GENERALIZATION OF I. VEKUA REDUCTION METHOD FOR PHYSICALLY AND GEOMETRICALLY NON-LINEAR AND NON-SHALLOW SHELLS

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Abstract

In the present paper, by means of Vekua's method, the system of differential equations for the Geometrically and Physically nonlinear theory non-shallow shells is obtained.

Key words and phrases: Non-shallow shells, metric tensor and tensor of curvature midsurface of the shell.

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I. Vekua constructed several versions of the refined linear theory of thin and shallow shells, containing, the regular processes by means of the method of reduction of 3-D problems of elasticity to 2-D ones.

By thin and shallow shells I.Vekua means 3-D shell type elastic bodies satisfying the following conditions

$$a_\alpha^\beta - x^3 b_\alpha^\beta \cong \alpha_\alpha^\beta \quad -h \le x^3 = x_3 \le b, \quad \alpha, \beta = 1, 2, \qquad (*)$$

where a_α^β and b_α^β are mixed components of the metric and curvature tensors of the midsurface of the shell, x^3 is the thickness coordinate and h is the semi-thickness.

In the sequel, under non-shallow shells we wean elastic bodies free from the assumption of the type (*) or, more exactly, the bodies with the conditions

$$a_\alpha^\beta - x_3 b_\alpha^\beta \neq a_\alpha^\beta \Rightarrow |hb\beta_\alpha| \le q < 1.$$

Such kind of shells are called shells with varying in thickness geometry, or non-shallow shells.

1 The Coordinate System in a Shell Normally Connected with a Surface

Let Ω denote a shall and a domain of the space occupied by the shell. Inside the shell, we consider a smooth surface S with respect to which the shell Ω lies symmetrically. The surface S is called the midsurface of the shell Ω. To construct the theory of shells, we use more convenient coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\vec{R}$ of any point of the domain Ω can be represented in the form

$$\vec{R}(x^1,x^2,x^3)=\vec{r}(x^1,x^2)+x^3\vec{n}(x^1,x^2)\ \ (x^3=x_3),$$

where $\vec{R}$ and $\vec{n}$ are respectively the radius-vector and the unit vector of the normal of the surface $S(x^3=0)$ and (x^1,x^2) are the Gaussian parameters of the midsurfaces S.

The covariant and contravariant basis vectors $\vec{R}_i$ and $\vec{R}^i$ of the surfaces $\hat{S}(x^3=\text{const})$, and the corresponding basis vectors $\vec{r}_i$ and $\vec{r}^{\,i}$ of the midsurface $S(x^3=0)$ are connected by the following relations:

$$\vec{R}_i=A_{i.}^{\cdot j}\vec{r}_j=A_{ij}\vec{r}^{j},\quad \vec{R}^i=A_{\cdot j}^{i\cdot}\vec{r}^{j}=A^{ij}\vec{r}_j,\quad (i,j=1,2,3),$$

where

$$A_{i.}^{\cdot j}=\begin{cases} a_\alpha^\beta-x_3b_\alpha^\beta, & i=\alpha,\ \ j=\beta\\ \delta_i^3, j=3\ \ (\alpha,\beta=1,2)\end{cases},\qquad \vec{r}_i,\vec{r}^{\,i}=\begin{cases}\vec{r}_\alpha,\vec{r}^{\,\alpha}, & i=\alpha,\\ \vec{n},\vec{n}, & i=3,\end{cases} \tag{1.1}$$

$$A_j^i=\begin{cases}\dfrac{(1-2Hx_3)a_\beta^\alpha+x_3b_\beta^\alpha}{1-2Hx_3+Kx_3^2}, & i=\alpha,\ j=\beta\\ \delta_i^3,\ \ j=3,\ (\alpha,\beta=1,2)\end{cases} \tag{1.2}$$

Here $(a_{\alpha\beta},a^{\alpha\beta},a_\alpha^\beta)$ and $(b_{\alpha,\beta,}b^{\alpha\beta},b_\alpha^\beta)$ are the components (covariant, contravariant and mixed) of the metric and curvature tensors of the midsurface S. By H and K we denote a middle and Gaussian curvature of the surface S, where

$$2H=b_\alpha^\alpha=b_1^1+b_2^2,\quad K=b_1^1b_2^2-b_2^1b_1^2.$$

It should be noted that for the refined theory of non-shallow shells (Koiter, Haghdi, Lurie) these relations have the form

$$\vec{R}^\alpha\cong(a_\beta^\alpha+x_3b_\beta^\alpha)\vec{r}^{\beta},\quad \vec{R}_\alpha=(a_\alpha^\beta-x_3b_\alpha^\beta)\vec{r}_\beta,\quad (x^3=x_3) \tag{1.2$_1$}$$

where

$$\vec{R}^\alpha=\{a_\beta^\alpha+x_3b_\beta^\alpha+\frac{1}{2}(4H^2-K)x_3^2+\cdots\}\vec{r}^{\beta}$$

