SPLITTING OF THE SEMI-DISCRETE SCHEMES OF SOLUTION OF THE EVOLUTIONARY EQUATION WITH VARIABLE OPERATOR ON TWO-LEVEL SCHEMES AND ESTIMATION OF THE APPROXIMATE SOLUTION ERROR

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Abstract

In the present paper, using a perturbation algorithm, purely implicit three-level semi-discrete scheme of abstract evolutionary equation with variable operator reduced to two-level schemes. Using the solutions of these two-level schemes an approximate solution to the original problem is constructed. Using the associated polynomials, approximate solution error are proved in the Hilbert space.

Key words and phrases: Evolutionary problem with variable operator, implicit semi-discrete schemes, perturbation algorithm, estimate approximate solution error.

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Introduction

In the present paper, purely implicit three-level semi-discrete schemes for an approximate solution of the Cauchy problem for an evolutionary equation with variable operator is considered in the Hilbert space. Using the perturbation algorithm, the considered scheme is reduced to two-level schemes. An approximate solution of the original problem is constructed by means of the solutions of these schemes. Note that the first two-level scheme gives an approximate solution to an accuracy of first order, whereas the solution of second two-level scheme is the refinement of the preceding solution by one order.

Questions connected with the construction and investigation of approximate solution algorithms of evolutionary problems are considered for example in the well-known books by S. K. Godunov and V. S. Ryabenki [1],

G. I. Marchuk [2], R. Richtmayer and K. Morton [3], A. A. Samarski [4], N. N. Yanenko [5].

The main difficulty which arises at realization of multi-layer schemes (especially for multidimensional problems), consists in use of large random access memory, which increases in proportion with growth of number of layers. One of opportunities of overcoming this problem is decomposition of multi-layer schemes.

The papers [6], [7], [8] are devoted to questions of splitting of the purely implicit three-level semi-discrete scheme for the evolutionary equation with constant operator. In these papers a purely implicit three-level semi-discrete scheme for evolutionary equation is reduced to two two-level schemes and explicit estimates for the approximate solution at rather general assumptions about data the tasks are proved in Banach space. Furthermore, in these works, by reducing with the aid of the perturbation algorithm the four-layer scheme to two-layer schemes we demonstrate the generality of the algorithm when it is applied to difference schemes.

We would note, in the present work for an estimate of the error of the approximate solution, we applied the approach offered in [9], where the stability of linear many-step methods is investigated by means of the properties of the class of polynomials of many variables (which are called associated polynomials). They are a natural generalization of classical Chebyshev polynomials of second kind.

We would emphasize that the application of the perturbation algorithm to difference schemes for differential equations was considered in [10]. The perturbation algorithm is widely used for solving problems of mathematical physics (e.g., see [11]).

1 Splitting of the three-level scheme

In the Hilbert space H, consider the evolution problem

$$\frac{du(t)}{dt} + A(t)u\,(t) = f(t), \quad t \in \left]0, T\right], \tag{1.1}$$

$$u\,(0) = u_0, \tag{1.2}$$

where $A(t)$ –is the self-adjoint positively defined operator in H with the domain of definition $D(A)$ does not depend on t, $f(t)$ – is a continuously differentiable abstract function taking values in H, u_0 –is a given vector in H and $u(t)$ –is the function to be found.

On the interval $[0, T]$, we define the grid $t_k = k\tau$, $k = 0, 1, \ldots, n$, with the step $\tau = T/n$. We will use the approximation of the first-order derivative

$$\left.\frac{du}{dt}\right|_{t=t_{k+1}} = \frac{\frac{3}{2}u(t_{k+1}) - 2u(t_k) + \frac{1}{2}u(t_{k-1})}{\tau} + \tau^2 R_{k+1}(\tau, u),$$

$$R_k(\tau, u) \in H.$$

As a result, at the point $t = t_{k+1}$, Eq.(1.1) can be represented in the form

$$\begin{gathered}\frac{\frac{3}{2}u(t_{k+1}) - 2u(t_k) + \frac{1}{2}u(t_{k-1})}{\tau} + A(t_{k+1})u\,(t_{k+1}) \\ = f(t_{k+1}) - \tau^2 R_{k+1}(\tau, u), \quad k = 1, \ldots, n-1.\end{gathered} \tag{1.3}$$

Write system (1.3) in the form

$$\frac{u(t_{k+1}) - u(t_k)}{\tau} + A(t_{k+1})u\,(t_{k+1})$$

$$+\frac{\tau}{2}\left(\frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2}\right) = f(t_{k+1}) - \tau^2 R_{k+1}(\tau, u).$$

It is obvious that expression in brackets in case of $\frac{\tau}{2}$ is $u''(t_k) + \tau^2 R_{1,k}$, $R_{1,k} \in H$.

