SOLUTION OF A NONCLASSICAL PROBLEMS OF STATICS OF MICROSTRETCH MATERIALS WITH MICROTEMPERATURES

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Abstract

The paper considers the static of the theory of linear thermoelasticity of microstretch materials with microtemperatures. The boundary value problem of statics is investigated when the normal components of displacement and the microtemperature vectors and tangent components of rotation vectors are given on the spherical surfaces. Uniqueness theorems are proved. Explicit solutions are constructed in the form of absolutely and uniformly convergent series.

Key words and phrases: Microtemperature, Fourier-Laplace series, Microstretch materials.

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1 Introduction

The mathematical models describing the chiral properties of linear thermoelasticity of materials with microtemperatures were proposed by Ieşan [8], [9], and recently extended to a more general case where the material points admit a micropolar structure, see Ieşan and Quintanilla [10].

The Dirichlet, Neumann and mixed type boundary value problems corresponding to these models were well investigated for general domains of arbitrary shape; uniqueness and existence theorems were proved, and the regularity of solutions was established using both potential and variational methods (see [1], [12], [15], [16] and the references therein).

The main goal of the present paper consists in deriving general formulas for representation of displacement, microtemperature vectors and temperature functions in terms of harmonic and metaharmonic functions. This means that solutions of a complicated coupled system of simultaneous differential equations of thermoelasticity can be represented with the aid of solutions of simple canonical equations.

In particular, the derived representation formulas are used to construct explicit solutions of Dirichlet and Neumann type boundary value problems for the ball. Solutions are represented in the form of Fourier–Laplace series and their absolute and uniform convergence, together with their first order derivatives, is proved when the boundary data satisfy appropriate smoothness conditions. Methods of fulfilling the boundary conditions are investigated in A. Ulitko [18], F. Mors and H. Feshbah [14], L. Giorgashvili [2], [3], L. Giorgashvili and K. Skhvitaridze [4], L. Giorgashvili and D. Natroshvili [5], L. Giorgashvili, G. Karseladze and G. Sadunishvili [6], L. Giorgashvili, A. Jagmaidze, K. Skhvitaridze [7] and other papers.

2 Basic Equations and Auxiliary Theorems

The equations which govern the thermoelastic deformations of microstretch materials withmicrotemperatures are [9]

$$\mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad\,div}\,u(x) + \eta\,\mathrm{grad}\,v(x) - \gamma\,\mathrm{grad}\,\vartheta(x) = 0, \tag{2.1}$$
$$\varkappa_6\Delta w(x) + (\varkappa_5+\varkappa_4)\,\mathrm{grad\,div}\,w(x) - \varkappa_3\,\mathrm{grad}\,\vartheta(x) - \varkappa_2 w(x) = 0, \tag{2.2}$$
$$\varkappa\Delta\vartheta(x) + \varkappa_1\,\mathrm{div}\,w(x) = 0, \tag{2.3}$$
$$\eta_1\Delta v(x) - \eta\,\mathrm{div}\,u(x) - \eta_2 v(x) - \varkappa_7\,\mathrm{div}\,w(x) + \gamma_1\vartheta(x) = 0, \tag{2.4}$$

where Δ is three-dimensional Laplace operator, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $w=(w_1,w_2,w_3)$ is a microtemperature vector, v is a microstretch material, η is temperature variation and $\lambda,\mu,\gamma,\gamma_1,\eta,\eta_1,\eta_2,\varkappa,\varkappa_j,\ j=1,2,...,7$ are physical constants which satisfy the following inequalities

$$3\varkappa_4+\varkappa_5+\varkappa_6>0,\ \ \varkappa_5+\varkappa_6>0,\ \ \varkappa_6-\varkappa_5>0,\ \ \varkappa>0,\ \ \mu\eta_2>\eta^2,$$

$$\eta_1>0,\ \ \mu>0,\ \ 3\lambda+2\mu>0,\ \ (\varkappa_1+T_0\varkappa_3)^2<4T_0\varkappa\varkappa_2,$$

$T_0>0$ is the initial temperature, $\top$ is the transposition symbol.

Definition. The vector $U=(u,w,\theta,v)^\top$, is called regular in a domain $\Omega\subset\mathbb{R}^3$ if $U\in C^2(\Omega)\cap C^1(\overline{\Omega})$.

The following theorem is true [11]

Theorem 2.1. *For the vector $U=(u,w,\theta,v)^\top$ to be a regular solution of system (2.1)-(2.4) in a domain $\Omega\subset\mathbb{R}^3$, it is necessary and sufficient that it be represented in the form*