The main quadratic forms of the midsurface $S(x_3 = 0)$ have the forms

$$I = ds^2 = a_{\alpha\beta}dx^\alpha dx^\beta, \quad II = K_s ds^2 = b_{\alpha\beta}dx^\alpha dx^\beta, \tag{1.3}$$

where k_s is the normal courvative of the S and

$$a_{\alpha\beta} = \vec{r}_\alpha \vec{r}_\beta, \;\; b_{\alpha\beta} = -\vec{n}_\alpha \vec{r}_\beta, \;\; k_s = b_{\alpha\beta}s^\alpha s^\beta, \;\; \vec{r}_\alpha = \partial_\alpha \vec{r}, \;\; s^\alpha = \frac{dx^\alpha}{ds} \tag{1.4}$$

To construct the theory of non-shallow shells, it is necessary to obtain formulas for a family of surfaces $\hat{S}(x^3 = \text{const})$ analogous to (1.3), (1.4), which have the forms

$$I = d\hat{s}^2 = g_{\alpha\beta}dx^\alpha dx^\beta, \;\; II = \hat{K}_{\hat{S}}ds^2 = \hat{b}_{\alpha\beta}dx^\alpha dx^\beta, \tag{1.5}$$

where

$$\begin{aligned} g_{\alpha\beta} &= \vec{R}_\alpha \vec{R}_\beta = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2(2Hb_{\alpha\beta} - Ka_{\alpha\beta}), \\ \hat{b}_{\alpha\beta} &= (1 - 2Hx_3)b_{\alpha\beta} + x_3 K a_{\alpha\beta} \end{aligned} \tag{1.6}$$

and $\hat{k}_s = \hat{b}_{\alpha\beta}s^\alpha s^\beta$ is the normal courvature of the $\hat{S}$.

The expressions for the unit tangent vector $\hat{\vec{S}}$ and the tangential normal $\hat{\vec{l}}$ of the surface $\hat{S}$ have the forms

$$\begin{aligned} &\hat{\vec{S}} = \frac{d\vec{R}}{d\hat{s}} = \Big[(1 - x_3 k_s)\hat{\vec{s}} + x_3 \tau_s \hat{\vec{l}}\Big]\frac{ds}{d\hat{s}}, \\ &\hat{\vec{l}} = \hat{\vec{S}} \times \vec{n} = \Big[(1 - x_3 k_s)\hat{\vec{l}} - x_3 \tau_s \hat{s}\Big]\frac{ds}{d\hat{s}}, \\ &d\hat{s} = \sqrt{1 - 2x_3 k_s + (k_s^2 + \tau_s^2)x_3^2}\,ds, \;\; \hat{\vec{l}} \times \hat{\vec{s}} = \vec{n} \end{aligned} \tag{1.7}$$

where $d\hat{s}$ and ds are linear elements of the surfaces $\hat{S}$ and S, τ_s is the geodesic torsion of the surface S.

The formula

$$\hat{\vec{l}}\vec{R}_\alpha = (1 - 2Hx_3 + kx_3)(\vec{l}\vec{r}_\alpha)\frac{ds}{d\hat{s}}, \tag{1.8}$$

which is necessary in writing the reduced basic boundary-value problems in stresses, is also valid.

2 System Geometrically and Physically Nonlinear Equation for Non-Shallow Shells

We write the equation of equilibrium of on elastic shell-type body in a vector form which is convenient for reduction to the 2-D equations

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\vec{\sigma}^i}{dx^i} + \vec{\Phi} = 0 \Rightarrow \hat{\nabla}_i \vec{\sigma}^i + \vec{\Phi} = 0, \tag{2.1}$$

where g is the discriminant of the metric quadratic form of the 3-D domain $\Omega, \hat{\nabla}_i$ are covariant derivatives with respect to the space coordinates x^i, $\vec{\Phi}$ is an external force, $\vec{\sigma}^i$ are the contravariant constituents of the stress vector $\vec{\sigma}^i_{(\overset{*}{\vec{l}})}$ acting in the area with the normal $\overset{*}{\vec{l}}$ and representable as the Cauchy formulas as follows

$$\vec{\sigma}_{(\overset{*}{\vec{l}})} = \vec{\sigma}^i \overset{*}{l}_i, \quad \overset{*}{l}_i = \overset{*}{\vec{l}} \vec{R}_i$$