By analogy with this system, we consider the one-parameter family of equations

$$\begin{gathered}\frac{u_{k+1} - u_k}{\tau} + A(t_{k+1})u_{k+1} + \frac{\varepsilon}{2}\left(\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2}\right) \\ = f_{k+1} + \varepsilon^2 R_{k+1},\end{gathered} \tag{1.4}$$

$$f_{k+1} = f(t_{k+1}), \;\; R_k \in H,$$

in the Hilbert space H.

Assume that u_k is analytic in ε,

$$u_k = \sum_{j=0}^{\infty} \varepsilon^j u_k^{(j)}. \tag{1.5}$$

Plug (1.5) into (1.4) and equate the coefficients of the equal powers of ε to obtain

$$\begin{gathered}\frac{u_{k+1}^{(0)} - u_k^{(0)}}{\tau} + A(t_{k+1})u_{k+1}^{(0)} = f_{k+1}, \\ u_0^{(0)} = u_0, \quad k = 0, \ldots, n-1,\end{gathered} \tag{1.6}$$

$$\begin{gathered}\tfrac{u_{k+1}^{(1)} - u_k^{(1)}}{\tau} + A(t_{k+1})u_{k+1}^{(1)} = -\tfrac{1}{2}\,\tfrac{\Delta^2 u_{k-1}^{(0)}}{\tau^2}, \\ k = 1, \ldots, n-1,\end{gathered} \tag{1.7}$$

$$\frac{u_{k+1}^{(2)} - u_k^{(2)}}{\tau} + A(t_{k+1})u_{k+1}^{(2)} = -\frac{1}{2}\frac{\Delta^2 u_{k-1}^{(1)}}{\tau^2} + R_{k+1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $\Delta u_k = u_{k+1} - u_k$.

Introduce the notation

$$v_k = u_k^{(0)} + \tau u_k^{(1)}, \quad k = 2, \ldots, n. \tag{1.8}$$

Consider the vector v_k as an approximate value of exact solution of problem (1.1), (1.2) at $t = t_k$, $u(t_k) \approx v_k$.

Note that the initial vector $u_1^{(1)}$ in scheme (1.7) is found from the equation $v_1 = u_1^{(0)} + \tau u_1^{(1)}$, where $u_1^{(0)}$ is found by scheme (1.6), and v_1 –the approximate value of $u(t_1)$ with accuracy of $O(\tau^2)$.

We prove the following result (let below everywhere c be a positive constant).

Theorem 1.1. *Let $A(t)$ be a self-adjoint positively defined operator in H with the domain of definition $D(A)$ does not depend on t and let the solution $u(t)$ of the problem* (1.1), (1.2) *be a smooth enough function. Then, if*

(a) $D(A^m(t)) = D(A^m(0)), \quad m = 2, 3;$

(b) $\|(A^m(t') - A^m(t''))A^{-m}(s)\| \le c|t' - t''|, \quad \forall\, t', t'', s \in [0, T], \quad m = 1, 2;$

(c) $\left\|(A(t_{k+1}) - 2A(t_k) + A(t_{k-1}))A^{-1}(t_{k+1})\right\| \le c\tau^2, \;\; k = 1, \ldots, n-1.$

Then, it holds that

$$\|u(t_k) - v_k\| = O(\tau^2), \qquad k = 1, \ldots, n.$$

Remark 1.2. From condition (a) of theorem 1 for any $t, s \in [0, T]$ we have

$$\|A(t)^m A^{-m}(s)\| \le c, \quad m = 1, 2, 3. \tag{1.9}$$

2 Estimate of the residual

We will estimate the residual of scheme (1.3) by substituting the solution v_k, which is determined by the scheme (1.6)-(1.8).

Multiplying (1.7) by τ and summing the result with (1.6), we obtain that v_k is a solution of the following system of equations:

$$\frac{v_{k+1}-v_k}{\tau}+A(t_{k+1})v_{k+1}=f_{k+1}-\frac{\tau}{2}\frac{\Delta^2 u_{k-1}^{(0)}}{\tau^2}, \qquad k=1,\dots,n-1. \quad (2.1)$$

Rewrite this system in the form

$$\frac{v_{k+1}-v_k}{\tau}+A(t_{k+1})v_{k+1}+\frac{\tau}{2}\frac{\Delta^2 v_{k-1}}{\tau^2}=f_{k+1}+\widetilde{R}_{k+1}(\tau), \quad (2.2)$$

where $k \geq 2$,

$$\widetilde{R}_{k+1}(\tau)=\frac{\tau}{2}\frac{\Delta^2 v_{k-1}}{\tau^2}-\frac{\tau}{2}\frac{\Delta^2 u_{k-1}^{(0)}}{\tau^2}.$$

