SOLUTION OF BOUNDARY VALUE CONTACT PROBLEMS OF THERMOELASTIC EQUILIBRIUM OF A MULTILAYER PIECCEWISE NON-HOMOGENEOUS ISOTROPIC RECTANGULAR PARALLELEPIPED

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Abstract

An analytical solution for some boundary value problems of the thermoelastic equilibrium of an isotropic rectangular parallelepiped with non-homogeneity of the type $\mu = const, \;\; \lambda = \lambda(x, y, z)$ (μ is a shift modulus; λ is material amenability, x, y, z are Cartesian coordinates) and an analytical solution of boundary value contact problems on the thermoelastic equilibrium of a multilayer piecewise non-homogeneous rectangular parallelepiped each layer of which has its own non-homogeneity of the form $\mu = const, \;\; \lambda = \lambda(z)$ .are constructed. At the end of the paper examples of problem solution are given.

Key words and phrases: Thermoelasticity, Non-homogeneous body, Boundary value problem, Boundary value contact problem.

AMS subject classification: 74B05, 74F05.

1 Introduction

There are quite a number of scientific papers dealing with elastic equilibrium of non-homogeneous bodies. The number of scientific publications on thermoelastic equilibrium of non-homogeneous bodies is significantly smaller, while those that study thermoelastic equilibrium of finite non-homogeneous bodies when non-homogeneity is a function of not just a single coordinate, but of two or three of them are quite few.

In the present paper we consider some boundary value problems of thermoelastic equilibrium of an isotropic rectangular parallelepiped with non-homogeneity of the type $\mu = const, \;\; \lambda = \lambda(x, y, z)$ (μ, λ are elastic characteristics of the body, μ is called shift modulus and λ is medium amenability) and boundary value contact problems for a multilayer piecewise non-homogeneous rectangular parallelepiped. Each layer of the multilayer parallelepiped has its own non-homogeneity of the type $\mu = const, \;\; \lambda =$

$\lambda(z)$ (where z is an applicata of the Cartesian system of coordinates x, y, z). Layers with different elastic characteristics are located along the coordinate z with contact planes perpendicular to this axis.

Among scientific publications devoted to the elastic and thermoelastic equilibrium of bodies we should mention monographs [1]-[5] and papers [7]-[11]. Although all the above-mentioned publications deal with problems similar to ours/statement and discussions of the problems considered in our paper can be found only in monographs [1] and [2]. To be more exact, these monographs state and analyze problems of elasticity, but not thermoelasticity.

A section of work [1] deals with elastic equilibrium of an isotropic medium with a constant shift modulus μ and variability amenability λ. The authors after some transformations of the Lamé equations express components of the stress tensor through three harmonic and one biharmonic functions. Unfortunately, it is not clear which of the indicated functions are defined in a general form and which of them have a partial form. This question arises due to the fact that three arbitrary harmonic functions (or one biharmonic and one harmonic function) are sufficient for the construction of the so called general solution. It is not clear either and not defined for what kind of bodies and what kind of boundary conditions the proposed general condition is convenient. In other scientific papers dealing with problems studied in the given paper we could not find an algorithm leading to the solution of particular problems with particular boundary conditions (especially as far as analytical solutions are concerned) either.

In the given work an analytical solution is constructed for some boundary value problems of thermoelastic equilibrium of an isotropic rectangular parallelepiped with non-homogeneity of the type $\mu = const$, $\lambda = \lambda(x, y, z)$ and an analytical solution of boundary value contact problems of thermoelastic equilibrium of a multilayer piecewise non-homogeneous rectangular parallelepiped each layer of which has its own non-homogeneity of the type $\mu = const$, $\lambda = \lambda(z)$. The solutions are constructed using the method of separation of variables the displacements and components of the stress tensor being represented as absolutely and uniformly converging double series.

At the end of the paper examples of problem solution are given.