Using relation (1.7), for the stress vector acting on the area with normal $\overset{*}{\vec{l}}$ we obtain

$$\vec{\sigma}_{(\hat{\vec{l}})} = \vec{\sigma}^\alpha (\hat{\vec{l}} \vec{R}_\alpha) = \sqrt{\frac{g}{a}} (\vec{l} \vec{r}_\alpha) \frac{ds}{d\hat{s}} \tag{2.2}$$

where

$$\sqrt{\frac{g}{a}} = 1 - 2Hx_3 + Kx_3^2 \quad (-h \le x_3 = x^3 \le h)$$

A material is said to be hyper-elastic if the stresses are obtained by means of the strain energy function

$$\sigma^{ij} = \frac{\partial \exists}{\partial e_{ij}},$$

where σ^{ij} are contravariant components of the stress tensor, $\exists$ is the strain energy function, and e_{ij} are covariant components of the strain tensor.

The theory of hyper-elasticity of the second order has the form

$$\begin{aligned} &\exists = \frac{1}{2} E^{ijpq} e_{ij} e_{pq} + \frac{1}{3} E^{ijpqsk} e_{ij} e_{pq} e_{sk}, \\ &e_{ij} = \frac{1}{2} (\vec{R}_i \partial_j \vec{U} + \vec{R}_j \partial_i \vec{U} + \partial_i \vec{U} \partial_j \vec{U}) \\ &\sigma^{ij} = E^{ijpq} e_{pq} + E^{ijpqsk} e_{pq} e_{sk}, \quad \vec{\sigma}^i = \sigma^{ij} (\vec{R}_j + \partial_j \vec{U}) \end{aligned} \tag{2.3}$$

where E^{ijpq} and E^{ijpqsk} are coefficients of elasticity of the first and second order and $\vec{u}$ is the displacement vector.

Coefficients of elasticity of the first order for isotropic elastic bodies are expressed by the two Lame coefficients.

$$E^{ijpq} = \lambda g^{ij} g^{\mu q} + \mu (g^{ip} g^{jq} + g^{iq} g^{jp}) (g^{ij} = \vec{R}^i \vec{R}^j) \tag{2.4}$$

and coefficients of elasticity of the second order are defined by the formula

$$E^{ijpqsk} = (E_1 + E_2) g^{ij} g^{pq} g^{sk} - E_2 g^{ij} g^{pk} g^{qs} + E_3 g^{ip} g^{jq} g^{sk} + E_4 g^{is} g^{pq} g^{jk}, \tag{2.5}$$

where E_1, E_2, E_3 and E_4 are modules of elasticity of the second order for isotropic elastic bodies.

To reduce the 3-D problems of the theory of elasticity to 2-D problems, it is necessary the rewrite the relation (2.1-2.5) in terms of the midsurface S of the shell Ω.

Relation (2.1) can be written as follows:

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\theta\vec{\sigma}^\alpha}{\partial x^\alpha}+\frac{\partial\theta\vec{\sigma}^3}{\partial x^3}+\theta\vec{\Phi}=0, \quad (\theta=1-2Hx_3+kx_3). \tag{2.6}$$

from (2.3)-(2.5) we obtain

$$\begin{aligned}&\vec{\sigma}^i=\sigma^{ij}(\vec{R}_j+\partial_j\vec{U})=(E^{ijpq}+E^{ijpqsk}e_{sk})e_{pq}(\vec{R}_j+\partial_j\vec{U})\\ &\Rightarrow\vec{\sigma}^i=\tfrac{1}{2}A^i_{i_1}[M^{i_1j_1p_1q_1}+\tfrac{1}{2}M^{i_1j_1p_1q_1s_1k_1}\\ &\times(A^k_{k_1}\vec{r}_{s_1}\partial_k\vec{U}+A^s_{s_1}A^k_{k_1}\partial_s\vec{U}\partial_k\vec{U})]\\ &\times(A^p_{p_1}\vec{r}_{q_1}\partial_p\vec{U}+A^q_{q_1}\vec{r}_{p_1}\partial_q\vec{U}+A^p_{p_1}A^q_{q_1}\partial_p\vec{U}\partial_q\vec{U})(\vec{r}_{j_1}+A^j_{j_1}\partial_j\vec{U}),\end{aligned} \tag{2.7}$$

where

$$M^{i_1j_1p_1q_1}=\lambda a^{i_1j_1}a^{p_1q_1}+\mu(a^{i_1p_1}a^{j_1q_1}+a^{i_1q_1}a^{j_1p_1}) \tag{2.8}$$

$$\begin{aligned}M^{i_1j_1p_1q_1s_1k_1}&=(E_1+E_2)a^{i_1j_1}a^{p_1q_1}-E_2a^{i_1j_1}a^{p_1k_1}q^{q_1s_1}\\ &+E_3a^{i_1p_1}a^{j_1q_1}a^{s_1k_1}+E_4a^{i_1s_1}a^{p_1q_1}a^{j_1k_1},\end{aligned} \tag{2.9}$$

$$(a^{ij}=\vec{r}^i\vec{r}^j).$$

3 Vekua's Method of Reduction

There are many different methods of reducing 3-D problems of the theory of elasticity to 2-D one of the theory shells (Reissner, Mindlin, Koiter, Naghdi, A. Lurie ...)