It is obvious that (2.2) can be represented in the following form:

$$\begin{gathered}\frac{\frac{3}{2}v_{k+1}-2v_k+\frac{1}{2}v_{k-1}}{\tau}+A(t_{k+1})v_{k+1}=f_{k+1}+\widetilde{R}_{k+1}(\tau),\\ k=2,\dots,n-1.\end{gathered} \quad (2.3)$$

Therefore $\widetilde{R}_{k+1}(\tau)$ is residual of purely implicit three-level scheme for solutions of scheme (1.6)-(1.8) (see. (1.3)).

It obvious that

$$\widetilde{R}_{k+1}(\tau)=\frac{\tau}{2}\frac{\Delta^2 v_{k-1}}{\tau^2}-\frac{\tau}{2}\frac{\Delta^2 u_{k-1}^{(0)}}{\tau^2}=\frac{\tau^2}{2}\frac{\Delta^2 u_{k-1}^{(1)}}{\tau^2}. \quad (2.4)$$

Note that representation (2.4) is true for $k>1$.

We will estimate the difference relation (2.4).

From (1.6) follows

$$u_{k+1}^{(0)}=S_{k+1}u_k^{(0)}+\tau S_{k+1}f_{k+1}, \ \ k=0,\dots,n-1. \quad (2.5)$$

where $S_k=(I+\tau A_k)^{-1}, \quad A_k=A(t_k)$.

We introduce the notation $T(k,l)=S_kS_{k-1}\dots S_l, \ k\geq l$.

Then from recurrence relation (2.5) we have

$$\begin{aligned}u_{k+1}^{(0)}=&\,T(k+1,1)u_0+\tau\left(T(k+1,1)f_1+T(k+1,2)f_2+\dots\right.\\ &\left.+T(k+1,k-1)f_{k-1}+S_{k+1}S_kf_k+S_{k+1}f_{k+1}\right).\end{aligned}$$

From this equality we have

$$\begin{gathered}\Delta^2 u_{k-1}^{(0)} = u_{k+1}^{(0)} - 2u_k^{(0)} + u_{k-1}^{(0)} = (T(k+1,1) - 2T(k,1) \\ +T(k-1,1))\, u_0 + \tau\left[(T(k+1,1) - 2T(k,1)\right. \\ +T(k-1,1))\, f_1 + (T(k+1,2) - 2T(k,2) + T(k-1,2))\, f_2 \\ + \ldots + (T(k+1,k-1) - 2T(k,k-1) + T(k-1,k-1))\, f_{k-1} \\ \left.+ (S_{k+1}S_k - 2S_k)\, f_k + S_{k+1} f_{k+1}\right].\end{gathered} \tag{2.6}$$

As

$$\begin{gathered}T(k+1,l) - 2T(k,l) + T(k-1,l) \\ = \left(I - 2S_{k+1}^{-1} + S_k^{-1}S_{k+1}^{-1}\right) T(k+1,l) \\ = \tau\left(A_k - A_{k+1} + \tau A_k A_{k+1}\right) T(k+1,l), \\ l = 1, \ldots, k-1.\end{gathered} \tag{2.7}$$

and

$$\begin{gathered}(S_{k+1}S_k - 2S_k)\, f_k + S_{k+1} f_{k+1} = -\tau A_{k+1} S_{k+1} S_k f_k \\ +(S_{k+1} - S_k) f_{k+1} + S_k(f_{k+1} - f_k) = -\tau A_{k+1} S_{k+1} S_k f_k \\ +\tau S_k (A_k - A_{k+1}) S_{k+1} f_{k+1} + S_k(f_{k+1} - f_k),\end{gathered} \tag{2.8}$$

that (2.6) will take the following form

$$\begin{gathered}\Delta^2 u_{k-1}^{(0)} = \tau\left(A_k - A_{k+1} + \tau A_k A_{k+1}\right)\left[T(k+1,1) u_0\right. \\ \left.+\tau\left(T(k+1,1) f_1 + T(k+1,2) f_2 + \ldots + T(k+1,k-1) f_{k-1}\right)\right] \\ -\tau^2 A_{k+1} S_{k+1} S_k f_k - \tau^2 S_k (A_{k+1} - A_k) S_{k+1} f_{k+1} \\ +\tau S_k (f_{k+1} - f_k).\end{gathered} \tag{2.9}$$