$$\begin{aligned}
u(x) &= \mathrm{grad}\,\Phi_1(x) + b\,\mathrm{grad}\,r^2\left(r\frac{\partial}{\partial r}+1\right)\Phi_2(x) + \mathrm{rot\,rot}(xr^2\Phi_2(x))\\
&\quad + \mathrm{rot}(x\Phi_3(x)) + a_5\,\mathrm{grad}\,r^2\Phi_4(x) + a_6\,\mathrm{grad}\,\Phi_5(x) - \eta\,\mathrm{grad}\,\Phi_8(x),\\
w(x) &= a_1\,\mathrm{grad}(2r\frac{\partial}{\partial r}+3)\Phi_4(x) + a_2\,\mathrm{grad}\,\Phi_5(x) + \mathrm{rot\,rot}(x\Phi_6(x))\\
&\quad + \mathrm{rot}(x\Phi_7(x)),\\
\theta(x) &= 2(\lambda+2\mu)(2r\frac{\partial}{\partial r}+3)\Phi_4(x) + (\lambda+2\mu)\lambda_1^2\Phi_5(x),\\
v(x) &= a_3(2r\frac{\partial}{\partial r}+3)\Phi_4(x) + a_4\Phi_5(x) + (\lambda+2\mu)\lambda_3^2\Phi_8(x)\\
&\quad + \frac{2\eta a}{\eta_1\lambda_3^2}(2r\frac{\partial}{\partial r}+3)(r\frac{\partial}{\partial r}+1)\Phi_2(x),
\end{aligned} \tag{2.5}$$

where

$$\Delta\Phi_j(x)=0,\ \ j=1,2,3,4,\ \ (\Delta-\lambda_1^2)\Phi_5(x)=0,\ \ (\Delta-\lambda_2^2)\Phi_j(x)=0,\ \ j=6,7,$$
$$(\Delta-\lambda_3^2)\Phi_8(x)=0,\ \ \lambda_1^2=\frac{\varkappa\varkappa_2-\varkappa_1\varkappa_3}{\varkappa(\varkappa_4+\varkappa_5+\varkappa_6)}>0,\ \ \lambda_2^2=\frac{\varkappa_2}{\varkappa_6}>0,$$
$$\lambda_3^2=\frac{\eta_2(\lambda+2\mu)-\eta^2}{\eta_1(\lambda+2\mu)}>0,\ \ a=\frac{\mu}{\lambda+2\mu},\ \ a_1=-\frac{2(\lambda+2\mu)\varkappa_3}{\varkappa_2},$$
$$a_2=-\frac{\varkappa\lambda_1^2(\lambda+2\mu)}{\varkappa_1},\quad a_3=\frac{2(\gamma_1(\lambda+2\mu)-\eta\gamma)}{\eta_1\lambda_3^2},$$
$$a_4=\frac{\lambda_1^2}{\eta_1(\lambda_1^2-\lambda_3^2)}(\eta\gamma-\gamma_1(\lambda+2\mu)+a_2\varkappa_7),$$
$$a_5=\gamma-\frac{\eta a_3}{2(\lambda+2\mu)},\ \ a_6=\gamma-\frac{\eta a_4}{(\lambda+2\mu)\lambda_1^2},\ \ b=-a-\frac{\eta^2 a}{\eta_1\lambda_3^2(\lambda+2\mu)},$$
$$r=|x|,\quad x=(x_1,x_2,x_3)^\top,\quad r\frac{\partial}{\partial r}=x\cdot\mathrm{grad}\,.$$

Assume that r,ϑ,φ $(0\le r<+\infty,\ \ 0\le\vartheta\le\pi,\ \ 0\le\varphi<2\pi)$ are the spherical coordinates of a point $x\in R^3$. Denote by $\sum_1$ the sphere with unit radius and center at the origin.

Let us consider, in the space $L_2(\sum_1)$, the following complete system of orthonormal vectors [2], [14], [18]

$$\begin{aligned}
&X_{mk}(\vartheta,\varphi)=e_rY_k^{(m)}(\vartheta,\varphi),\ \ k\ge 0,\\
&Y_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(e_\vartheta\frac{\partial}{\partial\vartheta}+\frac{e_\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\Big)Y_k^{(m)}(\vartheta,\varphi),\ \ k\ge 1,\\
&Z_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(\frac{e_\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi}-e_\varphi\frac{\partial}{\partial\vartheta}\Big)Y_k^{(m)}(\vartheta,\varphi),\ \ k\ge 1,
\end{aligned}\tag{2.6}$$

where $|m|\le k$, $e_r,\ e_\vartheta,\ e_\varphi$ are the orthonormal vectors in $\mathbb{R}^3$,

$$\begin{aligned}
&e_r=(\cos\varphi\sin\vartheta,\ \sin\varphi\sin\vartheta,\ \cos\vartheta)^\top,\\
&e_\vartheta=(\cos\varphi\cos\vartheta,\ \sin\varphi\cos\vartheta,\ -\sin\vartheta)^\top,\\
&e_\varphi=(-\sin\varphi,\ \cos\varphi,\ 0)^\top,
\end{aligned}$$
$$Y_k^{(m)}(\vartheta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\frac{(k-m)!}{(k+m)!}}P_k^{(m)}(\cos\theta)e^{im\varphi},$$

$P_k^{(m)}(\cos\vartheta)$ is the adjoint Legendre polynomial. Let $f^{(j)}=(f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)})^\top$, $j=1,2$, be a the vector-function and represent the function f_j $j=4,5,6,7$ as the following Fourier-Laplace series

$$\begin{aligned}
f^{(j)}(\vartheta,\varphi)=&\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{\alpha_{mk}^{(j)}X_{mk}(\vartheta,\varphi)\\
&+\sqrt{k(k+1)}\Big[\beta_{mk}^{(j)}Y_{mk}(\vartheta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\vartheta,\varphi)\Big]\Big\},\ \ j=1,2,
\end{aligned}\tag{2.7}$$

$$f_j(\vartheta,\varphi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\alpha_{mk}^{(j)}Y_k^{(m)}(\vartheta,\varphi),\quad j=4,5,6,7,\tag{2.8}$$

where $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j=1,2$, $\alpha_{mk}^{(l)}$, $l=4,5,6,7$, are Fourier-Laplace coefficients.