In the present paper, we realize the reduction by the method suggested by I. Vekua. Since the system of Legender polynomials $\{P_m(\frac{x_3}{a})\}_{m=0}^{\infty}$ is complete in the interval [-h,h]for equation (2.6) we obtain the equivalent infinite system of 2-D equations

$$\int\limits_{-h}^{h}\left[\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\theta\vec{\sigma}^\alpha}{\partial x^\alpha}+\frac{\partial\theta\vec{\sigma}^3}{\partial x^3}+\theta\vec{\Phi}\right]P_m\left(\frac{x_3}{h}\right)dx_3=0, \quad (x_3=x^3),$$

or in the form

$$\nabla_\alpha\overset{(m)}{\vec{\sigma}}{}^\alpha-\frac{2m+1}{2h}\left(\overset{(m-1)}{\vec{\sigma}_3}+\overset{(m-3)}{\vec{\sigma}_3}+\cdots\right)+\overset{(m)}{\vec{F}}=0, \tag{3.1}$$

where

$$\left(\overset{(m)}{\vec{\sigma}^i}, \overset{(m)}{\vec{\Phi}}\right) = \frac{2m+1}{2h}\int_{-h}^{h}(\theta\vec{\sigma}^i, \theta\vec{\Phi})P_m\frac{x_3}{h}dx_3,$$

$$\overset{(m)}{\vec{F}} = \overset{(m)}{\vec{\Phi}} + \frac{2m+1}{2h}\sqrt{\frac{g^+}{a}} + \overset{(+)}{\vec{\sigma}_3} - (-1)^m\sqrt{\frac{g^-}{a}}\overset{(-)}{\vec{\sigma}_3}, \qquad (3.1_1)$$

$$\sqrt{\frac{g^\pm}{a}} = \theta(\pm h) = 1 \mp 2Hh + Kh^2.$$

and ∇_α are covariant derivatives on the surface $S(x^3 = 0)$.

Thus, we have obtained the infinite system of 2-D equations of the theory of shells for which the boundary conditions on the face surface ($x_3 = \pm h$) are satisfied, i.e. $\vec{\sigma}_3{}^{(+)} = \vec{\sigma}_3(x^1, x^2, \pm h)$ is the preassigned vector field.

The relation (2.7) can be written as follows

$$\begin{aligned}
\overset{(m)}{\vec{\sigma}^i} &= \frac{2m+1}{2h}\int_{-h}^{h}\theta\vec{\sigma}^i P_m\left(\frac{x_3}{h}\right)dx_3 = M^{i_1j_1p_1q_1}\sum_{m_1=0}^{\infty} \\
&\left\{\underset{(m_1)}{\overset{(m)}{A}}{}^{ip}_{i_1p_1}(\vec{r}_{q_1}D_p\overset{(m_1)}{\vec{U}})\vec{r}_{j_1} + \sum_{m_2=0}^{\infty}\left[\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ijp}_{i_1j_1p_1}\left(\vec{r}_{q_1}D_p\overset{(m_1)}{\vec{U}}\right)D_j\overset{(m_2)}{\vec{U}}\right.\right. \\
&+\frac{1}{2}\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ipq}_{i_1p_1q_1}\left(D_p\overset{(m_1)}{\vec{U}}D_q\overset{(m_2)}{\vec{U}}\right)\vec{r}_{j_1} \\
&\left.\left.+\frac{1}{2}\sum_{m_3=0}^{\infty}\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{ipq}_{i_1p_1q_1}\left(D_p\overset{(m_1)}{\vec{U}}D_q\overset{(m_2)}{\vec{U}}\right)D_j\overset{(m_3)}{\vec{U}}\right]\right\} \\
&+M^{i_1j_1p_1q_1s_1k_1}\sum_{m_1,m_2=0}^{\infty}\left\{\underset{(m_1m_2)}{\overset{(m)}{A}}{}^{ipk}_{i_1p_1k_1}\left(\vec{r}_{q_1}D_p\overset{(m_1)}{\vec{U}}\right)\left(\vec{r}_{s_1}D_k\overset{(m_2)}{\vec{U}}\right)\vec{r}_{j_1}\right. \qquad (3.2) \\
&+\sum_{m_3=0}^{\infty}\left[\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{ijpk}_{i_1j_1p_1k_1}\left(\vec{r}_{q_1}D_p\overset{(m_1)}{\vec{U}}\right)\left(\vec{r}_{s_1}D_k\overset{(m_2)}{\vec{U}}\right)D_j\overset{(m_3)}{\vec{U}}\right. \\
&+\frac{1}{2}\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{ipsk}_{i_1p_1s_1k_1}\left(\vec{r}_{q_1}D_p\overset{(m_1)}{\vec{U}}\right)\left(D_s\overset{(m_2)}{\vec{U}}D_k\overset{(m_3)}{\vec{U}}\right)\vec{r}_{j_1} \\
&+\frac{1}{2}\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{ipqk}_{i_1p_1q_1k_1}\left(D_p\overset{(m_1)}{\vec{U}}D_q\overset{(m_2)}{\vec{U}}\right)\left(\vec{r}_{s_1}D_k\overset{(m_3)}{\vec{U}}\right)\vec{r}_{j_1} \\
&+\frac{1}{2}\sum_{m_4=0}^{\infty}\left(\underset{(m_1,\ldots,m_4)}{\overset{(m)}{A}}{}^{ijpqk}_{i_1j_1p_1q_1k_1}\left(D_p\overset{(m_1)}{\vec{U}}D_q\overset{(m_2)}{\vec{U}}\right)D_j\overset{(m3)}{\vec{U}}\left(r\vec{_{s_1}}D_k\overset{(m_4)}{\vec{U}}\right)\right.
\end{aligned}$$

