NORMED MOMENTS METHOD FOR NON-SHALLOW SHELLS

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(Received: 10.05.2015; accepted: 29.06.2015)

Abstract

In this paper we consider non-shallow shells. By means of I. Vekua's method of normed moments we get the approximate expression of the stress tensor which is compatible with boundary data on face surfaces.

Key words and phrases: Non-shallow shells, the method of normed moments.
AMS subject classification: 74K25.

1 Equations of Equilibrium of an Elastic Medium

I.Vekua has obtained the equations of shallow shells [1], [2]. It means that the interior geometry of the shell does not vary in thickness. This method for non-shallow shells in case of geometrical and physical non-linear theory was generalized by T.Meunargia [3].

The vector form of the equilibrium equation of the elastic shells have the following form

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\,\vec{\sigma}}{\partial x^i}+\vec{\Phi}=0,\quad (i=1,2,3),\tag{1}$$

where x^i $(i=1,2,3)$ are curvilinear coordinates, $g=\det(g_{ik})$ is the discriminant of the metric tensor of the space, $\vec{\Phi}$ is the volume force, $\vec{\sigma}$ are contravariant stress vectors.

Hooke's generalized law for isotropic and homogeneous shells has the form

$$\vec{\sigma}=C^{ij}\partial_j\vec{U},\tag{2}$$

$$C^{ij}=\lambda\Big(\vec{R}^i\otimes\vec{R}^j\Big)+\mu\Big(\vec{R}^j\otimes\vec{R}^i\Big)+\mu\Big(\vec{R}^i\cdot\vec{R}^j\Big)\mathrm{E}$$

where $\vec{U}$ is the displacement vector, $\vec{R}^i$ and $\vec{R}_i$ are covariant and contravariant base vectors of the space, λ and μ are Lamé's constants

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)},$$

where E and σ are Young's modulus and Poisson's ratio, respectively, E is the operator of identical transformation.

Let Ω denote a shell and the domain of the space occupied by this shell. Inside the shell, we consider a smooth surface S with respect to which the shell Ω lies symmetrically. The surface S is called the midsurface of the shell Ω. To construct the theory of shells, we use the more convenient coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\vec{R}$ of any point M of the domain Ω may be expressed by the formula

$$\vec{R}(x^1, x^2, x^3) = \vec{r}(x^1, x^2) + x^3\vec{n}(x^1, x^2),$$

where $\vec{r}$ and $\vec{n}$ are the radius-vector and the unit vector of the normal of the surface S $(x^3 = 0)$, respectively, (x^1, x^2) are the Gaussian parameters of the midsurfaces [1].

The covariant and contravariant basis vectors $\vec{R}_i$ and $\vec{R}^i$ of the surface $\hat{S}$ $(x_3 = const)$, and the corresponding basis vectors and $\vec{r}_i$ of $\vec{r}^i$ the midsurface are connected by the following relations

$$\vec{R}_i = A_{ij}\vec{r}^j = A_{i\cdot}^{\cdot j}\vec{r}_j, \;\; \vec{R}^i = A^{ij}\vec{r}_j = A_{\cdot j}^{i\cdot}\vec{r}^j,$$

where

$$A_{i\cdot}^{\cdot j} = \begin{cases} a_\alpha^\beta - x_3 b_\alpha^\beta, & (\alpha, \beta = 1, 2) \\ \delta_i^3 = \delta_3^i, \end{cases}$$

$$A_{\cdot j}^{i\cdot} = \begin{cases} \dfrac{(1-2Hx_3)a_\beta^\alpha + x_3 b_\beta^\alpha}{1 - 2Hx_3 + Kx_3^2}, & (\alpha, \beta = 1, 2) \\ \delta_i^3 = \delta_3^i, \end{cases} \tag{3}$$

Here a_β^α and b_β^α are the mixed components of the metric and curvature tensors of the midsurface $S(x_3 = 0)$, H and K are the middle and Gaussian curvature of the surface S and h is a semithickness of the shell.

