A PROBLEM OF PLANE ELASTICITY FOR A RECTANGULAR DOMAIN WITH A CURVILINEAR QUADRANGULAR HOLE

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Abstract

We consider a plane problem of elasticity for a rectangular domain with a curvilinear quadrangular hole, which is composed of rectilinear segments (parallel to the abscissa axis) and arcs of one and the same circumference. The problem is solved by the methods of conformal mappings and boundary value problems of analytic functions. The sought complex potentials are constructed effectively (in the analytical form). Estimates of the obtained solutions are derived in the neighborhood of angular points.

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1 Introduction

As is known (see [1]), the application of the methods of conformal mappings and boundary value problems of analytic functions has proved to be the most effective way of solving boundary value problems of elasticity and plate bending. However, if for a simply-connected domain these methods yield effective results (especially for domains mapped onto the circle by rational functions), they still remain poorly adapted to the use for multiply-connected domains. The difficulty consists in the effective construction of a conformally mapping function in general form. Nevertheless, for some practically important classes of doubly-connected domains bounded by polygons (including the rectangular domain with a curvilinear quadrangular hole considered here) we may succeed in constructing effectively (in the analytical form) functions conformally mapping this domain onto the circular ring. In addition to this, the Kolosov-Muskhelishvili methods make it possible to decompose these problems into two Riemann-Hilbert problems for the circular ring and by solving the latter problems to construct the sought complex potentials in the analytical form.

Analogous problems of plane elasticity and plate bending for finite doubly-connected domains bounded by polygons are considered in [2, 3].

2 Statement of the problem

Let the median surface of an isotropic elastic plate on the plane $z = x + iy$ occupy a finite doubly-connected domain S, the external boundary L_0 of which is a rectangle with vertices A_k $(k = 1, ..., 4)$ (i.e. $L_0 = \bigcup_{k=1}^{4} L_k^{(0)}$, $L_k^{(0)} = A_k A_{k+1}$, $k = 1, ..., 4$, $A_5 = A_1$), while the internal boundary L_1 is a curvilinear rectangle composed of segments $L_1^{(1)} = B_1B_2$, $L_1^{(2)} = B_3B_4$ (parallel to the Ox-axis) and arcs of one and the same circumference $L_1^{(3)} = B_2B_3$, $L_1^{(4)} = B_4B_1$. For better clearness, we consider the symmetric case. We denote by $\beta^0\pi$ the value of internal (with respect to the domain S) vertex angles $B_k (k = 1, ..., 4)$ (we mean the angles between the segments $L_1^{(1)}$, $L_1^{(2)}$ and the tangent arcs $L_1^{(3)}$ and $L_1^{(4)}$). It is assumed that the sides $L_0^{(2)}$ and $L_0^{(4)}$ (parallel to the Ox-axis) are under the action of constant normal tensile forces with a given principal vector P (or that the normal displacements $v_n(t) = v_n^{(k)} = const$, $t \in L_0^{(k)}$, $k = 2, 4$ are the given ones), while the remaining part of the boundary $L = L_0 \cup L_1$ is free from external forces (see Fig. 1).

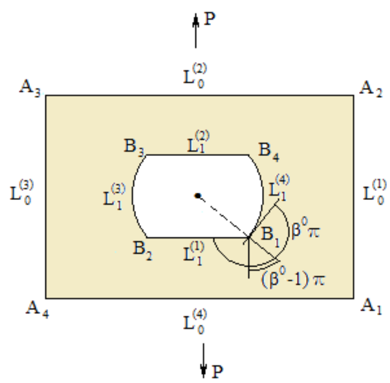

Fig. 1

The problem consists in defining the elastic equilibrium of the plate and establishing the situation in which the concentration of stresses occurs near the angular points and which in turn depends on the behavior of Kolosov-Muskhelishvili potentials at these points.

3 Solution of the problem

The problem is solved by the methods of conformal mappings and the theory of boundary value problems of analytic functions.