$$+\underset{(m_1,\dots,m_4)}{\overset{(m)}{A}}{}^{ipqsk}_{i_1p_1q_1s_1k_1}\left(D_p\overset{(m_1)}{\vec{U}}D_q\overset{(m_2)}{\vec{U}}\right)\left(D_k\overset{(m_3)}{\vec{U}}D_s\overset{(m_4)}{\vec{U}}\right)\vec{r}_{j_1}$$

$$+\underset{(m_1,\dots,m_4)}{\overset{(m)}{A}}{}^{ijpks}_{i_1j_1p_1k_1s_1}\left(\vec{r}_{q_1}D_p\overset{(m_1)}{\vec{U}}\right)D_j\overset{(m_2)}{\vec{U}}\left(D_s\overset{(m_3)}{\vec{U}}D_k\overset{(m_4)}{\vec{U}}\right)\Bigg)\Bigg]$$

$$+\frac{1}{4}\sum_{m_3,m_4,m_5}^{\infty}\underset{(m_1,\dots,m_5)}{\overset{(m)}{A}}{}^{ijpqsk}_{i_1j_1p_1q_1s_1k_1}\left(D_p\overset{(m_1)}{\vec{U}}D_q\overset{(m_2)}{\vec{U}}\right)D_j\overset{(m_3)}{\vec{U}}\left(D_s\overset{(m_4)}{\vec{U}}D_k\overset{(m_5)}{\vec{U}}\right)\Bigg\},$$

where

$$D_i\overset{(m)}{U}=\delta_i^{\beta}\partial_p\overset{(m)}{U}+\delta_i^3\overset{(m)}{U'},\qquad \overset{(m)}{U'}=\frac{2m+1}{h}\left(\overset{(m+1)}{\vec{U}}\overset{(m+3)}{\vec{U}}+\cdots\right) \tag{3.3}$$

$$\begin{aligned}
&\underset{(m_1)}{\overset{(m)}{A}}{}^{ij}_{i_1j_1}=\frac{2m+1}{2h}\int_{-h}^{h}\theta A^i_{i_1}A^j_{j_1}P_{m_1}P_m dx_3,\\
&\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ijp}_{i_1j_1p_1}=\frac{2m+1}{2h}\int_{-h}^{h}\theta A^i_{i_1}A^j_{j_1}A^p_{p_1}P_{m_1}P_{m_2}P_m dx_3,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\underset{(m_1,\dots,m_5)}{\overset{(m)}{A}}{}^{ijpqsk}_{i_1j_1p_1q_1s_1k_1}=\frac{2m+1}{2h}\int_{-h}^{h}\theta A^i_{i_1}A^j_{j_1}\cdots A^k_{k_1}P_{m_1}\cdots P_{m_5}P_m dx_3.
\end{aligned} \tag{3.4}$$

By virtue of (2.2) we have

$$\frac{2m+1}{2h}\int_{-h}^{h}\sigma_{\vec{(l)}}\frac{d\vec{s}}{ds}p_m dx_3=\overset{(m)}{\sigma^{\alpha}}l_\alpha=\overset{(m)}{\sigma}_{(ll)}\vec{l}+\overset{(m)}{\sigma}_{(ls)}\vec{s}+\overset{(m)}{\sigma}_{(ln)}\vec{n} \tag{3.5}$$