Using formulae (2) and (3) we get

$$\vec{\sigma} = A_{\cdot p}^{m\cdot} A_{\cdot q}^{i\cdot} c^{pq} \partial_m \vec{U}, \tag{4}$$

$$c^{pq} = \lambda\Big(\vec{r}^p \otimes \vec{r}^q\Big) + \mu\Big(\vec{r}^q \otimes \vec{r}^p\Big) + \mu\Big(\vec{r}^p \cdot \vec{r}^q\Big)\mathrm{E}.$$

2 I. Vekua's reduction method

Multiplying both sides of equation (1) by factors $\dfrac{2m+1}{2h}P_m\left(\dfrac{x_3}{h}\right)$ $(m = 0, 1, ...)$ and then integrating with respect to x_3 from $-h$ to h, we have

$$\begin{aligned}&\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\overset{(m)}{\vec{\sigma}}{}^{\alpha}}{\partial x^{\alpha}} - \frac{2m+1}{h}\left(\overset{(m-1)}{\vec{\sigma}}{}^{3} + \overset{(m-3)}{\vec{\sigma}}{}^{3} + \cdots\right)\\ &+\frac{2m+1}{2h}\left[\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{\sigma}}{}^{3} - (-1)^m\sqrt{\frac{g_-}{a}}\overset{(-)}{\vec{\sigma}}{}^{3}\right] + \overset{(m)}{\vec{\Phi}},\end{aligned} \tag{5}$$

where

$$\left(\overset{(m)}{\vec{\sigma}}{}^{i}, \overset{(m)}{\vec{\Phi}}\right) = \frac{2m+1}{2h}\int_{-h}^{h}\left(\sqrt{\frac{g}{a}}\vec{\sigma}^{i}, \sqrt{\frac{g}{a}}\vec{\Phi}\right)P_m\left(\tfrac{x_3}{h}\right)dx_3.$$

$\overset{(+)}{\vec{\sigma}}{}^{3} = \vec{\sigma}^3(x^1, x^2, h)$ and $\overset{(-)}{\vec{\sigma}}{}^{3} = \vec{\sigma}^3(x^1, x^2, -h)$ denote $\vec{\sigma}^3$ on face surfaces S^+ and S^-, respectively.

For moments $\overset{(m)}{\vec{\sigma}}{}^{i}$ we get the expressions

$$\overset{(m)}{\vec{\sigma}}{}^{i} = \sum_{m_1=0}^{\infty}\overset{(m,m_1)}{A}{}^{ij}_{i_1j_1}c^{i_1j_1}D_j\overset{(m_1)}{\vec{u}}, \tag{6}$$

where

$$\overset{(m_1)}{\vec{u}} = \frac{2m_1+1}{2h}\int_{-h}^{h}\vec{U}P_{m_1}\left(\frac{x_3}{h}\right)dx_3,$$

$$\overset{(m,m_1)}{A}{}^{ij}_{i_1j_1} = \frac{2m_1+1}{2h}\int_{-h}^{h}\sqrt{\frac{g}{a}}A^{i\cdot}_{\cdot i_1}A^{j\cdot}_{\cdot j_1}P_m\left(\frac{x_3}{h}\right)P_{m_1}\left(\frac{x_3}{h}\right)dx_3, \tag{7}$$

$$D_j\overset{(m_1)}{\vec{u}} = \begin{cases} D_\alpha\overset{(m_1)}{\vec{u}}, & (j = \alpha),\\ \dfrac{2m_1+1}{2h}\left(\overset{(m_1+1)}{\vec{u}} + \overset{(m_1+3)}{\vec{u}} + \cdots\right), & j = 3.\end{cases}$$

If we use expressions (6) in to equations (5) we obtain an infinite system of second-order partial differential equations with respect to the moments $\overset{(0)}{\vec{u}}, \overset{(0)}{\vec{u}}, ...$ of the unknown displacement vector $\vec{U}$. Each equation of this system contains an infinite number of desired moments $\overset{(0)}{\vec{u}}, \overset{(0)}{\vec{u}}, ...$

An infinite system of equations (5) has the advantage that it contains two independent variables Gaussian coordinates x^1, x^2 of the surface S. But the decrease in the number of independent variables by one is achieved by increasing the number of equations to infinity, which, naturally, has an obvious practical inconvenience. Therefore it is necessary to make the next step a further simplification of the problem. Our aim is to reduce the problem to a finite system of equations in two independent variables.