Note that in the formula (2.7) and in the analogous series below the summation index k varies from 1 to $+\infty$ in the summands contain the vectors $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$.

Let us introduce several important lemmas [3], [13]

Lemma 2.2. *Let $f^{(j)}\in C^l(\Sigma_1)$, $l\geq 1$, then the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$ $j=1,2$, admit the following estimates*

$$\alpha_{mk}^{(j)}=O(k^{-l}),\ \ \beta_{mk}^{(j)}=O(k^{-l-1}),\ \ \gamma_{mk}^{(j)}=O(k^{-l-1}).\ \ j=1,2.$$

Lemma 2.3. *If $f_j\in C^l(\Sigma_1)$, $l\geq 1$, then the coefficients $\alpha_{mk}^{(j)}$, $j=4,5,6,7$ admit the following estimates*

$$\alpha_{mk}^{(j)}=O(k^{-l}),\ \ j=4,5,6,7.$$

Lemma 2.4. *The vectors $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$, defined by equalities* (2.6) *admit the estimates:*

$$\begin{aligned}|X_{mk}(\vartheta,\varphi)|&\leq\sqrt{\frac{2k+1}{4\pi}},\ \ k\geq 0,\\ |Y_{mk}(\vartheta,\varphi)|&<\sqrt{\frac{k(k+1)}{2k+1}},\ \ k\geq 1,\\ |Z_{mk}(\vartheta,\varphi)|&<\sqrt{\frac{k(k+1)}{2k+1}},\ \ k\geq 1,\end{aligned}\tag{2.9}$$

Note that

$$|Y_{mk}(\vartheta,\varphi)|\leq\sqrt{\frac{2k+1}{4\pi}},\ \ k\geq 0,$$

Theorem 2.5. *The vector $U=(u,w,\theta,v)^\top$ represented as* (2.5) *will be uniquely defined in the domain $\Omega^+:=B(O,R)$ by the functions $\Phi_j(x)$, $j=1,2,...,8$, if the following conditions is fulfilled*

$$\int\limits_{\Sigma_r}\Phi_j(x)d\Sigma_r=0,\qquad j=1,3,6,7,\qquad r=|x|\leq R,\tag{2.10}$$

which means that to the zero value of the vector $U=(u,w,\theta,v)^\top$ there corresponds the zero value of the vector $(\Phi_1,\ \Phi_2,...,\Phi_8)^\top$ and vise versa.

3 Statement of the problem. The uniqueness theorem

Let Ω^+ be a ball with center at the origin and radius R and $\Sigma_R = \partial\Omega$.

Problem$(N)^+$ Find,a regular solution of system (2.1)–(2.4) in the domain Ω^+,which on the boundary $\partial\Omega$, satisfies the conditions:

$$\begin{aligned} \{n(z)\cdot u(z)\}^+ = f_4(z), \quad \{n(z)\times \operatorname{rot} u(z)\}^+ = f^{(1)}(z),\\ \{n(z)\cdot w(z)\}^+ = f_5(z), \quad \{n(z)\times \operatorname{rot} w(z)\}^+ = f^{(2)}(z), \end{aligned} \tag{3.1}$$

(problem) $(N.I)$:$^+$

$$\{\theta(z)\}^+ = f_6(z), \quad \{v(z)\}^+ = f_7(z); \tag{3.2}$$

(problem) $(N.II)$:$^+$

$$\{\vartheta(z)\}^+ = f_6(z), \quad \left\{\frac{\partial v(z)}{\partial n(z)}\right\}^+ = f_7(z), \tag{3.3}$$

where $f^{(j)} = (f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)})$, $j = 1,2,$ $f_j = 4,5,6,7,$ $f_k^{(j)}$, $\quad j = 1,2,$ $k = 1,2,3$ are function given on $\partial\Omega$, $n(z)$ is the outward unit normal with respect to Ω^+. at a point $z\in\partial\Omega$, $\frac{\partial}{\partial n(x)} = \sum_{k=1}^3 n_k(x)\frac{\partial}{\partial x_k}$, the symbol $a\cdot b$ denote the scalar and symbol $a\times b$ denote the vector prodacts of two vectors in R^3.

Theorem 3.1. *If Problem $(N)^+$ have solutions, these solutions are unique.*

Proof. The theorem will be proved if we show that the homogeneous problems $(N)_0^+$ $(f^{(j)} = 0,\ \ j = 1,2,\ \ f_j = 0,\ \ j = 4,5,6,7)$ have only trivial solution.