Taking into account (2.9), from (1.7) we get:

$$u_{k+1}^{(1)} = S_{k+1} u_k^{(1)} + \tau S_{k+1} g_{k+1}, \quad k = 1, \ldots, n-1 \tag{2.10}$$

where

$$\begin{gathered}g_{k+1} = \left(A_k - A_{k+1} + \tau A_k A_{k+1}\right)\left[-\frac{1}{2\tau} T(k+1,1) u_0\right. \\ \left.-\frac{1}{2}\left(T(k+1,1) f_1 + T(k+1,2) f_2 + \ldots + T(k+1,k-1) f_{k-1}\right)\right] \\ +\frac{1}{2} A_{k+1} S_{k+1} S_k f_k + \frac{1}{2} S_k \left((A_{k+1} - A_k) A_{k+1}^{-1}\right) S_{k+1} A_{k+1} f_{k+1} \\ -\frac{1}{2\tau} S_k (f_{k+1} - f_k).\end{gathered} \tag{2.11}$$

From (2.10) we get

$$u_{k+1}^{(1)} = T(k+1,2)u_1^{(1)} + \tau\left(T(k+1,2)g_2 + T(k+1,3)g_3 + \cdots \right. \\ \left. + T(k+1,k-1)g_{k-1} + S_{k+1}S_kg_k + S_{k+1}g_{k+1}\right). \tag{2.12}$$

From here, taking into account (2.7) and (2.8), we get

$$\begin{aligned}
&\Delta^2 u_{k-1}^{(1)} = \tau\left(A_k - A_{k+1} + \tau A_kA_{k+1}\right)\Big[T(k+1,2)u_1^{(1)} \\
&+\tau\left(T(k+1,2)g_2 + T(k+1,3)g_3 + \ldots + T(k+1,k-1)g_{k-1}\right)\Big] \\
&-\tau^2 A_{k+1}S_{k+1}S_kg_k - \tau^2 S_k(A_{k+1}-A_k)S_{k+1}g_{k+1} \\
&+\tau S_k(g_{k+1}-g_k) = \tau\left(A_k - A_{k+1} + \tau A_kA_{k+1}\right)\Big[T(k+1,2)u_1^{(1)} \\
&+\tau\left(T(k+1,2)g_2 + T(k+1,3)g_3 + \ldots + T(k+1,k-1)g_{k-1}\right)\Big] \\
&-\tau^2 S_{k+1}(A_{k+1}A_k^{-1})S_kA_kg_k - \tau^2 S_k((A_{k+1}-A_k)A_{k+1}^{-1})S_{k+1} \\
&\times A_{k+1}g_{k+1} + \tau S_k(g_{k+1}-g_k).
\end{aligned} \tag{2.13}$$

Due to condition (b) of Theorem 1.1 from (2.11) it follows that norm of g_k is uniformly bounded. Taking this fact and condition (b) of Theorem 1 into account, it is obvious that all components in expression (2.13), behind an exception $\tau S_k(g_{k+1}-g_k)$, are order of $O(\tau^2)$.

From (2.11), for $g_{k+1}-g_k$ we have

$$\begin{aligned}
&g_{k+1} - g_k = -\frac{1}{2\tau}Q_{k,1}u_0 - \frac{1}{2}\left(Q_{k,1}f_1 + Q_{k,2}f_2\right. \\
&\left. + \ldots + Q_{k,k-1}f_{k-1}\right) + \frac{1}{2}\left(A_{k+1}S_{k+1}S_kf_k - A_kS_kS_{k-1}f_{k-1}\right) \\
&+\frac{1}{2}\left(S_k\left((A_{k+1}-A_k)A_{k+1}^{-1}\right)S_{k+1}A_{k+1}f_{k+1}\right. \\
&\left.-S_{k-1}\left((A_k - A_{k-1})A_k^{-1}\right)S_kA_kf_k\right) \\
&-\frac{1}{2\tau}\left(S_k(f_{k+1}-f_k) - S_{k-1}(f_k - f_{k-1})\right)
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
Q_{k,l} &= (A_k - A_{k+1} + \tau A_kA_{k+1})T(k+1,l) \\
&\quad -(A_{k-1} - A_k + \tau A_{k-1}A_k)T(k,l) \\
&= \left(A_k - A_{k+1} + \tau A_kA_{k+1}\right)\left(T(k+1,l) - T(k,l)\right) \\
&\quad + \left(A_k - A_{k+1} + \tau A_kA_{k+1} - A_{k-1} + A_k - \tau A_{k-1}A_k\right)T(k,l) \\
&= -\tau(A_k - A_{k+1} + \tau A_kA_{k+1})A_{k+1}T(k+1,l)
\end{aligned}$$