3 I. Vekua's method of normed moments

Let us fix a non-negative integer N and out of an infinite system (8) consider only the first equations. Then we obtain the system of equations of finite order

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\overset{(m)}{\vec{\sigma}}{}^{\alpha}}{\partial x^{\alpha}}-\frac{2m+1}{h}\left(\overset{(m-1)}{\vec{\sigma}}{}^{\alpha}+\overset{(m-3)}{\vec{\sigma}}{}^{\alpha}+\cdots\right)+\overset{(m)}{\vec{F}}=0, \qquad (8)$$

$$\overset{(m)}{\vec{F}}=\overset{(m)}{\vec{\Phi}}+\frac{2m+1}{2h}\left[\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{\sigma}}{}^{3}-(-1)^m\sqrt{\frac{g_+}{a}}\overset{(-)}{\vec{\sigma}}{}^{3}\right],$$

$$(m=0,1,...N)$$

where according to (6)

$$\overset{(m)}{\vec{\sigma}}{}^{i}=\sum_{m_1=0}^{N}\left[\overset{(m,m_1)}{A}{}^{ij}_{i_1j_1}c^{i_1j_1}D^N_i\overset{(m_1)}{\vec{u}}+\overset{(m,m_1)}{A}{}^{i3}_{i_13}c^{i_13}\frac{2m+1}{2h}(\overset{(+)}{\vec{V}}-(-)^m_1\overset{(-)}{\vec{V}})\right], \qquad (9)$$

here $\overset{(+)}{\vec{V}}$ and $\overset{(-)}{\vec{V}}$ denote the vectors

$$\overset{(+)}{\vec{V}}=\sum_{p=N+1}^{\infty}\overset{(p)}{\vec{u}},\quad \overset{(-)}{\vec{V}}=\sum_{p=N+1}^{\infty}(-1)^p\overset{(p)}{\vec{u}},$$

$$D^N_j\overset{(m_1)}{\vec{u}}=\begin{cases}D_\alpha\overset{(m_1)}{\vec{u}}, & j=\alpha,\\ \dfrac{2m_1+1}{2h}\displaystyle\sum_{p=m_1}^{N}(1-(-1)^{p+m_1})\overset{(p)}{\vec{u}}, & j=3,\end{cases}$$

which obviously depend on N.

For (9) we used the following equation

$$\frac{2m_1+1}{h}\left(\overset{(m_1+1)}{\vec{u}}+\overset{(m_1+3)}{\vec{u}}+\cdots\right)=$$

$$= \frac{2m_1+1}{2h} \sum_{p=m_1}^{N} \left(1-(-1)^{p+m_1}\right) \overset{(p)}{\vec{u}} + \frac{2m+1}{2h} \left(\overset{(+)}{\vec{V}} - (-)_1^m \overset{(-)}{\vec{V}} \right).$$

Expressions (9) contain $N+3$ vectors $\overset{(0)}{\vec{u}}, \overset{(1)}{\vec{u}}, \overset{(N)}{\vec{u}}, \overset{(+)}{\vec{V}}, \overset{(-)}{\vec{V}}$. Our purpose is to express the vectors $\overset{(+)}{\vec{V}}$ and $\overset{(-)}{\vec{V}}$ by means of vectors $\overset{(0)}{\vec{u}}, \overset{(1)}{\vec{u}}, ..., \overset{(N)}{\vec{u}}$, ensuring the satisfaction of boundary conditions on face surfaces. If we represent contravariant vectors of the stress forces by the approximate formula

$$\vec{\sigma}^i(x_1, x^2, x^3) = \sum_{m=0}^{N} \overset{(m)}{\vec{\sigma}}{}^i (x^1, x^2) P_m \left(\frac{x_3}{h}\right),$$

we can write boundary conditions on S^+ and S^- in the form

$$\sum_{m=0}^{N} \overset{(m)}{\vec{\sigma}}{}^3 = \overset{(+)}{\vec{P}}, \qquad \sum_{m=0}^{N} (-1)^m \overset{(m)}{\vec{\sigma}}{}^3 = \overset{(-)}{\vec{P}}. \tag{10}$$