Let us recall some results (see [4]) concerning the conformal mapping of a doubly-connected domain $S^{(0)}$ bounded by the convex polygons (A) and (B) with vertices A_k $(k=\overline{1,n})$ and B_k $(k=\overline{1,p})$ and internal (with respect to the domain $S^{(0)}$) vertex angles $\pi\alpha_k^0$ and $\pi\beta_k^0$ onto the circular ring $D_0\{1<|\zeta|<R_0\}$. The existence condition of the function $z=\omega_0(\zeta)$ is expressed by the formula

$$\prod_{k=1}^{n}\left(\frac{a_k}{R}\right)^{\alpha_k^0-1}\prod_{m=1}^{p}(b_m)^{\beta_m^0-1}=1,$$

a_k and b_k are the inverse images of the points A_k and B_k, while the derivative of this function has the form

$$\omega_0'(\zeta)=K^0\prod_{j=-\infty}^{\infty}G\left(R_0^{2j}\zeta\right)g\left(R_0^{2j}\zeta\right)R^{2\delta_j}, \tag{1}$$

where

$$G(\zeta)=\prod_{k=1}^{n}(\zeta-a_k)^{\alpha_k^0-1},\quad g(\zeta)=\prod_{m=1}^{p}(\zeta-b_m)^{\beta_m^0-1},$$

$$\delta_j=\begin{cases}0, & j\ge 0,\\ 1, & j\le -1,\end{cases}$$

with K^0 as an arbitrary real constant.

Note that in the derivation of formula (1) we used the values of the following integrals

$$I_1=\frac{1}{2\pi i}\int_{l_0}\frac{\ln R_0^2}{\sigma-R_0^{2j}\zeta}d\sigma=\begin{cases}\ln R_0^2, & j\le 0,\\ 0, & j\ge 1,\end{cases}$$

$$I_2=\frac{1}{2\pi i}\int_{l_0}\frac{\ln\sigma}{\sigma-R_0^{2j}\zeta}d\sigma=\begin{cases}\ln(R_0^{2j}\zeta-a_1), & j\le 0,\\ \ln(R_0^{2j}\zeta-a_1)-\ln(R_0^{2j}\zeta), & j\ge 1,\end{cases}$$

$$I_3=\frac{1}{2\pi i}\int_{l_0}\frac{i\alpha_0(\sigma)}{\sigma-R_0^{2j}\zeta}d\sigma=\frac{1}{2\pi}\sum_{k=2}^{n}(\alpha_0^{(k-1)}-\alpha_0^{(k)})\ln(a_k-R_0^{2j}\zeta)+$$

$$+\frac{1}{2\pi}(\alpha_0^{(n)} - \alpha_0^{(1)})\ln(a_1 - R_0^{2j}\zeta),$$

where $\alpha_0^{(k)}(\sigma) = \alpha^{(k)}[\omega(\sigma)]$, $\sigma \in l_0^{(k)}$ ($l_0^{(k)}$ ($k = \overline{1,n}$) are the arcs of the circumference l_0 which correspond to the segments $L_0^{(k)}$).

Let us now assume that the regular open polygons with sides δ_n are inscribed in the arcs $L_1^{(3)}$ and $L_1^{(4)}$ and denote the obtained doubly-connected domain by $S^{(n)}$. Applying the results obtained above for the domain $S^{(n)}$ and treating the domain S as a limit case of the domain $S^{(n)}$ as $n \to \infty$ (i.e. $\delta_n \to 0$), we represent the derivative of the function mapping conformally the domain S onto the circular ring $D = \{1 < |\zeta| < R\}$ by the formula

$$\omega'(\zeta) = K\, e^{\gamma(\zeta)} A(\zeta), \tag{2}$$

where

$$\gamma(\zeta) = \frac{1}{2\pi i}\int\limits_{l_1} \frac{\ln[\sigma^{-2}e^{2i\Delta_0(\sigma)}]}{\sigma - \zeta}d\sigma + \frac{1}{2\pi i}\sum_{j=-\infty}^{\infty}{}' \int\limits_{l_1} \frac{\ln[\sigma^{-2}e^{2i\Delta_0(\sigma)}]}{\sigma - R^{2j}\zeta}d\sigma,$$

$$\Delta_0(\sigma) = \begin{cases} \arg\sigma, & \sigma \in l_1^{(k)}, \quad k = 3,4, \\ (-1)^{m-1}\dfrac{\pi}{2}, & \sigma \in l_1^{(m)}, \quad m = 1,2, \end{cases}$$

K is an arbitrary real constant, $\sum'$ indicates that $j = 0$ is omitted, $A(\zeta) = \prod\limits_{j=-\infty}^{\infty}\prod\limits_{k=1}^{4}(R^{2j}\zeta - a_k)^{\alpha_k^0 - 1}$.