2 Equilibrium Equations, Physical Equations and Some Properties of an Isotropic Medium with a Constant Shift Modulus and Variable Amenability

In the rectangular Cartesian system of coordinates x, y, z, equilibrium equations and expressions for the components of stress tensor through displacements in the case of a non-homogeneous elastic isotropic body will have the following form [12]

$$a)\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=0,$$

$$b)\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}=0, \tag{1}$$

$$c)\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}=0;$$

$$a)\ X_x=\lambda\theta+2\mu\frac{\partial u}{\partial x}-(3\lambda+2\mu)\,kT, \qquad d)\ Y_z=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),$$

$$b)\ Y_y=\lambda\theta+2\mu\frac{\partial v}{\partial y}-(3\lambda+2\mu)\,kT, \qquad e)\ Z_x=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \tag{2}$$

$$c)\ Z_z=\lambda\theta+2\mu\frac{\partial w}{\partial z}-(3\lambda+2\mu)\,kT, \qquad f)\ X_y=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

In Equalities (1) and (2) $\overrightarrow{U}(u,v,w)$ is a displacement vector with components u,v,w, respectively, along the coordinate lines x,y,z; $\theta=div\overrightarrow{U}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$; $\lambda=\frac{\nu E}{(1-2\nu)(1+\nu)}$, $\mu=\frac{E}{2(1+\nu)}$, where E is elasticity modulus and ν is Poisson's ratio; k is the coefficient of linear thermal expansion, $T(x,y,z)$ is the change in the temperature in a point of the body [4],[6], and

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\Delta T=0. \tag{3}$$

We will also write the following formulas here:

$$\overrightarrow{K}(K_x,K_y,K_z)=rot\overrightarrow{U}, \text{ and } K_x=\mu\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial x}\right),$$

$$K_y=\mu\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\ K_z=\mu\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right).$$

The given paper studies a multilayer rectangular parallelepiped each layer of which has its own non-homogeneity of the following form:

$$\mu_l=const, \qquad \lambda_l=\lambda_l(z) \tag{4}$$

In the case when the isotropic parallelepiped is characterized by the following form of non-homogeneity

$$\mu=const, \qquad \lambda=\lambda(x,y,z) \tag{5}$$

we will consider thermoelastic equilibrium only of a one-layer parallelepiped, i.e. we will consider boundary value problems, not boundary value contact problems of thermoelasticity.

In Formulas (4) the index l shows the number of the layer and implies that the l-th layer has a constant shift modulus μ_l and variable amenability $\lambda_l = \lambda_l\,(z)$.

Now we will write down one more form of the equilibrium equation different of Equations (1). To do this we will use the following formulas:

$$\begin{aligned} &X_x = (\lambda + 2\mu)\,(\theta - 3kT) - 2\mu \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - 2kT \right), \\ &X_y = -K_z + 2\mu \frac{\partial v}{\partial x}, \;\; X_z = K_y + 2\mu \frac{\partial w}{\partial x}; \end{aligned} \tag{6}$$

$$\begin{aligned} &Y_y = (\lambda + 2\mu)\,(\theta - 3kT) - 2\mu \left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} - 2kT \right), \\ &Y_x = K_z + 2\mu \frac{\partial u}{\partial y}, \;\; Y_z = -K_x + 2\mu \frac{\partial w}{\partial y}. \end{aligned} \tag{7}$$

$$\begin{aligned} &Z_z = (\lambda + 2\mu)\,(\theta - 3kT) - 2\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - 2kT \right), \\ &Z_x = -K_y + 2\mu \frac{\partial u}{\partial z}, \;\; Z_y = K_x + 2\mu \frac{\partial v}{\partial z}. \end{aligned} \tag{8}$$

If Formulas (6) are substituted in Equation ($1a$), Formulas (7) - in Equation ($1b$) and Formulas (8) - in Equation ($1c$) and add identity div rot $\overrightarrow{U} = 0$ to the obtained equations (in our case this identity will take the form $\frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z} = 0$), we will have

$$\begin{aligned} &a) \quad \frac{\partial D}{\partial x} - \frac{\partial K_z}{\partial y} + \frac{\partial K_y}{\partial z} + 4\mu k \frac{\partial T}{\partial x} = 0, \\ &b) \quad \frac{\partial D}{\partial y} - \frac{\partial K_x}{\partial z} + \frac{\partial K_z}{\partial x} + 4\mu k \frac{\partial T}{\partial y} = 0, \\ &c) \quad \frac{\partial D}{\partial z} - \frac{\partial K_y}{\partial x} + \frac{\partial K_x}{\partial y} + 4\mu ;k \frac{\partial T}{\partial z} = 0, \\ &d) \quad \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z} = 0, \\ &e) \quad \Delta T = 0; \end{aligned} \tag{9}$$

$$\begin{aligned}
&a) \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=3kT+g\,(x,y,z)\,D,\\
&b) \quad \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=\frac{1}{\mu}K_x,\\
&c) \quad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=\frac{1}{\mu}K_y,\\
&d) \quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{1}{\mu}K_z.
\end{aligned} \tag{10}$$

In Equality $(10a)$ $g=\frac{1}{\lambda+2\mu}$.