Then

$$\overset{(m)}{\vec{U}}=\frac{2m+1}{2h}\int_{-h}^{h}\vec{U}P_m\left(\frac{x_3}{h}\right)dx_3=\overset{(m)}{U}_{(l)}\vec{l}+\overset{(m)}{U}_{(s)}\vec{s}+\overset{(m)}{U}_{(n)}\vec{n} \tag{3.6}$$

$$(m=0,1,\cdots)$$

The passage to the finite system can be realized by various methods one of which consists of considering of a finite series, i.e.,

$$\left(\sqrt{\frac{g}{a}}\vec{\sigma}^i, \vec{U}\sqrt{\frac{g}{a}}\vec{\Phi}\right) = \sum_{m=0}^{N} \Big(\overset{(m)}{\vec{\sigma}_j^i} \ \overset{(m)}{\vec{U}}, \ \overset{(m)}{\vec{\Phi}} \Big) P_m\Big(\frac{x_3}{h}\Big),$$

where N is a fixed nonnegative number. In other words, in the previous equations it is assumed that

$$\overset{(m)}{\vec{U}} = 0, \quad \overset{(m)}{\vec{\sigma^i}} = 0, \quad if \ \ m > N.$$

The approximation of such a type will be called an approximation of order N.

The second difficulty (no less important) consists in that integrals of the type (3.4) should be calculated explicitly.

By the F. Neumann and J. Adams formulas

$$\int_{-1}^{1} \frac{P_m(y)dy}{x-y} = 2Q_m(x), \quad |x| > 1, \ \ (Neumann)$$

and

$$P_m(x)P_s(x) = \sum_{r=0}^{min(m,s)} \alpha_{msr} P_{m+s-2r}(x), \ \ (Adams)$$

respectively, where

$$\alpha_{msr} = \frac{A_{m-r}A_rA_{s-r}}{A_{m+s-r}} \frac{2(m+s)-4r+1}{2(m+s)-2r+1}, \quad A_m = \frac{1\cdot 3\cdots(2m-1)}{m!},$$

and $Q_m(x)$ are the Legendre functions of the second kind, it is not to difficult obtain expression of these integrals explicitly

$$(I_1) \underset{(m_1)}{\overset{(m)}{A}}{}^{\alpha_1\alpha_2}_{\beta_1\beta_2} \frac{2m+1}{2h} \int_{-h}^{h} \sqrt{\frac{g}{a}} A^{\alpha_1}_{\beta_1} A^{\alpha_2}_{\beta_2} P_{m_1} P_m dx_3$$

$$= \begin{cases} \dfrac{2m+1}{2\sqrt{E}h}\left[B^{\alpha_1}_{\beta_1}(hy)B^{\alpha_2}_{\beta_2}(hy)\begin{pmatrix} P_{m_1}(y)Q_m(y), \ m_1 \le m \\ Q_{m_1}(y)P_m(y), \ m_1 gem \end{pmatrix}\right]_{y_1}^{y_2} \\ +K^{-1}L^{\alpha_1}_{\beta_1}L^{\alpha_2}_{\beta_2}\delta^m_{m_1}\begin{pmatrix} E \pm 0 \\ k \pm 0 \end{pmatrix} \\ a^{\alpha_1}_{\beta_1}a^{\alpha_2}_{\beta_2}\delta^m_{m_1}, \quad E = H^2 - K = 0, \end{cases} \tag{3.7}$$

where δ_s^m is the Kroneker symbol, E is Euler difference, $B_\beta^\alpha(x) = a_\beta^\alpha + xL_\beta^\alpha$, $L_\beta^\alpha = b_\beta^\alpha - 2Ha_\beta^\alpha$. The essence of the square brackets consists of the following

$$[f(y)]_{y_1}^{y_2} = f(y_2) - f(y_1), \quad y_{1,2} = [(H \mp \sqrt{k}h]^{-1}.$$

The explicit expression for the product $P_m(y)Q_s(y)$ has the form

$$P_m(y)Q_s(y) = \sum_{r=0}^{[\frac{m}{2}]} \sum_{p=0}^{\infty} \overset{m,s}{M}_{rp} y^{m-s-2(r+p)-1} \quad (m \leq s)$$

where

$$\overset{(m,s)}{M_{rp}} = 2^{s-m} \frac{(-1)^r}{r!} \frac{(2m-2r)!}{(m-r)!(m-2r)!} \frac{(s+p)!(s+2p!)}{p!(2s+2p+1)!}$$