Substituting expressions (9) into these equalities we get

$$\sum_{m=0}^{N} \sum_{m_1=0}^{N} \left[\overset{(m,m_1)}{A}{}^{3j}_{3j_1} c^{3j_1} D^N_{j_1} \overset{(m_1)}{\vec{u}} + \overset{(m,m_1)}{A}{}^{33}_{33} c^{33} \frac{2m+1}{2h} \right.$$
$$\left. \times \left(\overset{(+)}{\vec{V}} - (-1)^{m_1} \overset{(-)}{\vec{V}} \right) \right] = \overset{(+)}{\vec{P}},$$
$$\sum_{m=0}^{N} \sum_{m_1=0}^{N} (-1)^m \left[\overset{(m,m_1)}{A}{}^{3j}_{3j_1} c^{3j_1} D^N_{j_1} \overset{(m_1)}{\vec{u}} + \overset{(m,m_1)}{A}{}^{33}_{33} c^{33} \frac{2m+1}{2h} \right.$$
$$\left. \times \left(\overset{(+)}{\vec{V}} - (-1)^{m_1} \overset{(-)}{\vec{V}} \right) \right] = \overset{(-)}{\vec{P}},$$

Assuming that

$$\overset{(m,m_1)}{A}{}^{33}_{33} = \frac{2m+1}{2h} \int_{-h}^{h} (1-k_1x_3)(1-k_2x_3) P_m\left(\frac{x_3}{h}\right) P_{m_1}\left(\frac{x_3}{h}\right) dx_3$$
$$= \delta^m_{m_1} - 2hH \left(\frac{m+1}{2m+3} \delta^{m+1}_{m_1} + \frac{m}{2m-1} \delta^{m-1}_{m_1} \right)$$
$$+ h^2 K \left(\frac{(m+1)(m+2)}{(2m+3)(2m+5)} \delta^{m+2}_{m_1} + \frac{2m^2+2m-1}{(2m+3)(2m-1)} \delta^m_{m_1} \right.$$
$$\left. + \frac{m(m-1)}{(2m-3)(2m-1)} \delta^{m-2}_{m_1} \right),$$

$$\sum_{m=0}^{N}(2m+1)=(N+1)^2,$$

$$\sum_{m=0}^{N}(-1)^m(2m+1)=(-1)^N(N+1),$$

we get

$$\begin{aligned}\overset{(+)}{c}\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{V}}-\overset{(-)}{c}\sqrt{\frac{g_+}{a}}\overset{(-)}{\vec{V}}&=\overset{(+)}{A}_N\\ \overset{(-)}{c}\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{V}}-\overset{(+)}{c}\sqrt{\frac{g_+}{a}}\overset{(-)}{\vec{V}}&=\overset{(-)}{A}_N\end{aligned}\tag{11}$$

where

$$\overset{(+)}{c}=\frac{(N+1)^2}{2h}c^{33},\quad \overset{(-)}{c}=\frac{(-1)^N(N+1)^2}{2h}c^{33},$$

$$\overset{(+)}{A}_N=\overset{(+)}{\vec{P}}-\sum_{m=0}^{N}\sum_{m_1=0}^{N}\overset{(m,m_1)}{A}{}^{3j}_{3j_1}c^{3j_1}D^N_{j_1}\overset{(m_1)}{\vec{u}},$$

$$\bar{A}_N=\overset{(-)}{\vec{P}}-\sum_{m=0}^{N}\sum_{m_1=0}^{N}(-1)^m\overset{(m,m_1)}{A}{}^{3j}_{3j_1}c^{3j_1}D^N_{j_1}\overset{(m_1)}{\vec{u}}.$$

Then we can write the system of equations (11) as

$$\begin{aligned}c^{33}\left[(N+1)^2\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{V}}-(-1)^N(N+1)\sqrt{\frac{g_+}{a}}\overset{(-)}{\vec{V}}\right]&=2h\overset{(+)}{A}_N,\\ c^{33}\left[(-1)^N(N+1)\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{V}}-(N+1)^2\sqrt{\frac{g_+}{a}}\overset{(-)}{\vec{V}}\right]&=2h\overset{(-)}{A}_N,\end{aligned}\tag{12}$$