Remark 1. Equations $(9a)$, $(9b)$, $(9c)$ and Identity $(9d)$ imply

$$\operatorname{grad}\left[(\lambda+2\mu)\left(\operatorname{div}\overrightarrow{U}-3kT\right)\right]-\operatorname{rot}\left(\mu\operatorname{rot}\overrightarrow{U}\right)+4\mu k\operatorname{grad}T=0, \tag{11}$$

$$\operatorname{div}\left(\mu\operatorname{rot}\overrightarrow{U}\right)=0 \tag{12}$$

or

$$\operatorname{grad}D-\operatorname{rot}\overrightarrow{K}+4\mu k\operatorname{grad}T=0, \tag{13}$$

$$\operatorname{div}\overrightarrow{K}=0. \tag{14}$$

Hence, in any curvilinear orthogonal system of coordinates for an isotropic specially non-homogeneous elastic body of the form

$$\mu=const, \qquad \lambda=\lambda\,(x,y,z)$$

Equations (11) and, naturally, Identity (12) are true.

3 Boundary Conditions for the Parallelepiped under Consideration

Now we give boundary conditions [13] for the parallelepiped $\Omega=\{0<x<x_1,\ 0<y<y_1,\ 0<z<z_1\}$ under consideration (Fig. 1):

For $x = x_j \quad (j = 0,\, 1; \;\; x_0 = 0)$:

$$a)\; u = 0,\;\; X_y = 0, rm\;\; X_z = 0,\;\; \frac{\partial T}{\partial x} = 0 \;\; \Rightarrow$$

$$\Rightarrow \frac{\partial D}{\partial x} = 0,\;\; \frac{\partial K_x}{\partial x} = 0,\;\; \frac{\partial v}{\partial x} = 0,\;\; \frac{\partial w}{\partial x} = 0,\;\; K_y = 0,\;\; K_z = 0$$

or (15)

$$b)\; X_x = 0, \quad v = 0, \quad w = 0, \quad \mathrm{T} = 0 \;\; \Rightarrow$$

$$\Rightarrow \;\; D = 0, \quad K_{\mathrm{x}} = 0, \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial K_y}{\partial x} = 0, \quad \frac{\partial K_z}{\partial x} = 0$$

For $y = y_j \quad (j = 0, 1; \;\; y_0 = 0)$:

$$a)\; v = 0,\;\; Y_z = 0,\;\; Y_x = 0,\;\; \frac{\partial T}{\partial y} = 0 \;\; \Rightarrow$$

$$\Rightarrow \;\; \frac{\partial D}{\partial y} = 0,\;\; \frac{\partial K_y}{\partial y} = 0,\;\; \frac{\partial w}{\partial y} = 0,\;\; \frac{\partial u}{\partial y} = 0,\;\; K_z = 0,\;\; K_x = 0$$

or (16)

$$b)\; Y_y = 0, \quad w = 0, \quad u = 0, \quad \mathrm{T} = 0 \;\; \Rightarrow$$

$$\Rightarrow \;\; D = 0, \quad K_{\mathrm{y}} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial K_z}{\partial y} = 0, \quad \frac{\partial K_x}{\partial y} = 0$$

For $z = z_j \quad (j = 0,\, 1; \;\; z_0 = 0)$:

$$a) \quad w = f_{j1},\;\; Z_x = f_{j2},\;\; Z_y = f_{j3},\;\; \tau_0 T + \tau_1 \frac{\partial T}{\partial z} = f_{j4}\;;$$

from these three first conditions on the face $z = z_j$
are defined functions

$$\frac{\partial}{\partial x} D, \quad \frac{\partial}{\partial z} K_z, \quad \frac{\partial u}{\partial z}, \quad \frac{\partial v}{\partial z}, \quad K_x, \quad K_y$$

or (17)

$$b)\; Z_z = f_{j1},\;\; u = f_{j2},\;\; v = f_{j3},\;\; \tau_0 T + \tau_1 \frac{\partial T}{\partial z} = f_{j4}\;;$$

from these three first conditions on the face $z = z_j$
are defined functions

$$D, \;\; K_z, \;\; \frac{\partial w}{\partial z}, \;\; \frac{\partial K_z}{\partial z}, \;\; \frac{\partial K_y}{\partial z},$$

where τ_0 and τ_1 are arbitrary real numbers and $f_{jh}(x,y)$ $(h=1,2,3,4)$ are functions that can be expanded into a uniformly converging double trigonometric Fourier series.