$$-(A_{k+1}-2A_k+A_{k-1}+\tau(A_{k-1}A_k-A_kA_{k+1}))T(k,l)$$
$$=\tau\left(\left((A_{k+1}-A_k)A_{k+1}^{-1}\right)-\tau\left(A_kA_{k+1}^{-1}\right)A_{k+1}^2\right)A_{k+1}T(k+1,l)$$
$$-\left(\left((A_{k+1}-2A_k+A_{k-1})A_{k+1}^{-1}\right)A_{k+1}\right.$$
$$\left.-\tau\left((A_{k-1}(A_{k+1}-A_k))+((A_k-A_{k-1})A_{k+1})\right)\right)T(k,l)$$
$$=\tau\left(\left((A_{k+1}-A_k)A_{k+1}^{-1}\right)\right.$$
$$\left.-\tau\left(A_kA_{k+1}^{-1}\right)A_{k+1}^2\right)A_{k+1}T(k+1,l)-\left(\left((A_{k+1}-2A_k+A_{k-1})A_{k+1}^{-1}\right)A_{k+1}\right.$$
$$+\tau\left((A_{k-1}A_k^{-1})\left((A_k^2-A_{k+1}^2)A_{k+1}^{-2}+(A_{k+1}-A_k)A_k^{-1}\right)A_k^2\right.$$
$$\left.\left.+\left((A_k-A_{k-1})A_{k+1}^{-1}\right)A_{k+1}^2\right)\right)T(k,l) \tag{2.15}$$

Here and for following transformation we take into account the following equality

$$A_{k-1}(A_{k-1}-A_k)=A_{k-1}^2-A_{k-1}A_k$$
$$=A_{k-1}^2-A_k^2+A_k^2-A_{k-1}A_k=A_{k-1}^2-A_k^2+(A_k-A_{k-1})A_k\ .$$

Further we have

$$A_{k+1}S_{k+1}S_kf_k-A_kS_kS_{k-1}f_{k-1}=A_{k+1}S_{k+1}S_k(f_k-f_{k-1})$$
$$+(A_{k+1}S_{k+1}S_k-A_kS_kS_{k-1})f_{k-1}=S_{k+1}\left(A_{k+1}A_k^{-1}\right)S_kA_k$$
$$\times(f_k-f_{k-1})+(A_{k+1}S_{k+1}(S_k-S_{k-1})+A_{k+1}S_{k+1}S_{k-1}$$
$$-A_kS_kS_{k-1})f_{k-1}=S_{k+1}\left(A_{k+1}A_k^{-1}\right)S_kA_k(f_k-f_{k-1})$$
$$+\left(\tau A_{k+1}S_{k+1}S_{k-1}(A_{k-1}-A_k)S_k+A_{k+1}(S_{k+1}-S_k)S_{k-1}\right.$$
$$\left.+A_{k+1}S_kS_{k-1}-A_kS_kS_{k-1}\right)f_{k-1}=S_{k+1}\left(A_{k+1}A_k^{-1}\right)S_kA_k$$
$$\times(f_k-f_{k-1})+\left(\tau S_{k+1}\left(A_{k+1}A_{k-1}^{-1}\right)S_{k-1}A_{k-1}(A_{k-1}-A_k)S_k\right. \tag{2.16}$$
$$\left.+\tau A_{k+1}S_k(A_k-A_{k+1})S_{k+1}S_{k-1}+(A_{k+1}-A_k)S_kS_{k-1}\right)f_{k-1}$$
$$=S_{k+1}\left(A_{k+1}A_k^{-1}\right)S_kA_k(f_k-f_{k-1})+\left(\tau S_{k+1}\left(A_{k+1}A_{k-1}^{-1}\right)S_{k-1}\right.$$
$$\left((A_{k-1}^2-A_k^2)A_k^{-2}+(A_k-A_{k-1})A_k^{-1}\right)S_kA_k^2$$
$$+\tau(A_{k+1}A_k^{-1})S_k\left((A_k^2-A_{k+1}^2)A_{k+1}^{-2}\right.$$
$$\left.+(A_{k+1}-A_k)A_{k+1}^{-1}\right)S_{k+1}\left(A_{k+1}^2A_{k-1}^{-2}\right)S_{k-1}A_{k-1}^2$$
$$\left.+(A_{k+1}-A_k)A_k^{-1}S_k(A_kA_{k-1}^{-1})S_{k-1}A_{k-1}\right)f_{k-1}.$$

$$\begin{aligned}
&S_k(f_{k+1}-f_k)-S_{k-1}(f_k-f_{k-1})=(S_k-S_{k-1})(f_{k+1}-f_k)\\
&+S_{k-1}(f_{k+1}-2f_k+f_{k-1})=-\tau S_k(A_k-A_{k-1})S_{k-1}(f_{k+1}-f_k)\\
&+S_{k-1}(f_{k+1}-2f_k+f_{k-1})=-\tau S_k\left((A_k-A_{k-1})A_{k-1}^{-1}\right)\\
&\times S_{k-1}A_{k-1}(f_{k+1}-f_k)+S_{k-1}(f_{k+1}-2f_k+f_{k-1}).
\end{aligned} \tag{2.17}$$