Based on the results given in [5, §78], we conclude that the function $e^{\gamma(\zeta)}$ near the points b_k ($k = \overline{1,4}$) can be written in the form

$$e^{\gamma(\zeta)} = \prod_{m=1}^{4}(\zeta - b_m)^{\beta^0 - 1}\Omega(\zeta),$$

where $\beta^0 = \beta_1^0 = \cdots = \beta_4^0$, $\Omega(\zeta)$ is the function holomorphic near the point b_k and tending to definite nonzero limits as $\zeta \to b_k$.

Thus, for a conformally mapping function bounded at the points ($k = \overline{1,4}$) (i.e. of the class $h(b_1, ..., b_4)$ (see [5])), from (2) we obtain the formula

$$\omega'(\zeta) = K\prod_{m=1}^{4}(\zeta - b_m)^{\beta^0 - 1}\Omega(\zeta)\prod_{j=-\infty}^{\infty}\prod_{k=1}^{4}(R^{2j}\zeta - a_k)^{-\frac{1}{2}}R^{2\delta_j}. \tag{3}$$

Now, the boundary conditions for $\omega'(\zeta)$ are written in the form

$$\begin{aligned} &Re[i\sigma e^{-i\alpha_0(\sigma)}\omega'(\sigma)] = 0, \quad \sigma \in l_0, \\ &Re[i\sigma e^{-i\Delta_0(\sigma)}\omega'(\sigma)] = 0, \quad \sigma \in l_1. \end{aligned} \tag{4}$$

Let us now return to the considered problem. By virtue of the well known Kolosov-Muskhelishvili formulas (see [1], §41), for finding the complex potentials $\varphi(z)$ and $\psi(z)$ we obtain the boundary conditions

$$\begin{aligned} &Re\left[\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\right]=D_1, \\ &Re\left[\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}\right]=0, \end{aligned} \quad t\in L_0^{(1)},$$

$$\begin{aligned} &Im\left[\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\right]=D_2, \\ &Im\left[\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}\right]=2\mu v_n^{(2)}, \end{aligned} \quad t\in L_0^{(2)},$$

$$\begin{aligned} &Re\left[\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\right]=-P+D_1, \\ &Re\left[\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}\right]=0, \end{aligned} \quad t\in L_0^{(3)},$$

$$\begin{aligned} &Im\left[\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\right]=D_2, \\ &Im\left[\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}\right]=-2\mu v_n^{(2)}, \end{aligned} \quad t\in L_0^{(4)},$$

$$\begin{aligned} &\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=C_1+iC_2, \\ &\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}=0, \end{aligned} \quad t\in L_1.$$

These conditions are in turn divided for two problems

$$Re\left[e^{-i\alpha(t)}\varphi(t)\right]=F_0^*(t),\ \ t\in L_0,\ \ \varphi(t)=F_1^*(t),\ \ t\in L_1, \tag{5}$$

and

$$\begin{aligned} &Re\left[e^{-i\alpha(t)}\left(\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\right)\right]=\Gamma_0(t),\ \ t\in L_0, \\ &\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=\Gamma_1(t),\ \ t\in L_1, \end{aligned} \tag{6}$$

where

$$\alpha(t)=\begin{cases} 0,\ \ t\in L_0^{(1)}, \\ \dfrac{\pi}{2},\ \ t\in L_0^{(2)}, \\ \pi,\ t\in L_0^{(3)}, \\ \dfrac{3}{2}\pi,\ t\in L_0^{(4)}, \end{cases} \qquad F_0^*(t)=\begin{cases} \varkappa_0 D_1,\ \ t\in L_0^{(1)}, \\ \varkappa_0\left[D_2+2\mu v_n^{(2)}\right],\ \ t\in L_0^{(2)}, \\ \varkappa_0\left(-P+D_1\right),\ \ t\in L_0^{(3)}, \\ \varkappa_0\left[D_2-2\mu v_n^{(2)}\right],\ \ t\in L_0^{(4)}, \end{cases}$$

$$F_1^*(t)=\varkappa_0\left(C_1+iC_2\right)\ \ t\in$$

$$\Gamma_0(t) = \begin{cases} D_1, & t \in L_0^{(1)}, \\ D_2, & t \in L_0^{(2)}, \\ -P + D_1, & t \in L_0^{(3)}, \\ D_2, & t \in L_0^{(4)}, \end{cases}$$

$$\Gamma_1(t) = C_1 + iC_2, \quad t \in L_1,$$

where $\varkappa_0 = (\varkappa + 1)^{-1}$, C_1, C_2, D_1, D_2 are arbitrary real constants.

