

we have from (3.11)

$$\begin{gathered}
\|z^{j+1}\|^2-\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2+\frac{1}{2}\|z^{j+1}-z^j\|^2 \\
+\|w^{j+1}\|^2-\frac{1}{2}\|w^{j+1}\|^2-\frac{1}{2}\|w^j\|^2+\frac{1}{2}\|w^{j+1}-w^j\|^2 \\
+hp\sum_{i=1}^{M}\int_0^1 \left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right] d\mu \\
+\tau^2 hp\sum_{i=1}^{M}\int_0^1 \left(1+s_i^{j+1}(\mu)\right)^{p-1}\left(\xi_{t,\,i}^j\right)^2 d\mu \\
+\tau h\sum_{i=1}^{M}\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right] \\
\leq -\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).
\end{gathered} \tag{3.12}$$

From (3.12) we arrive at

$$\frac{1}{2}\|z^{j+1}\|^2 - \frac{1}{2}\|z^j\|^2 + \frac{\tau^2}{2}\|z_t^j\|^2$$

$$+\frac{1}{2}\|w^{j+1}\|^2 - \frac{1}{2}\|w^j\|^2 + \frac{\tau^2}{2}\|w_t^j\|^2$$

$$+hp\sum_{i=1}^{M}\int_0^1 \left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu \tag{3.13}$$

$$+\tau\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)$$

$$\le \frac{\tau}{2}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right)+\frac{\tau}{2}\left(\|z^{j+1}\|^2+\|w^{j+1}\|^2\right).$$

Using discrete analogue of Poincare inequality [29]

$$\|r^{j+1}\|^2 \le \|r_{\bar{x}}^{j+1}]|^2$$

from (3.13) we get

$$\|z^{j+1}\|^2-\|z^j\|^2+\tau^2\|z_t^j\|^2+\|w^{j+1}\|^2-\|w^j\|^2+\tau^2\|w_t^j\|^2$$

$$+2hp\sum_{i=1}^{M}\int_0^1 \left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu \tag{3.14}$$

$$+\tau\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\tau\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right).$$

Summing (3.14) from $j=0$ to $j=n-1$ we arrive at

$$\|z^n\|^2+\tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2+\|w^n\|^2+\tau^2\sum_{j=0}^{n-1}\|w_t^j\|^2$$

$$+2hp\sum_{j=0}^{n-1}\sum_{i=1}^{M}\int_0^1 \left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu \tag{3.15}$$

$$+\tau\sum_{j=0}^{n-1}\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\tau\sum_{j=0}^{n-1}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right).$$

Note, that since $s_i^{j+1}(\mu) \geq s_i^j(\mu)$ and $p \leq 1$, for the second line of the last formula we have

$$\sum_{j=0}^{n-1} \left(1 + s_i^{j+1}(\mu)\right)^{p-1} \left[\left(\xi_i^{j+1}\right)^2 - \left(\xi_i^j\right)^2\right]$$

$$= \left(1 + s_i^1(\mu)\right)^{p-1} \left(\xi_i^1\right)^2 - \left(1 + s_i^1(\mu)\right)^{p-1} \left(\xi_i^0\right)^2$$

$$+ \left(1 + s_i^2(\mu)\right)^{p-1} \left(\xi_i^2\right)^2 - \left(1 + s_i^2(\mu)\right)^{p-1} \left(\xi_i^1\right)^2$$

$$+ \cdots + (1 + s_i^n(\mu))^{p-1} \left(\xi_i^n\right)^2 - (1 + s_i^n(\mu))^{p-1} \left(\xi_i^{n-1}\right)^2$$

$$= (1 + s_i^n(\mu))^{p-1} (\xi_i^n)^2 + \sum_{j=1}^{n-1} \left[\left(1 + s_i^j(\mu)\right)^{p-1} - \left(1 + s_i^{j+1}(\mu)\right)^{p-1}\right] \left(\xi_i^j\right)^2 \geq 0.$$

Taking into account the last relation and (3.16) one can deduce

$$\begin{gathered}\|z^n\|^2 + \|w^n\|^2 + \tau^2 \sum_{j=0}^{n-1} \|z_t^j\|^2 + \tau^2 \sum_{j=0}^{n-1} \|w_t^j\|^2 \\ +\tau \sum_{j=0}^{n-1} \left(\|z_{\bar{x}}^{j+1}]|^2 + \|w_{\bar{x}}^{j+1}]|^2\right) \leq \tau \sum_{j=0}^{n-1} \left(\|\psi_1^j\|^2 + \|\psi_2^j\|^2\right).\end{gathered} \tag{3.16}$$

From (3.16) we get (3.5), and Theorem 3.1 is proved.

Some numerical experiments for different initial and boundary data are carried out. All experiments were performed by using software FreeFem++ [8]. The results of numerical experiments agree with theoretical ones.

Acknowledgments

The authors thank Shota Rustaveli National Science Foundation and France National Center for Scientific Research (grant N CNRS/SRNSF 2013, 04/26) for the financial support.

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