From (2.14) taking into account (2.15), (2.16) and (2.17) we have the inequality

$$\|g_{k+1}-g_k\|\le c\tau. \tag{2.18}$$

Note that due to condition of Theorem 1.1 we get

$$\begin{gathered}
\|A_{k+1}^m T(k,l)u\|=\|\left(A_{k+1}^m A_k^{-m}\right)S_k\left(A_k^m A_{k-1}^{-m}\right)S_{k-1}\\
\dots S_{l+1}\left(A_{l+1}^m A_l^{-m}\right)S_l A_l^m u\|\\
\le\|\left(A_{k+1}^m A_k^{-m}\right)\|\|S_k\|\|\left(A_k^m A_{k-1}^{-m}\right)\|\|S_{k-1}\|\\
\dots\|S_{l+1}\|\|\left(A_{l+1}^m A_l^{-m}\right)\|\|S_l\|\|A_l^m u\|\\
\le c\|A_l^m u\|,\quad m=1,3,\ \ k\ge l.
\end{gathered}$$

Then from (2.13) taking into account (2.18) and due to the conditions of theorem 1.1, in the case of the sufficient smoothness of the initial data the following estimate holds:

$$\left\|\frac{\Delta^2 u_{k-1}^{(1)}}{\tau^2}\right\|\le c,\qquad c=const>0. \tag{2.19}$$

Hence, taking into account (2.19), from (2.4) for residual $\widetilde{R}_k(\tau)$, we obtain

$$\left\|\widetilde{R}_{k+1}(\tau)\right\|\le c\tau^2,\qquad c=const>0. \tag{2.20}$$

3 The estimate for the approximate solution error

Taking into account (1.3) and (2.3), for error $z_k=u(t_k)-v_k$ we have:

$$\frac{\frac{3}{2}z_{k+1}-2z_k+\frac{1}{2}z_{k-1}}{\tau}+A(t_{k+1})z_{k+1}=r_{k+1}(\tau),\quad k=2,\dots,n-1, \tag{3.1}$$

where $r_k(\tau)=-\left(\tau^2 R_k(\tau,u)+\widetilde{R}_k(\tau)\right)$.

Remark 3.1. Taking into account (2.20) we conclude that if the solution of problem (1.1)–(1.2) is sufficiently smooth, then $\|r_k(\tau)\| = O(\tau^2)$.

The following result holds.

Theorem 3.2. *Let $A(t)$ be a self-adjoint positively defined operator in H with domain of definition $D(A)$ does not depend on t. Let it holds that*

$$\left\|(A(t') - A(t''))A^{-1}(s)\right\| \leq c|t' - t''|,$$
$$\forall t', t'', s \in [0, T], \quad c = const > 0.$$

Then, it holds that

$$\begin{gathered}\|z_{k+1}\| \leq c(\|z_0\| + \|z_1\| + \tau \sum_{i=1}^{k} e^{c(t_k - t_i)} \|r(t_{i+1})\|), \\ k = 2, \ldots, n-1, \quad c = const > 0.\end{gathered} \tag{3.2}$$

For the proof of Theorem 3.2, we need some auxiliary proposition.

Lemma 3.3. Let the operator satisfy the conditions of Theorem 3.2. Then, it holds that

$$\|(A(t_{k+j}) - A(t_j))U_k(\frac{4}{3}L_j, \frac{1}{3}L_j)\| \leq c, \quad c = const > 0, \tag{3.3}$$

where $k = 1, \ldots, n-1, \quad j = 0, \ldots, n-1,$

$$L_j = (I + \frac{2}{3}\tau A(t_j))^{-1}.$$

Proof. The scalar polynomials of two variables $U_k(x, y)$ we will be defined by the following recurrence relation:

$$U_k(x, y) = xU_{k-1}(x, y) - yU_{k-2}(x, y),$$

$$U_0(x, y) = 1, \quad U_{-1}(x, y) = 0.$$

The following formula holds (see [1])

$$U_k(x, y) = \sqrt{y^k} U_k\left(\frac{x}{\sqrt{y}}, 1\right). \tag{3.4}$$

As $A(t)$ is a self-adjoint positively defined operator, then $Sp(L_j) \subset [0, 1]$. Then according to the formula (3.4) we have:

$$U_k\left(\frac{4}{3}L_j, \frac{1}{3}L_j\right) = L_j^{\frac{k}{2}} U_k\left(\frac{4}{3}L_j^{\frac{1}{2}}, \frac{1}{3}I\right).$$