Further,

$$(I_2) \underset{(m_1)}{\overset{(m)}{A}}{}_{\beta_1 3}^{\alpha_1 3} = \frac{2m+1}{2h} \int_{-h}^{h} \sqrt{\frac{g}{a}} A_{\beta_1}^{\alpha_1} P_{m_1} P_m dx_3$$

$$= a_{\beta_1}^{\alpha_1} \delta_{m_1}^m + h \left(\frac{m}{2m-1} \delta_{m_1}^{m-1} + \frac{m+1}{2m+3} \delta_{m_1}^{m+1} \right) L_{\beta_1}^{\alpha_1},$$

$$(\mathrm{I}_3) \underset{(m_1)}{\overset{(m)}{A}}{}_{33}^{33} = \frac{2m+1}{2h} \int_{-h}^{h} \sqrt{\frac{g}{a}} P_{m_1} P_m dx_3$$

$$= \delta_{m_1}^m - 2Hh \left(\frac{m}{2m+1} \delta_m^{m-1} + \frac{m+1}{2m+3} \delta_m^{m+1} \right)$$

$$+Kh^2 \left[\frac{m+1}{2m+3} \frac{m+2}{2m+5} \delta_{m_1}^{m+2} + \left(\frac{(m+1)^2}{2m+3} \frac{m^2}{2m-1} \right) \frac{\delta_{m_1}^m}{2m+1} + \frac{m}{2m-1} \frac{m-1}{2m-3} \delta_{m_1}^{m-2} \right]$$

Consider the integrals containing the product of three Legendre polynomials:

$$(\mathrm{II}_1) \underset{(m_1,m_2)}{\overset{(m)}{A}}{}_{\beta_1\beta_2\beta_3}^{\alpha_1\alpha_2\alpha_3} = \frac{2m+1}{2h} \int_{-h}^{h} \frac{B_{\beta_1}^{\alpha_1}(x_3) B_{\beta_2}^{\alpha_2}(x_3) B_{\beta_3}^{\alpha_3}(x_3)}{(1-2Hx_3+kx_3^2)^2}$$

$$\times P_{m_1}\left(\frac{x_3}{h}\right) P_{m_2}\left(\frac{x_3}{h}\right) P_m\left(\frac{x_3}{h}\right) dx_3.$$

Introduce the notation

$$B_{\beta_1}^{\alpha_1}(x_3) B_{\beta_2}^{\alpha_2}(x_3) B_{\beta_3}^{\alpha_3}(x_3) = \sum_{n=0}^{3} \overset{n}{\mathbb{C}}{}_{\beta_1\beta_2\beta_3}^{\alpha_1\alpha_2\alpha_3} x^n,$$

where

$$\overset{0}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = a^{\alpha_1}_{\beta_1}a^{\alpha_2}_{\beta_2}a^{\alpha_3}_{\beta_3}, \quad \overset{1}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = a^{\alpha_1}_{\beta_1}a^{\alpha_2}_{\beta_2}L^{\alpha_3}_{\beta_3} + a^{\alpha_1}_{\beta_1}L^{\alpha_2}_{\beta_2}a^{\alpha_3}_{\beta_3} + L^{\alpha_1}_{\beta_1}a^{\alpha_2}_{\beta_2}a^{\alpha_3}_{\beta_3},$$

$$\overset{2}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = a^{\alpha_1}_{\beta_1}L^{\alpha_2}_{\beta_2}L^{\alpha_3}_{\beta_3}, + L^{\alpha_1}_{\beta_1}a^{\alpha_2}_{\beta_2}L^{\alpha_3}_{\beta_3} + L^{\alpha_1}_{\beta_1}L^{\alpha_2}_{\beta_2}a^{\alpha_3}_{\beta_3}, \quad \overset{3}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = L^{\alpha_1}_{\beta_1}L^{\alpha_2}_{\beta_2}L^{\alpha_3}_{\beta_3},$$

$$(L^{\beta}_{\alpha} = b^{\alpha}_{\beta} - 2Ha^{\alpha}_{\beta})$$

Then we have

$$\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = \frac{2m+1}{k^2h^4}\sum_{r=0}^{min(m_1,m_2)} \alpha_{m_1m_2r}\sum_{n=0}^{3}\overset{n}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}\, h^n \frac{\partial^2}{\partial y_1 \partial y_2}$$

$$\times\left[\frac{y^n}{y_1-y_2}\begin{pmatrix} P_s(y)Q_m(y), & s\le m \\ Q_s(y)P_m(y), & s\ge m\end{pmatrix}\right]_{y_1}^{y_2}$$

where

$$y_{1,2} = [(H\mp\sqrt{E})h]^{-1} \Rightarrow \frac{1}{y_1-y_2} = \frac{kh}{2\sqrt{E}}, \quad (s = m_1+m_2-2r),$$