Let the vector $U = (u, w, \theta, v)^\top$ be a solution of system (2.1)–(2.4). We multiply both sides of equality (2.2) by the vector $w(x)$ and equations (2.3) by $\theta(x)$. and summation give

$$\begin{aligned} w(x)\cdot A^{(2)}(\partial)w(x) - \varkappa_3 w(x)\cdot \operatorname{grad}\vartheta(x) - \varkappa_2 w^2(x)\\ +\varkappa\vartheta(x)\Delta\vartheta(x) + \varkappa_1\vartheta(x)\operatorname{div} w(x) = 0, \end{aligned} \tag{3.4}$$

where

$$A^{(2)}(\partial)w(x) := \varkappa_6\Delta w(x) + (\varkappa_5+\varkappa_4)\operatorname{grad}\operatorname{div} w(x).$$

Note that [4]

$$\begin{aligned} w(x)\cdot\Delta w(x) &= \operatorname{div}(w(x)\operatorname{div} w(x)) + \operatorname{div}[w(x)\times \operatorname{rot} w(x)]\\ &\quad - (\operatorname{div} w(x))^2 - (\operatorname{rot} w(x))^2,\\ w(x)\cdot \operatorname{grad}\operatorname{div} w(x) &= \operatorname{div}\left(w(x)\operatorname{div} w(x)\right) - \left(\operatorname{div} w(x)\right)^2,\\ \vartheta(x)\operatorname{div} w(x) &= \operatorname{div}\left(\vartheta(x) w(x)\right) - w(x)\cdot\operatorname{grad}\vartheta(x),\\ \vartheta(x)\Delta\vartheta(x) &= \operatorname{div}\left(\vartheta(x)\operatorname{grad}\vartheta(x)\right) - \left(\operatorname{grad}\vartheta(x)\right)^2. \end{aligned} \tag{3.5}$$

Substituting these equalities into (3.4), we have

$$\begin{aligned}\operatorname{div}[lw(x)\operatorname{div} w(x) + \varkappa_6(w(x)\times \operatorname{rot} w(x)) + \varkappa_1\vartheta(x)w(x) +\\ + \varkappa\vartheta(x)\operatorname{grad}\vartheta(x)] - E(U',U'),\end{aligned} \tag{3.6}$$

where $l = \varkappa_4 + \varkappa_5 + \varkappa_6$, $U' = (w, \vartheta)^\top$,

$$\begin{aligned}E(U',U') = l(\operatorname{div} w)^2 + \varkappa_6(\operatorname{rot} w)^2 + (\varkappa_1+\varkappa_3)w\cdot\operatorname{grad}\vartheta\\ + \varkappa_2 w^2 + \varkappa(\operatorname{grad}\vartheta)^2.\end{aligned} \tag{3.7}$$

Applying the Gauss-Ostrogradski theorem, from (3.6), we obtain

$$\int\limits_{\partial\Omega}\{U'(z)\}^+\cdot\{P(\partial,n)U'(z)\}^+\ ds = \int\limits_{\Omega^+} E(U',U')\,dx, \tag{3.8}$$

where

$$\begin{aligned}U'(z)\cdot P(\partial,n(z))U'(z) = l(n(z)\cdot w(z))\operatorname{div} w(z)\\ - \varkappa_6 w(z)\cdot[n(z)\times\operatorname{rot} w(z)] + \varkappa_1\vartheta(z)(n(z)\cdot w(z)) + \varkappa\vartheta(z)\frac{\partial\vartheta(z)}{\partial n(z)}.\end{aligned} \tag{3.9}$$

Here we have used the identity

$$n\cdot[w\times\operatorname{rot} w] = -w[n\times\operatorname{rot} w].$$

Applying the boundary conditions of problem $(N)_0^+$, we obtain

$$\{U'(z)\}^+\cdot\{P(\partial,n)U'(z)\}^+ = 0, \quad z = \partial\Omega.$$

Using this equality in (2.8), we have

$$\int\limits_{\Omega^+} E(U',U')dx = 0. \tag{3.10}$$

$E(U',U')$ can be rewritten as follow

$$\begin{aligned}E(U',U') = l(\operatorname{div} w)^2 + \varkappa_6(\operatorname{rot} w)^2 + \frac{4\varkappa\varkappa_2 - (\varkappa_1+\varkappa_3)^2}{4\varkappa}w^2(x)\\ + \frac{1}{4\varkappa}[(\varkappa_1+\varkappa_3)w(x) + 2\varkappa\operatorname{grad}\vartheta(x)]^2.\end{aligned} \tag{3.11}$$

According to inequalities $l > 0$, $\varkappa > 0$, $\varkappa_6 > 0$, $4\varkappa\varkappa_2 - (\varkappa_1+\varkappa_3)^2 > 0$, we have $E(U',U') \geq 0$, $x \in \Omega^+$. By virtue of this fact, (3.10) implies

$$E(U'(x),U'(x)) = 0, \quad x\in\Omega^+.$$

Hence, taking into account (3.11), we obtain

$$\operatorname{div} w(x) = 0, \quad \operatorname{rot} w(x) = 0, \quad w(x) = 0, \quad \operatorname{grad}\vartheta(x) = 0.$$

Hence in follows that $w(x) = 0$, $\vartheta(x) = c = const$, $x \in \Omega^+$.