Let us consider problem (5). After the conformal mapping of the domain S onto the circular ring D, this problem for the function

$$\varphi_*(\zeta) = \frac{\varphi[\omega(\zeta)]}{\zeta} \equiv \frac{\varphi_0(\zeta)}{\zeta}$$

reduces to the Riemann-Hilbert problem for a circular ring [5]

$$\begin{aligned} Re\left[\sigma e^{-i\alpha_0(\sigma)}\varphi_*(\sigma)\right] &= F_0(\sigma), \quad \sigma \in l_0, \\ Re\left[\sigma e^{-i\Delta_0(\sigma)}\varphi_*(\sigma)\right] &= F_1(\sigma), \quad \sigma \in l_1, \end{aligned} \tag{7}$$

where

$$F_0(\sigma) = F_0^*[\omega(\sigma)], \quad \sigma \in l_0,$$

$$F_1(\sigma) = \begin{cases} \varkappa_0 C_2, & \sigma \in l_1^{(1)}, \\ -\varkappa_0 C_2, & \sigma \in l_1^{(2)}, \\ Re\left[\bar{\sigma}(C_1 + iC_2)\right], & \sigma \in l_1^{(3)} \cup l_1^{(4)}. \end{cases}$$

We easily observe that from the boundary conditions (4) we obtain the factorization of the coefficient of problem (4) in the following form

$$e^{2i\alpha_0(\sigma)}R^2\sigma^{-2} = \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}, \quad \sigma \in l_0,$$

$$e^{2i\Delta_0(\sigma)}\sigma^{-2} = \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}, \quad \sigma \in l_1,$$

where $\omega'(\zeta)$ is defined by formula (3).

With the obtained results taken into account, from the boundary conditions (7) for the function

$$\Omega(\zeta) = \frac{\varphi_*(\zeta)}{\omega'(\zeta)} = \frac{\varphi_0(\zeta)}{\zeta\omega'(\zeta)} \tag{8}$$

we obtain the Dirichlet problem for a circular ring

$$\begin{aligned} Re\,[\Omega(\sigma)] &= \frac{F_0(\sigma)e^{i\alpha_0(\sigma)}}{\sigma\omega'(\sigma)}, \quad \sigma \in l_0, \\ Re\,[\Omega(\sigma)] &= \frac{F_1(\sigma)e^{i\Delta_0(\sigma)}}{\sigma\omega'(\sigma)}, \quad \sigma \in l_1. \end{aligned} \tag{9}$$

A solvability condition of problem (9) has the form (see [4])

$$\int_{l_0} \frac{F_0(\sigma)e^{i\alpha_0(\sigma)}}{\sigma\omega'(\sigma)}\frac{d\sigma}{\sigma} = \int_{l_1} \frac{F_1(\sigma)e^{i\Delta_0(\sigma)}}{\sigma\omega'(\sigma)}\frac{d\sigma}{\sigma}, \tag{10}$$

and its solution is given by the formula

$$\Omega(\zeta) = \Theta(\zeta),$$

where

$$\Theta(\zeta) = \frac{1}{\pi i}\sum_{j=-\infty}^{\infty}\left[\int_{l_0} \frac{F_0(t)e^{i\alpha_0(t)}}{(t-R^{2j}\zeta)t\omega'(t)}dt + \int_{l_1} \frac{F_1(t)e^{i\Delta_0(t)}}{(t-R^{2j}\zeta)t\omega'(t)}dt\right] + iE_1,$$

where E_1 is an arbitrary real constant.

Thus, using (8), for the function $\varphi_0(\zeta)$ we obtain the formula

$$\varphi_0(\zeta) = \zeta\omega'(\zeta)\Theta(\zeta). \tag{11}$$

Taking into account the form of the function $\omega'(\zeta)$ in the neighborhood of the point a_k ($k = \overline{1,4}$), we conclude that for the continuous extension of the function $\varphi_0(\zeta)$ in the domain $D + l$ it is necessary that the conditions

$$\Theta(a_k) = 0, \quad k = \overline{1,4} \tag{12}$$

be fulfilled.