where

$$c^{33}=(\lambda+\mu)\vec{n}\otimes\vec{n}+\mu\mathrm{E},$$

Since the solution of the equation

$$c^{33}\vec{U}=[(\lambda+\mu)\vec{n}\otimes\vec{n}+\mu\mathrm{E}]\vec{U}=\vec{V},$$

where $\vec{U}$ is the unknown vector and $\vec{V}$ is the given vector field, is expressed by

$$\vec{U}=\frac{1}{\mu}\left(\vec{V}-\frac{\lambda+\mu}{\lambda+2\mu}V_3\vec{n}\right)=c^{-1}\vec{V},$$

the equality (12) may be rewritten as

$$(N+1)^2\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{V}}-(-1)^N(N+1)\sqrt{\frac{g_+}{a}}\overset{(-)}{\vec{V}}=2hc^{-1}\overset{(+)}{A}_N,$$

$$(-1)^N(N+1)\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{V}}-(N+1)^2\sqrt{\frac{g_+}{a}}\overset{(-)}{\vec{V}}=2hc^{-1}\overset{(-)}{A}_N.$$

Thus we have

$$\begin{aligned}\sqrt{\frac{g_+}{a}}\overset{(+)}{\vec{V}}&=\frac{2h}{N(N+2)}c^{-1}\left[\overset{(+)}{A}_N-\frac{(-1)^N}{N+1}\overset{(-)}{A}_N\right],\\ \sqrt{\frac{g_-}{a}}\overset{(+)}{\vec{V}}&=\frac{2h}{N(N+2)}c^{-1}\left[\frac{(-1)^N}{N+1}\overset{(+)}{A}_N-\overset{(-)}{A}_N\right].\end{aligned}\tag{13}$$

If we substitute these expressions into equalities (9) we obtain relations which connect the vector-functions $\overset{(m)}{\vec{\sigma}}{}^i$ and $\overset{(0)}{\vec{u}},\overset{(1)}{\vec{u}},...,\overset{(N)}{\vec{u}}$. If we substitute the expression $\overset{(m)}{\vec{\sigma}}{}^i$ into (10) we get the approximate expression of the stress tensor which is compatible with boundary data on face surfaces for any vector fields $\overset{(0)}{\vec{u}},\overset{(1)}{\vec{u}},...,\overset{(N)}{\vec{u}}$, representing moments of the unknown vector field $\vec{U}$.

The expressions (9), which are compatible with boundary conditions on face surfaces, will be called normed moments of order m of contravariant vectors of the stress forces, or more briefly, normed moments of the stress field.

If we substitute the obtained expressions for normed moments $\overset{(m)}{\vec{\sigma}}{}^i$ into equations (8) we get the system consisting $N+1$ vector equations. Each of these equations contains $N+1$ vectors $\overset{(0)}{\vec{u}},\overset{(1)}{\vec{u}},...,\overset{(N)}{\vec{u}}$ and their second-order partial derivatives with respect to Gaussian parameters of the $6N+6$th order for defining $3N+3$ functions $\overset{(0)}{u}{}^1,\overset{(0)}{u}{}^2,\overset{(0)}{u}{}^3,...,\overset{(N)}{u}{}^1,\overset{(N)}{u}{}^2,\overset{(N)}{u}{}^3$ which represent contravariant components of the unknown moments $\overset{(0)}{\vec{u}},\overset{(1)}{\vec{u}},...,\overset{(N)}{\vec{u}}$.

Acknowledgment. The designated project has been fulfilled by a financial support of Shota Rustaveli National Science Foundation (Grant SRNSF/FR /358/5-109/14). Any idea in this publication is possessed by the author and may not represent the opinion of Shota Rustaveli National Science Foundation itself.

References

1. Vekua I.N. *Shell Theory: General Methods of construction*. Pitman Advanced Publishing Program, Boston-London-Melbourne (1985).

2. Vekua I.N. *Theory on Thin and Shallow Shells with Variable Thickness.* (Russian) Tbilisi, Metsniereba, 1965.

3. Meunargia T.V. On one method of construction of geometrically and physically nonlinear theory of non-shallow shells. *Proc. A. Razmadze Math. Inst.*, **119** (1999), 133-154.

4. Meunargia T.V. Ensuring the boundary condition of face surfaces for non-shallow shells. *Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics,* **26** (2012), 42-45.