From the boundary conditions $\{\vartheta(z)\}^+ = 0$ we have $c = 0$, i.e. $\vartheta(x) = 0$, $x \in \Omega^+$.

Substituting the values $\vartheta(x) = 0$, $w(x) = 0$ in equalities (2.1) and (2.4), we obtain

$$\mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x) + \eta\,\mathrm{grad}\,v(x) = 0, \tag{3.12}$$

$$\eta_1\Delta v(x) - \eta\,\mathrm{div}\,u(x) - \eta_2 v(x) = 0, \tag{3.13}$$

we multiply both sides of equality (3.12) by the vector $u(x)$ and equations (3.13) by $v(x)$ and summation give

$$\begin{aligned} u(x)\cdot A^{(1)}(\partial)u(x) - \eta u(x)\cdot \mathrm{grad}\,v(x) + \eta_1 v(x)\Delta v(x)& \\ - \eta v(x)\,\mathrm{div}\,u(x) - \eta_2 v^2(x) = 0&, \end{aligned} \tag{3.14}$$

where

$$A^{(1)}(\partial)u(x) := \mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x).$$

Substituting the equalities (3.5) into (3.14), we have

$$\begin{aligned} \mathrm{div}[(\lambda+2\mu)u(x)\,\mathrm{div}\,u(x) + \mu\,(u(x)\times \mathrm{rot}\,u(x)) + \eta u(x)v(x)& \\ + \eta_1 v(x)\,\mathrm{grad}\,v(x)] - \widetilde{E}(U'',U'')&, \end{aligned} \tag{3.15}$$

where $U'' = (u,v)^\top$,

$$\begin{aligned} \widetilde{E}(U'',U'') = (\lambda+2\mu)(\mathrm{div}\,u(x))^2 + \mu(\mathrm{rot}\,u(x))^2 + 2\eta v(x)\,\mathrm{div}\,u(x)& \\ + \eta_2 v^2(x) + \eta_1(\mathrm{grad}\,v(x))^2&. \end{aligned} \tag{3.16}$$

Applying the Gauss-Ostroradski theorem, from (3.15), we obtain

$$\int\limits_{\partial\Omega} \{U''(z)\}^+\cdot\{\widetilde{P}(\partial,n)U''(z)\}^+\,ds = \int\limits_{\Omega^+} \widetilde{E}(U'',U'')dx \tag{3.17}$$

where

$$\begin{aligned} U''(z)\cdot\widetilde{P}(\partial,n(z))U''(z) = (\lambda+2\mu)(n(z)\cdot u(z))\,\mathrm{div}\,u(z)& \\ - \mu u(z)\cdot[n(z)\times \mathrm{rot}\,u(z)] + \eta v(z)(n(z)\cdot u(z)) + \eta_1 v(z)\frac{\partial v(z)}{\partial n(z)}&. \end{aligned} \tag{3.18}$$

Applying the boundary conditions of problem $(N)_0^+$, we obtain

$$\{U''(z)\}^+ \cdot \left\{\widetilde{P}(\partial, n)U''(z)\right\}^+ = 0, \quad z \in \partial\Omega.$$

Using this equality in (3.17), we have

$$\int\limits_{\Omega^+} \widetilde{E}(U'', U'')dx = 0. \tag{3.19}$$

According to inequalities $\mu > 0, \;\; \lambda + 2\mu > 0, \;\; \eta_1 > 0, \;\; \mu\eta_2 > \eta^2$, we have

$$\begin{aligned} \widetilde{E}(U'', U'') &= (\lambda + \mu)(\operatorname{div} u(x))^2 + \mu(\operatorname{rot} u(x))^2 \\ &+ \frac{1}{\mu}(\eta v(x) + \mu \operatorname{div} u(x))^2 + \frac{\mu\eta_2 - \eta^2}{\mu} v^2(x) \geq 0, \quad x \in \Omega^+. \end{aligned} \tag{3.20}$$

By virtue of this fact, (3.19) implies

$$\widetilde{E}(U'', U'') = 0, \quad x \in \Omega^+.$$

Hence, taking into account (3.20), we obtain

$$\operatorname{div} u(x) = 0, \quad \operatorname{rot} u(x) = 0, \quad v(x) = 0.$$

A solution of system $\operatorname{div} u(x) = 0$, $\operatorname{rot} u(x) = 0$, $x \in \Omega^+$ has the form

$$u(x) = \operatorname{grad} \Psi(x), \;\; x \in \Omega^+, \tag{3.21}$$

where $\Psi(x)$ is an arbitrary harmonic function.

Since $\{n(z) \cdot u(z)\}^+ = 0$, the harmonic function $\Psi(x)$ satisfy, on the boundary $\partial\Omega$, the Neumann condition

$$\left\{\frac{\partial\Psi(z)}{\partial n(z)}\right\}^+ = 0, \;\; z \in \partial\Omega.$$

As is known, the homogeneous Neumann problem has the solution $\Psi(x) = c = const$. Substitutins this value of $\Psi(x)$ into (3.21), we obtain $u(x) = 0, \;\; x \in \Omega^+$.