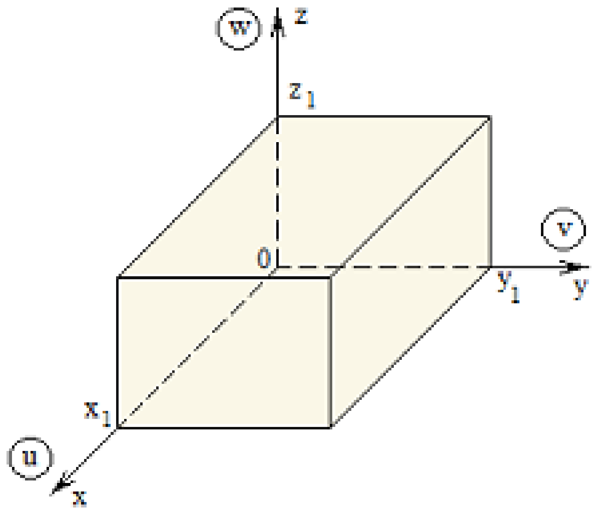

Fig. 1. Thermoelastic Rectangular Parallelepiped under Consideration

We assume that the given boundary conditions satisfy compatibility conditions on the edges of the parallelepiped

$$\Omega = \{0 < x < x_1,\ \ 0 < y < y_1,\ \ 0 < z < z_1\} \ .$$

Before we continue the discussion of boundary conditions we should point out that the following conditions:

$$\left(\frac{\partial g}{\partial x}\right)_{x=x_j} = 0, \qquad \left(\frac{\partial g}{\partial y}\right)_{y=y_j} = 0, \tag{18}$$

are imposed on the function $g(x,y,z)$ where $j = 0,\ 1$, and $x_0 = 0,\ y_0 = 0$.

Boundary conditions $(15a)$ and $(16a)$ will be called boundary conditions of continuous symmetrical extension of the solution through the facet $x = x_j$ or, correspondingly, through the facet $y = y_j$ or, briefly, boundary conditions of symmetry while Conditions $(15b)$ and $(16b)$ will be called boundary conditions of continuous antisymmetrical extension of the solution through the facet $x = x_j$ or, correspondingly, through the facet $y = y_j$ or, briefly, boundary conditions of antisymmetry.

Boundary conditions $(17a)$ for $\tau_0 = 0, \tau_1 = 1$ will be called non-homogeneous boundary conditions of symmetry and boundary conditions $(17b)$ for $\tau_0 = 1, \tau_1 = 0$ will be called non-homogeneous boundary conditions of antisymmetry.

4 Basic Equations by Means of Which Analytical Solutions of Boundary Value and Boundary Value Contact Problems of Thermoelasticity are Constructed

Let us go back to Equilibrium Equations (9). They easily give equations

$$a)\ \Delta D = 0, \quad b)\ \Delta K_z = 0, \quad \text{c)}\ \Delta T = 0. \tag{19}$$

From Equations (9c), (10c) and (10b) we have

$$\begin{aligned} &a)\ \Delta w = -\frac{\partial T}{\partial z} - \frac{1}{\mu}\frac{\partial D}{\partial z} - \frac{\partial\,(gD)}{\partial z} = 0 \quad \text{or} \\ &b)\ \Delta\left(w + \frac{z}{2\mu}D + \frac{z}{2}T\right) = \frac{\partial\,(gD)}{\partial z} \end{aligned} \tag{20}$$

If we introduce the notation $w + \frac{z}{2\mu}D + \frac{z}{2}T = \Psi$, we have

$$\Delta\Psi = \frac{\partial\,(gD)}{\partial z}. \tag{21}$$

Equation (21) implies that $\Psi = \Psi_0 + \Psi_*$, where Ψ_0 is a harmonic function and Ψ_* is a partial solution of Equation (21).