Further

$$(\mathrm{II}_2)\ \underset{(m_1m_2)}{\overset{(m)}{A}}{}^{\alpha_1\alpha_2 3}_{\beta_1\beta_2 3} = \frac{2m+1}{3h}\int_{-h}^{h}\frac{B^{\beta_1}_{\alpha_1}(x_3)B^{\beta_2}_{\alpha_2}(x_3)}{1-2Hx_3+kx_3^2}P_{m_1}P_{m_2}P_m x_3 = \sum_{r=0}^{min(m_1,m_2)}\alpha_{m_1m_2r}$$

$$\times\left\{\frac{L^{\alpha_1}_{\beta_1}L^{\alpha_2}_{\beta_2}}{K}\delta^m_s + \frac{2m+1}{2\sqrt{E}h}\left[B^{\alpha_1}_{\beta_1}(hy)B^{\alpha_2}_{\beta_2}h(y)\begin{pmatrix} P_s(y)Q_m(y), & s\le m \\ Q_s(y)P_m(y), & s\ge m^{y_2}_{y_1}\end{pmatrix}\right]_{y_1}^{y_2}\right\},$$

$$(\mathrm{II}_3)\ \underset{(m_1m_2)}{\overset{(m)}{A}}{}^{\alpha_1 33}_{\beta_1 33} = \frac{2m+1}{2h}\int_{-h}^{h}B^{\beta_1}_{\alpha_1}(x_3)P_{m_1}P_{m_2}P_m dx_3$$

$$= \sum_{r=0}^{min(m_1,m_2)}\alpha_{m_1m_2r}\left[a^{\alpha_1}_{\beta_1}\delta^m_s + h\left(\frac{m+1}{2m+3}\delta^{m+1}_s + \frac{m}{2m-1}\delta^{m-1}_s\right)L^{\alpha_1}_{\beta_1}\right]$$

$$(s = m_1+m_2-2r)$$

$$(\mathrm{II}_4)\ \underset{(m_1m_2)}{\overset{(m)}{A}}{}^{333}_{333} = \frac{2m+1}{2h}\int_{-h}^{h}(1-2Hx_3+kx_3^2)P_{m_1}p_{m_2}P_m dx_3$$

$$= \sum_{r=0}^{min(m_1,m_2)} \alpha_{m_1 m_2 r} \left\{ \delta_s^m - 2Hh \left(\frac{m+1}{2m+3} \delta_s^{m+1} + \frac{m}{2m-1} \delta_s^{m-1} \right) + Kh^2 \right.$$
$$\times \left[\frac{m+1}{2m+3} \frac{m+3}{2m+5} \delta_s^{m+1} + \left(\frac{(m+1)^2}{2m+3} + \frac{m^2}{2m-1} \right) \frac{\delta_s^m}{2m+1} \right.$$
$$\left.\left. + \frac{m}{2m-1} \frac{m-1}{2m+3} \delta_s^{m-2} \right] \right\}$$
$$(s = m_1 + m_2 - 2r)$$

The integrals

$$\text{(III)} \quad \underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{\alpha_1 \cdots \alpha_4}_{\beta_1 \cdots \beta_4} = \frac{2m+1}{2h} \int_{-h}^{h} \frac{B_{\beta_1}^{\alpha_1}(x_3) \cdots B_{\beta_4}^{\alpha_4}(x_3)}{(1 - 2Hx_3 + kx_3^2)^3} P_{m_1} P_{m_2} P_{m_3} P_m dx_3,$$

$$\text{(IV)} \quad \underset{(m_1,\cdots,m_4)}{\overset{(m)}{A}}{}^{\alpha_1 \cdots \alpha_5}_{\beta_1 \cdots \beta_5} = \frac{2m+1}{2h} \int_{-h}^{h} \frac{B_{\beta_1}^{\alpha_1}(x_3) \cdots B_{\beta_5}^{\alpha_5}(x_3)}{(1 - 2Hx_3 + kx_3^2)^4} P_{m_1} \cdots P_{m_4} P_m dx_3,$$

$$\text{(V)} \quad \underset{(m_1 \cdots m_5)}{\overset{(m)}{A}}{}^{\alpha_1 \cdots \alpha_6}_{\beta_1 \cdots \beta_6} = \frac{2m+1}{2h} \int_{-h}^{h} \frac{B_{\beta_1}^{\alpha_1}(x_3) \cdots B_{\beta_6}^{\alpha_6}(x_3)}{(1 - 2Hx_3 + kx_3^2)^5} P_{m_1} \cdots P_{m_5} P_m dx_3.$$

are calculated similarly.

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