Thus the homogeneous problem $(N)_0^+$ has only a trivial solution. Hance it follows that problem $(N)^+$ admits no more then regular solution.

4 Solution of Problems

A Solution of Problem $(N)^+$ will be sought for in the form (2.5), where the functions $\Phi_j(x)$, $j = 1, 2, ..., 8$, are represented as [17]

$$\begin{aligned}
\Phi_j(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \left(\frac{r}{R}\right)^k Y_k^{(m)}(\vartheta, \varphi) A_{mk}^{(j)}, \quad j = 1, 2, 3, 4, \\
\Phi_5(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} g_k(\lambda_1 r) Y_k^{(m)}(\vartheta, \varphi) A_{mk}^{(5)}, \\
\Phi_j(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} g_k(\lambda_2 r) Y_k^{(m)}(\vartheta, \varphi) A_{mk}^{(j)}, \quad j = 6, 7, \\
\Phi_8(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} g_k(\lambda_3 r) Y_k^{(m)}(\vartheta, \varphi) A_{mk}^{(8)},
\end{aligned} \tag{4.1}$$

where $A_{mk}^{(j)}$, $j = 1, 2, \ldots, 8$, are the constants to be defined, and

$$g_k(\lambda_j r) = \sqrt{\frac{R}{r}} \frac{I_{k+\frac{1}{2}}(\lambda_j r)}{I_{k+\frac{1}{2}}(\lambda_j R)}, \quad j = 1, 2, 3,$$

$I_{k+1/2}(\lambda_j r)$ is the Bessel function with an imaginary argument.

Substituting the values of $\Phi_j(x)$, $j = 1, 3, 6, 7$, from (4.1), in (2.10) and using equalities

$$\int\limits_{\partial\Omega} Y_k^{(m)}(\vartheta, \varphi) ds = \begin{cases} \sqrt{\pi} R^2, & k = o, \;\; m = 0, \\ 0, & in \;\; other \;\; cases, \end{cases}$$

we get that $A_{00}^{(j)} = 0$, $j = 1, 3, 6, 7$.

Substituting the values of the function $\Phi_j(x)$, $j = 1, 2, \ldots, 8$, defined by (4.1) in (2.5) and taking into consideration the equalities [3]

$$\begin{aligned}
\text{grad}\left[a(r) Y_k^{(m)}(\vartheta, \varphi)\right] &= \frac{da(r)}{dr} X_{mk}(\vartheta, \varphi) + \frac{\sqrt{k(k+1)}}{r} a(r) Y_{mk}(\vartheta, \varphi), \\
\text{rot}\left[x a(r) Y_k^{(m)}(\vartheta, \varphi)\right] &= \sqrt{k(k+1)}\, a(r) Z_{mk}(\vartheta, \varphi), \\
\text{rot}\,\text{rot}\left[x a(r) Y_k^{(m)}(\vartheta, \varphi)\right] &= \frac{k(k+1)}{r} a(r) X_{mk}(\vartheta, \varphi) \\
&\quad + \sqrt{k(k+1)} \left(\frac{d}{dr} + \frac{1}{r}\right) a(r) Y_{mk}(\vartheta, \varphi), \\
x a(r) Y_k^{(m)}(\vartheta, \varphi) &= r a(r) X_{mk}(\vartheta, \varphi),
\end{aligned}$$