According to the theorem about an estimate of norm of an operator polynomial (see [2], p. 346) the following inequality holds:

$$\left\| U_k \left(\frac{4}{3} L_j^{\frac{1}{2}}, \frac{1}{3} I \right) \right\| \leq \max_{x \in [0,1]} |U_k(\frac{4}{3}x, \frac{1}{3})| \leq \frac{3}{2}. \tag{3.5}$$

Here we used the following estimation

$$|U_k(x,y)| \leq \frac{1}{1-y}, \quad 0 \leq y < 1, \quad |x| \leq 1 + y. \tag{3.6}$$

As

$$\tau A(t_j) L_j^{\frac{k}{2}} = \frac{3}{2}(I - L_j) L_j^{\frac{k-2}{2}},$$

then we have

$$\|\tau A(t_j) L_j^{\frac{k}{2}}\| \leq \frac{3}{2} \max_{x \in [0;1]} (1-x) x^{\frac{k-2}{2}} \leq \frac{3}{k}, \quad k \geq 2. \tag{3.7}$$

Taking into account (3.5), (3.7) and due to the conditions of theorem 3.2 we obtain

$$\left\| (A(t_{k+j}) - A(t_j)) U_k \left(\frac{4}{3} L_j, \frac{1}{3} L_j \right) \right\| \leq$$

$$\left\| (A(t_{k+j}) - A(t_j)) A^{-1}(t_j) \right\| \left\| A(t_j) L_j^{\frac{k}{2}} \right\| \left\| U_k \left(\frac{4}{3} L_j^{\frac{1}{2}}, \frac{1}{3} I \right) \right\| \leq$$

$$\leq c_0 \frac{t_{k+j} - t_j}{\tau} \cdot \frac{3}{k} \cdot \frac{3}{2} = c.$$

Thus, Lemma 3.3 is proved.

Proceed to the proof of the Theorem 3.2.

Proof. From (3.1) we obtain

$$z_{k+1} = \frac{4}{3} L_{k+1} z_k - \frac{1}{3} L_{k+1} z_{k-1} + \frac{2}{3} \tau L_{k+1} r_{k+1}(\tau), \tag{3.8}$$

where

$$L_k = (I + \frac{2}{3} \tau A(t_k))^{-1}.$$

By induction, from (3.8) we obtain formula (see [3]):

$$z_{k+1} = U_k^1 U_k^1 z_1 - \frac{1}{3} U_{k-1}^2 L_2 z_0 + \frac{2}{3} \tau \sum_{i=1}^{k} U_{k-i}^{i+1} L_{i+1} r_{k+1}(\tau), \tag{3.9}$$

where operators U_k^i are determined by the following recurrence relation

$$U_k^i = \frac{4}{3} L_{k+i} U_{k-1}^i - \frac{1}{3} L_{k+i} U_{k-2}^i,$$

$$U_0^i = I, \quad U_{-1}^i = 0.$$

Consider the following homogeneous difference equation

$$\frac{3w_{k+1} - 4w_k + w_{k-1}}{2\tau} + Aw_{k+1} = 0, \quad i = 1, 2, ... \tag{3.10}$$

where A is self-adjoint and positive definite operator.

Obviously, from (3.10) we have

$$w_{k+1} = \frac{4}{3}Lw_k - \frac{1}{3}Lw_{k-1},$$

$$L = (I + \frac{2}{3}\tau A)^{-1}.$$

By induction we obtain

$$w_{k+1} = U_k\left(\frac{4}{3}L, \frac{1}{3}L\right)w_1 - \frac{1}{3}LU_{k-1}\left(\frac{4}{3}L, \frac{1}{3}L\right)w_0. \tag{3.11}$$

In the equation (3.10) we will replace operator A by the operator $A(t_j)$ (j-it is fixed) and we write the obtained equation in the form

$$\frac{3w_{k+1} - 4w_k + w_{k-1}}{2\tau} + A(t_{k+j})w_{k+1} = (A(t_{k+j}) - A(t_j))w_{k+1}.$$

From this equation we have

$$w_{k+1} = \frac{4}{3}L_{k+j}w_k - \frac{1}{3}L_{k+j}w_{k-1} + \frac{2}{3}\tau L_{k+j}(A(t_{k+j} - A(t_j))w_{k+1}.$$