The equations given above imply

$$w = -\frac{z}{2\mu}D - \frac{z}{2}T + \Psi_0 + \Psi_*. \tag{22}$$

We assume that boundary conditions of antisymmetry, i.e. Conditions (15b) and (16b), are satisfied on the lateral facets of the parallelepiped. Then using the method of separation of variables we can represent the harmonic functions D, T and Ψ_0 in the form of corresponding infinite series and for the displacement w we will have:

$$\begin{aligned} w = -\sum_{l=1}^{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\eta_l\left[A_{lmn}e^{-pz} + B_{lmn}e^{p(z-z_1)}\right] \\ \times\sin(\tilde{m}x)\sin(\tilde{n}y) + \tilde{\eta}_l\,\Psi_*, \end{aligned} \tag{23}$$

where $\eta_1 = \frac{z}{2\mu}$, $\eta_2 = 0$, $\eta_3 = \frac{z}{2}$, $\eta_4 = 1$, $\tilde{\eta}_1 = 1$, $\tilde{\eta}_2 = \tilde{\eta}_3 = \tilde{\eta}_4 = 0$; $\tilde{m} = \frac{\pi m}{x_1}$, $\tilde{n} = \frac{\pi n}{y_1}$.
It can be easily seen that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{lmn}e^{-pz} + B_{lmn}e^{p(z-z_1)}\right]\sin(\tilde{m}x)\sin(\tilde{n}y) = D,\ \ T,\ \ \Psi_0;$$

$$l = 1,\ 3,\ 4,$$

i.e. for $l = 1$ we have the function D, for $l = 3$ we have the function T, and for $l = 4$ we have the function Ψ_0.

With boundary conditions (15), (16), (17) in mind we can write the harmonic functions D, K_z, T and Ψ_0 in the following way:

$$\Phi_{l0}(z) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi_{lmn}(z)\,\varphi_{lmn}^{(s)}(x,y) = D\ ,\ K_z,\ T,\ \Psi_0; \qquad l = 1,2,3,4;\quad s = 1,2,3,4 \tag{24}$$

Formula (24) means that its left-hand side represents the function D for $l = 1$, the function K_z -for $l = 2$, the function T - for $l = 3$ and the function Ψ_0 - for $l = 4$ and

$$\begin{aligned} &a)\ \Phi_{l0} = a_l z + b_l;\quad \Phi_{lmn} = A_{lmn}e^{-pz} + B_{lmn}e^{p(z-z_1)},\\ &\quad \text{where } p = \sqrt{\tilde{m}^2+\tilde{n}^2},\ \text{ and } \tilde{m} = \tilde{m}(m),\ \ \tilde{n} = \tilde{n}(n);\\ &b)\ \Phi_{lmn}^{(1)} = \sin(\tilde{m}x)\sin(\tilde{n}y),\ \Phi_{lmn}^{(2)} = \cos(\tilde{m}x)\sin(\tilde{n}y),\\ &\quad \Phi_{lmn}^{(3)} = \sin(\tilde{m}x)\cos(\tilde{n}y),\ \Phi_{lmn}^{(4)} = \cos(\tilde{m}x)\cos(\tilde{n}y). \end{aligned} \tag{25}$$

a_l, b_l, A_{lmn} and B_{lmn} are constants here that are to be defined from boundary conditions (17), and formulas (24) (25) themselves represent harmonic functions D, K_z, T and Ψ_0 in the form of the corresponding infinite series. Naturally, these infinite series are obtained by applying the method of separation of variables to Laplace's equation.

Note that the function Φ_{l0} in the majority of cases is equal to zero. This will become evident after some particular boundary value problems of thermoelasticity, which are given in the forthcoming, have been considered. Specifically, as an example we consider some of the boundary conditions on the lateral surface of the parallelepiped Ω, i.e. some of boundary conditions (15) and (16).

Example 1. Boundary conditions of antisymmetry are defined on the lateral facets of the parallelepiped. In this case we have:

$$D, T, \Psi_0 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[A_{lmn}e^{-pz} + B_{lmn}e^{-p(z-z_1)}\right]\sin(\tilde{m}x)\sin(\tilde{n}y);$$
$$l = 1,3,4;\quad \tilde{m} = \tfrac{\pi m}{x_1},\quad \tilde{n} = \tfrac{\pi n}{y_1};$$

$$K_z = \Phi_{20}(z) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[A_{2mn}e^{-pz} + B_{2mn}e^{-p(z-z_1)}\right]\cos(\tilde{m}x)\cos(\tilde{n}y),$$
$$\tilde{m} = \tfrac{\pi m}{x_1},\quad \tilde{n} = \tfrac{\pi n}{y_1}.$$