where $a(r)$ is the function of r, we obtain

$$\begin{aligned} u(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ & u_{mk}^{(1)}(r) X_{mk}(\vartheta,\varphi) \\ & + \sqrt{k(k+1)} \Big[v_{mk}^{(1)}(r) Y_{mk}(\vartheta,\varphi) + w_{mk}^{(1)}(r) Z_{mk}(\vartheta,\varphi) \Big] \Big\}, \\ w(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ & u_{mk}^{(2)}(r) X_{mk}(\vartheta,\varphi) \\ & + \sqrt{k(k+1)} \Big[v_{mk}^{(2)}(r) Y_{mk}(\vartheta,\varphi) + w_{mk}^{(2)}(r) Z_{mk}(\vartheta,\varphi) \Big] \Big\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \theta(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} u_{mk}^{(3)}(r) Y_k^{(m)}(\vartheta,\varphi), \\ v(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} u_{mk}^{(4)}(r) Y_k^{(m)}(\vartheta,\varphi), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} u_{mk}^{(1)}(r) = {} & \frac{k}{R}\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(1)} + R(k+1)(b(k+2)+k)\left(\frac{r}{R}\right)^{k+1} A_{mk}^{(2)} \\ & + a_5(k+2)R\left(\frac{r}{R}\right)^{k+1} A_{mk}^{(4)} + a_6 \frac{d}{dr} g_k(\lambda_1 r) A_{mk}^{(5)} \\ & - \eta \frac{d}{dr} g_k(\lambda_3 r) A_{mk}^{(8)}, \quad k \geq 0, \\ v_{mk}^{(1)}(r) = {} & \frac{1}{R}\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(1)} + R(b(k+1)+k+3)\left(\frac{r}{R}\right)^{k+1} A_{mk}^{(2)} \\ & + a_5 R\left(\frac{r}{R}\right)^{k+1} A_{mk}^{(4)} + \frac{a_6}{r} g_k(\lambda_1 r) A_{mk}^{(5)} \\ & - \frac{\eta}{r} g_k(\lambda_3 r) A_{mk}^{(8)}, \quad k \geq 1, \\ u_{mk}^{(2)}(r) = {} & \frac{a_1 k(2k+3)}{R}\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(4)} + a_2 \frac{d}{dr} g_k(\lambda_1 r) A_{mk}^{(5)} \\ & + \frac{k(k+1)}{r} g_k(\lambda_2 r) A_{mk}^{(6)}, \quad k \geq 0, \\ v_{mk}^{(2)}(r) = {} & \frac{a_1(2k+3)}{R}\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(4)} + \frac{a_2}{r} g_k(\lambda_1 r) A_{mk}^{(5)} \\ & + \left(\frac{d}{dr} + \frac{1}{r}\right) g_k(\lambda_2 r) A_{mk}^{(6)}, \quad k \geq 1, \\ w_{mk}^{(1)}(r) = {} & \left(\frac{r}{R}\right)^{k} A_{mk}^{(3)}, \quad w_{mk}^{(2)}(r) = g_k(\lambda_2 r) A_{mk}^{(7)}, \quad k \geq 1, \\ u_{mk}^{(3)}(r) = {} & 2(\lambda+2\mu)(2k+3)\left(\frac{r}{R}\right)^{k} A_{mk}^{(4)} + (\lambda+2\mu)\lambda_1^2 g_k(\lambda_1 r) A_{mk}^{(5)}, \quad k \geq 0, \end{aligned}$$

$$u_{mk}^{(4)}(r) = \frac{2\eta a}{\eta_1 \lambda_3^2}\,(2k+3)(k+1)\Big(\frac{r}{R}\Big)^k A_{mk}^{(2)} + a_3(2k+3)\Big(\frac{r}{R}\Big)^k A_{mk}^{(4)}$$
$$+ a_4 g_k(\lambda_1 r) A_{mk}^{(5)} + (\lambda+2\mu)\lambda_3^2 g_k(\lambda_3 r) A_{mk}^{(8)}, \quad k \geq 0.$$

If we substitute the values of the vectors $u(x)$, $w(x)$ and the functions $\theta(x)$, $v(x)$ from (4.2) in (2.6) and take into account the equalities [2]

$$e_r \cdot X_{mk}(\vartheta,\varphi) = Y_k^{(m)}(\vartheta,\varphi), \quad e_r \cdot Y_{mk}(\vartheta,\varphi) = 0, \quad e_r \cdot Z_{mk}(\vartheta,\varphi) = 0,$$
$$e_r \times X_{mk}(\vartheta,\varphi) = 0, \quad e_r \times Y_{mk}(\vartheta,\varphi) = -Z_{mk}(\vartheta,\varphi),$$
$$e_r \times Z_{mk}(\vartheta,\varphi) = Y_k^{(m)}(\vartheta,\varphi);$$
$$\operatorname{rot}\big[a(r)X_{mk}(\vartheta,\varphi)\big] = \frac{\sqrt{k(k+1)}}{r}\,a(r)Z_{mk}(\vartheta,\varphi),$$
$$\operatorname{rot}\big[a(r)Y_{mk}(\vartheta,\varphi)\big] = -\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Z_{mk}(\vartheta,\varphi),$$
$$\operatorname{rot}\big[a(r)Z_{mk}(\vartheta,\varphi)\big] = \frac{\sqrt{k(k+1)}}{r}\,a(r)X_{mk}(\vartheta,\varphi) + \Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi),$$

then we get

$$n(x)\cdot u(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} u_{mk}^{(1)}(r)Y_k^{(m)}(\vartheta,\varphi),$$
$$n(x)\cdot w(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} u_{mk}^{(2)}(r)Y_k^{m}(\vartheta,\varphi),$$
$$n(x)\times \operatorname{rot} u(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} \sqrt{k(k+1)}\,\Big\{\Big[\frac{1}{r}u_{mk}^{(1)}(r) - (\frac{d}{dr}+\frac{1}{r})v_{mk}^{(1)}(r)\Big]$$
$$\times Y_{mk}(\vartheta,\varphi) - (\frac{d}{dr}+\frac{1}{r})w_{mk}^{(1)}(r)Z_{mk}(\vartheta,\varphi)\Big\},$$

$$n(x)\times \operatorname{rot} w(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} \sqrt{k(k+1)}\,\Big\{\Big[\frac{1}{r}u_{mk}^{(2)}(r) - \Big(\frac{d}{dr}+\frac{1}{r}\Big)v_{mk}^{(2)}(r)\Big]$$
$$\times Y_{mk}(\vartheta,\varphi) - \Big(\frac{d}{dr}+\frac{1}{r}\Big)w_{mk}^{(2)}(r)Z_{mk}(\vartheta,\varphi)\Big\}, \tag{4.4}$$
$$\frac{\partial v(x)}{\partial n(x)} = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} \frac{d}{dr}u_{mk}^{(4)}(r)Y_k^{(m)}(\vartheta,\varphi).$$