Since $\varphi'(z) = \varphi_0'(\zeta)[\omega'(\zeta)]^{-1}$, from (11) we have

$$\varphi'(z) = \frac{\varphi_0'(\zeta)}{\omega'(\zeta)} = \Theta(\zeta) + \zeta\frac{\omega''(\zeta)}{\omega'(\zeta)}\Theta(\zeta) + \zeta\Theta'(\zeta). \tag{13}$$

Based on the results obtained in [5] (§26) as to the behavior of a Cauchy type integral near the density discontinuity points, we conclude that near the points b_k ($k = \overline{1,4}$) the function $\Theta(\zeta)$ has the form

$$\Theta(\zeta) = \frac{K_1}{(\zeta - b_k)^{\beta^0 - 1}} + \Theta_k^0(\zeta), \quad k = \overline{1,4},$$

where $\Theta_k^0(\zeta)$ is the function that near the point b_k admits the following estimate

$$\left|\Theta_k^0(\zeta)\right| < \frac{C}{|\zeta - b_k|^{\delta_0}}, \ \ C = const, \ \ 0 < \delta_0 < \beta^0 - 1,$$

where K_1 is the well-defined constant.

Taking into account the behavior of the conformally mapping function near the angular points (see [6], §37), we obtain

$$\omega(\zeta) = B_k + (\zeta - b_k)^{\beta^0}\Omega_k(\zeta),$$

$$\zeta\frac{\omega''(\zeta)}{\omega'(\zeta)} = \frac{b_k(\beta^0 - 1)}{\zeta - b_k} + \Omega_k^*(\zeta), \ \ k = \overline{1,4},$$

where $\Omega_k(b_k) \neq 0$, $\Omega_k^*(\zeta)$ is the regular part of the Loran decomposition of the function $\zeta\dfrac{\omega''(\zeta)}{\omega'(\zeta)}$.

By the above reasoning, from (13) we obtain the estimate

$$\varphi'(z) = \frac{k_0}{\zeta - b_k} + \Theta_0^k(\zeta), \ \ k = \overline{1,4}, \ \ K_0 = -K_1(\beta^0 - 1),$$

and thus near a point B which is one of the points B_k $(k = \overline{1,4})$ we have the estimates

$$\left|\varphi'(z)\right| < M_1\left|z - B\right|^{\frac{1}{\beta^0}-1}, \ \ \left|\varphi''(z)\right| < M_2\left|z - B\right|^{\frac{1}{\beta^0}-2}, \ \ M_1, M_2 = const.$$

By a similar reasoning to the above, it is proved that $\varphi'(z)$ is almost bounded (i.e. has singularities of logarithmic type) near the points A_k $(k = \overline{1,4})$.

After finding the function $\varphi(z)$, the definition of the function $\psi(z)$ by (6) reduces to the following problem which is analogous to problem (5)

$$\begin{aligned} &Re\left[e^{i\alpha(t)}R(t)\right] = N_0(t), \ \ t \in L_0, \\ &Re\left[R(t)\right] = N_1(t), \ \ t \in L_1, \end{aligned} \tag{14}$$

where

$$R(z) = \psi(z) + P(z)\varphi'(z),$$

$$N_0(t) = \Gamma_0(t) - Re\left[e^{i\alpha(t)}\left(\overline{\varphi(t)} + (\bar{t} - P(t))\varphi'(t)\right)\right], \ \ t \in L_0,$$

$$N_1(t) = Re\left[\Gamma_1(t) - \overline{\varphi(t)} - (\bar{t} - P(t))\varphi'(t)\right], \ \ t \in L_1,$$

$P(z)$ is an interpolation polynomial satisfying the condition $P(B_k) = \overline{B}_k$ $(k = \overline{1,4})$ and having the form

$$P(z) = \frac{(z - B_2)\cdots(z - B_4)}{(B_1 - B_2)\cdots(B_1 - B_4)}\overline{B}_1 + \cdots + \frac{(z - B_1)\cdots(z - B_3)}{(B_4 - B_1)\cdots(B_4 - B_3)}\overline{B}_4.$$

The use of the polynomial $P(z)$ makes bounded the right-hand part of the boundary condition (14) so that the solution of this problem can be constructed in an analogous manner as above (see problem (5)), while the solvability condition (with the assumption that the function $\psi(z)$ is continuous up to the boundary) will be analogous to conditions (10) and (12). All these conditions will be represented as a non-homogeneous system with real coefficients with respect to unknown real constants. It is proved that the obtained system is uniquely solvable and therefore the problem posed has a unique solution.

Remark. The obtained results can be extended to the case of a rectangular domain with a circular hole under the assumption that $\beta^0 = 1$ and to a rectangular domain with a rectilinear cut under the assumption that $\beta^0 = 2$.

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