Example 2. On the facets $x = 0$ and $y = 0$ antisymmetry conditions are defined, while on the facets $x = x_1$ and $y = y_1$ symmetry conditions are given, then

$$D, T, \Psi_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A_{lmn} e^{-pz} + B_{lmn} e^{-p(z-z_1)} \right] \sin(\tilde{m}x) \sin(\tilde{n}y);$$
$$l = 1, 3, 4; \;\; \tilde{m} = \tfrac{\pi(2m-1)}{2x_1}, \;\; \tilde{n} = \tfrac{\pi(2n-1)}{2y_1};$$

$$K_z = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A_{2mn} e^{-pz} + B_{2mn} e^{-p(z-z_1)} \right] \cos(\tilde{m}x) \cos(\tilde{n}y) \quad ,$$
$$\tilde{m} = \tfrac{\pi(2m-1)}{2x_1}, \;\; \tilde{n} = \tfrac{\pi(2n-1)}{2y_1}.$$

Example 3. On the facet $x = 0$ antisymmetry conditions are defined, while on the three remaining facets symmetry conditions are given. In this case we have:

$$D, T, \Psi_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A_{lmn} e^{-pz} + B_{lmn} e^{-p(z-z_1)} \right] \sin(\tilde{m}x) \cos(\tilde{n}y);$$
$$l = 1, 3, 4; \;\; \tilde{m} = \tfrac{\pi(2m-1)}{2x_1}, \;\; \tilde{n} = \tfrac{\pi n}{y_1};$$

$$K_z = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A_{2mn} e^{-pz} + B_{2mn} e^{-p(z-z_1)} \right] \cos(\tilde{m}x) \sin(\tilde{n}y),$$
$$\tilde{m} = \tfrac{\pi(2m-1)}{2x_1}, \;\; \tilde{n} = \tfrac{\pi n}{y_1}.$$

Example 4. On the lateral facets of the parallelepiped symmetry conditions are defined, then we have:

$$D, T, \Psi_0 = \Phi_{l0}(z) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A_{lmn} e^{-pz} + B_{lmn} e^{-p(z-z_1)} \right]$$
$$\times \cos(\tilde{m}x) \cos(\tilde{n}y); \;\; l = 1, 3, 4; \;\; \tilde{m} = \tfrac{\pi m}{x_1}, \;\; \tilde{n} = \tfrac{\pi n}{y_1};$$

$$K_z = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A_{2mn} e^{-pz} + B_{2mn} e^{-p(z-z_1)} \right] \sin(\tilde{m}x) \sin(\tilde{n}y),$$
$$\tilde{m} = \tfrac{\pi m}{x_1}, \qquad \tilde{n} = \tfrac{\pi n}{y_1}.$$

It can be easily seen that the function Φ_{l0} appears only Examples 1 (for $l = 2$) and 4 (for $l = 1,\ 3,\ 4$) while in most of the other cases $\Phi_{l0} = 0$.

Now we will try to express the displacements u and v through the functions D, K_z, T and Ψ.

It is reasonable to define displacements u v for particular values of boundary conditions, selected from boundary conditions defined by formulas (15), (16) and (17). For example, choose the conditions given in Example 1. To be more definite, add Conditions ($17a$) to the conditions of Example 1.

Determining displacements u and v, in the case when particular boundary conditions are defined makes both the understanding of the present material significantly easier and the possibility of its extension for the case of other boundary conditions, different from conditions (15b), (16b), (17a).

Bearing in mind boundary conditions (15b), (16b) and (17a) that we have chosen we can write the harmonic functions D, K_z, T Ψ_0 in the following way:

$$D, K_z, T, \Psi_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{lmn}e^{-pz} + B_{lmn}e^{-p(z-z_1)}\right]\varphi_{lmn}(x,y), \tag{26}$$

where $l = 1,2,3,4$, $\varphi_{1mn}(x,y) = \varphi_{3mn}(x,y) = \varphi_{4mn}(x,y) = \sin(\tilde{m}x)\sin(\tilde{n}y)$, $\varphi_{2mn}(x,y) = \cos(\tilde{m}x)\cos(\tilde{n}y)$; $\tilde{m} = \frac{\pi m}{x_1}$, $\tilde{n} = \frac{\pi n}{y_1}$.