Since on the sphere Σ_1 the sets $\{Y_k^{(m)}(\vartheta,\varphi)\}_{|m|\leq k,\ k=\overline{0,\infty}}$ and $\{X_{mk}(\vartheta,\varphi), Y_{mk}(\vartheta,\varphi),\ Z_{mk}(\vartheta,\varphi)\}_{|m|\leq k,\ k=\overline{0,\infty}}$ form a complete ortonormal system in the space $L_2(\Sigma_1)$ and provided that the sufficient condition of smoothness is fulfilled, we can represent function $f_j(z)$, $j=4,5,6,7$ and the vector $f^{(j)}(z)$, $j=1,2$, as Fourier series (2.7)–(2.8).

Taking into account that $n(z)\cdot f^{(j)}(z)=0$, $j=1,2$, from (2.7) we obtain

$$f^{(j)}(\vartheta,\varphi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}\left[\beta_{mk}^{(j)}Y_{mk}(\vartheta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\vartheta,\varphi)\right], \tag{4.5}$$

$$j=1,2.$$

Passing on both sides of equalities (4.3), (4.4) to the limit as $x\to z\in\partial\Omega$ and taking into account the boundary condition of Problem $(N)^+$ and also formulas (2.8) and (4.5), for the unknown constants $A_{mk}^{(j)}$, $j=1,2,...,8$, we obtain the following systems of algebraic equations:

$$u_{00}^{(j)}(R)=\alpha_{00}^{(j+3)},\ \ j=1,2,\ \ u_{00}^{(4)}(R)=\alpha_{00}^{(7)},\ \ or\ \ \frac{d}{dR}u_{00}^{(4)}(R)=\alpha_{00}^{(7)}, \tag{4.6}$$

$$\begin{gathered}\left(\frac{d}{dR}+\frac{1}{R}\right)v_{mk}^{(j)}(R)=\alpha_{mk}^{(j+3)}-\beta_{mk}^{(j)},\ \ j=1,2,\ \ k\geq1,\\ \left(\frac{d}{dR}+\frac{1}{R}\right)w_{mk}^{(j)}(R)=-\gamma_{mk}^{(j)}\ \ j=1,2,\ \ k\geq1,\\ u_{mk}^{(j)}(R)=\alpha_{mk}^{(j+3)},\ \ j=1,2,3,\\ u_{mk}^{(4)}(R)=\alpha_{mk}^{(7)},\ \ or\ \ \frac{d}{dR}u_{mk}^{(4)}(R)=\alpha_{mk}^{(7)},\ \ k\geq1.\end{gathered} \tag{4.7}$$

It is assumed here that

$$\frac{d}{dR}\,g_k(\lambda_jR)=\lim_{r\to R}\frac{d}{dr}\,g_k(\lambda_jr),\ \ j=1,2.$$

Due to theorem 2.5 and theorem 3.1 this systems (4.6), (4.7) are uniquely solvable with respect to the unknowns $A_{mk}^{(j)}$, $j=1,2,...,8$. Thus we can construct explicity the formal solution of the problem $(N)^+$ in the form of series. Further we have to investigate the convergence of these formal series an their derivatives.

The following asymptotic representation are valid for $k\to+\infty$ [17]

$$g_k(k_jr)\approx\left(\frac{r}{R}\right)^k,\ \ g_k'(k_jr)\approx\frac{k}{R}\left(\frac{r}{R}\right)^k,\ \ r<R. \tag{4.8}$$

If $x\in\Omega^+$ $(r<R)$, then by the asymptotic (4.8), the series (4.2)– (4.4) convergent absolutely and uniformly convergent provided that the following majorized series

$$\sum_{k=k_0}^{\infty}k^{3/2}\Big[\sum_{j=1}^{2}k(|\beta_{mk}^{(j)}|+|\gamma_{mk}^{(j)}|)+\sum_{j=4}^{7}|\alpha_{mk}^{(j)}|\Big], \tag{4.9}$$

are convergent. Series (4.9) will be convergent if the coefficients $\alpha_{mk}^{(j)}$, $j=4,5,6,7$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j=1,2$ admit the estimates:

$$\begin{gathered}\beta_{mk}^{(j)}=O(k^{-4}),\ \ \ \gamma_{mk}^{(j)}=O(k^{-4}),\ \ j=1,2,\\ \alpha_{mk}^{(j)}=O(k^{-3})\ \ j=4,5,6,7.\end{gathered} \tag{4.10}$$

According to Lemma 2.2 and Lemma 2.3, estimates (4.10) will hold if we require of the boundary vector-functions to satisfy the following smoothness conditions

$$f^{(j)}(z) \in C^3(\partial\Omega), \quad j = 1, 2, \quad f_j(z) \in C^3(\partial\Omega), \quad j = 4, 5, 6, 7. \tag{4.11}$$

Therefore if the boundary vector-functions satisfy conditions (4.11), then the vector $U = (u, w, \theta, v)^\top$ represented by equalities (4.2)-(4.3) will be a regular solution of problem $(N)^+$.

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