Therefore, taking into account (3.10) we obtain

$$\begin{gathered} w_{k+1} = U_k^j w_1 - \frac{1}{3}U_{k-1}^{j+1}L_{j+1}w_0 \\ +\tfrac{2}{3}\tau \sum_{i=1}^{k} U_{k-i}^{i+j}L_{i+j}(A(t_{i+j}) - A(t_j))w_{i+1}. \end{gathered} \tag{3.12}$$

If $w_0 = 0$, then taking into account (3.11), from (3.12) we have

$$U_k(\frac{4}{3}L_j, \frac{1}{3}L_j)w_1 = U_k^j w_1$$

$$+\frac{2}{3}\tau \sum_{i=1}^{k} U_{k-i}^{i+j}L_{i+j}(A(t_{i+j}) - A(t_j))U_i(\frac{4}{3}L_j, \frac{1}{3}L_j)w_1.$$

As w_1 is an arbitrary vector, from last equality we obtain the formula

$$\begin{gathered} U_k^j = U_k(\frac{4}{3}L_j, \frac{1}{3}L_j) \\ -\tfrac{2}{3}\tau \sum_{i=1}^{k} U_{k-i}^{i+j} L_{i+j}(A(t_{i+j}) - A(t_j))U_i(\tfrac{4}{3}L_j, \tfrac{1}{3}L_j). \end{gathered} \tag{3.13}$$

Obviously, from here we get the following inequality

$$\begin{gathered} \|U_k^j| \leq \|U_k(\frac{4}{3}L_j, \frac{1}{3}L_j)\| \\ +\frac{2}{3}\tau \sum_{i=1}^{k} \|U_{k-i}^{i+j}\| \|L_{i+j}\| \|(A(t_{i+j} - A(t_j))U_i(\frac{4}{3}L_j, \frac{1}{3}L_j)\|. \end{gathered} \tag{3.14}$$

According to (3.6) we have

$$\begin{gathered} \|U_k(\frac{4}{3}L_j, \frac{1}{3}L_j)\| \leq \max_{0\leq y\leq 1} |U_k(\frac{4}{3}y, \frac{1}{3}y)| \\ = \max_{0\leq y\leq 1} (\sqrt{y^k}|U_k(\frac{4}{3}\sqrt{y}, \frac{1}{3})|) \leq \frac{3}{2}. \end{gathered} \tag{3.15}$$

Taking into account bounds (3.3) and (3.15), from (3.14) we have

$$\|U_k^j\| \leq c + c\cdot\tau \sum_{i=1}^{k} \|U_{k-i}^{i+j}\|, \quad c = const > 0.$$

If k we replace by $(n-j)$, we obtain

$$\|U_{n-j}^j\| \leq c + c\cdot\tau \sum_{i=1}^{n-j} \|U_{k-j-i}^{i+j}\|,$$

or that too most

$$\|U_{n-j}^j\| \leq c + c\cdot\tau \sum_{s=j+1}^{n} \|U_{n-s}^{s}\|.$$

Introduce the notations $\|U_{n-s}^s\| = x_s$, we obtain

$$x_j \leq c + c\tau(x_{j+1} + \cdots + x_n), \quad j = 0, 1, \cdots, n-1.$$

From here, by induction we obtain the following bound

$$x_{n-i} \leq c(1 + c\tau)^{i-1}(1 + \tau x_n). \tag{3.16}$$

As

$$x_n = \|U_0^n\| = 1,$$

from (3.16) we have

$$x_{n-i} \leq c(1+c\tau)^i, \quad i = 0, 1, \cdots, n.$$

Hence, we can write the bound

$$\|U_i^{n-i}\| \leq c(1+c\tau)^i \leq ce^{ct_i}.$$

If i we replace by the $(n-i-j)$, we obtain

$$\|U_{n-j-i}^{i+j}\| \leq ce^{ct_{n-j-i}}, \quad i = 0, 1, \cdots, n-j,$$

or that too most

$$\|U_{k-i}^{i+j}\| \leq ce^{ct_{k-i}}, \quad i = 0, 1, \cdots, k. \tag{3.17}$$

Taking into account (3.17), from the representation (3.9) we obtain the statement of the Theorem 3.2.

Proceed to the proof of the Theorem 1.1.

It must be noted that as representation (2.4) is right only at $k \geq 2$, the proof of the estimation (3.2) does not extend for the case $k = 1$, i.e. for vector v_2. Though it is simple to obtain the estimation

$$\|u(t_2) - v_2\| = O(\tau^2). \tag{3.18}$$

Then taking into account (3.18) and due to Remark 3.1 from inequality (3.2), in the case of the sufficient smoothness of the functions, follows the statement of the Theorem 1.1

$$\|u(t_k) - v_k\| = O(\tau^2), \quad k = 1, \ldots, n.$$

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