Equations (9a) and (9b) lead to

$$\begin{aligned} &a)\ \tfrac{\partial K_y}{\partial z} = -\sum_{l=1}^{3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{lmn}e^{-pz} + B_{lmn}e^{-p(z-z_1)}\right]\chi_{1l} \\ &\qquad\qquad \times\cos(\tilde{m}x)\sin(\tilde{n}y), \\ &b)\ \tfrac{\partial K_x}{\partial z} = \sum_{l=1}^{3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{lmn}e^{-pz} + B_{lmn}e^{-p(z-z_1)}\right]\chi_{2l} \\ &\qquad\qquad \times\sin(\tilde{m}x)\cos(\tilde{n}y), \end{aligned} \tag{27}$$

where $\chi_{11} = \tilde{m}$, $\chi_{12} = \tilde{n}$, $\chi_{13} = 4\mu\tilde{m}$, $\chi_{21} = \tilde{n}$, $\chi_{22} = -\tilde{m}$,$\chi_{23} = 4\mu\tilde{n}$; $\tilde{m} = \frac{\pi m}{x_1}$, $\tilde{n} = \frac{\pi n}{y_1}$.

After the integration of Equations (27), we have

$$\begin{aligned} &a)\ K_y = \sum_{l=1}^{3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tfrac{\chi_{1l}}{p}\left[A_{lmn}e^{-pz} - B_{lmn}e^{-p(z-z_1)}\right] \\ &\qquad\qquad \times\cos(\tilde{m}x)\sin(\tilde{n}y), \\ &b)\ K_x = -\sum_{l=1}^{3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tfrac{\chi_{2l}}{p}\left[A_{lmn}e^{-pz} - B_{lmn}e^{-p(z-z_1)}\right] \\ &\qquad\qquad \times\sin(\tilde{m}x)\cos(\tilde{n}y), \end{aligned} \tag{28}$$

where $\chi_{11} = \tilde{m}$, $\chi_{12} = \tilde{n}$, $\chi_{13} = 4\mu\tilde{m}$, $\chi_{21} = \tilde{n}$, $\chi_{22} = -\tilde{m}$,$\chi_{23} = 4\mu\tilde{n}$; $\tilde{m} = \frac{\pi m}{x_1}$, $\tilde{n} = \frac{\pi n}{y_1}$.

Equations (10b) and (10c) imply

$$a)\ \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} - \frac{1}{\mu}K_x, \qquad b)\ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} + \frac{1}{\mu}K_y. \tag{29}$$

(28), taking (22) into account, results in

$$\begin{aligned} &a)\ \frac{\partial v}{\partial z} = -\frac{z}{2\mu}\frac{\partial D}{\partial y} - \frac{z}{2}\frac{\partial T}{\partial y} + \frac{\partial \Psi_0}{\partial y} - \frac{1}{\mu}K_x + \frac{\partial \Psi_*}{\partial y}, \\ &b)\ \frac{\partial u}{\partial z} = -\frac{z}{2\mu}\frac{\partial D}{\partial x} - \frac{z}{2}\frac{\partial T}{\partial x} + \frac{\partial \Psi_0}{\partial x} + \frac{1}{\mu}K_y + \frac{\partial \Psi_*}{\partial x}. \end{aligned} \tag{30}$$

After the integration of Equations (30) we have:

$$\begin{aligned} a)\ v &= \sum_{l=1}^{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{lmn}\eta_{Al}e^{-pz} - B_{lmn}\eta_{Bl}e^{-p(z-z_1)}\right] \\ &\qquad \times \sin(\tilde{m}x)\cos(\tilde{n}y) + \tilde{\eta}_l \int \tfrac{\partial \Psi_*}{\partial y} dz, \\ b)\ \mathrm{u} &= \sum_{l=1}^{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{lmn}\bar{\eta}_{Al}e^{-pz} - B_{lmn}\bar{\eta}_{Bl}e^{-p(z-z_1)}\right] \\ &\qquad \times \cos(\tilde{m}x)\sin(\tilde{n}y) + \tilde{\eta}_l \int \tfrac{\partial \Psi_*}{\partial x} dz \end{aligned} \tag{31}$$

where $\eta_{A1} = \frac{(pz+3)\tilde{n}}{2\mu p^2}$, $\eta_{B1} = \frac{(pz-1)\tilde{n}}{2\mu p^2}$, $\eta_{A2} = \eta_{B2} = -\frac{\tilde{m}}{\mu p^2}$, $\eta_{A3} = \frac{(pz+9)\tilde{n}}{2p^2}$, $\eta_{B3} = \frac{(pz-7)\tilde{n}}{2p^2}$, $\eta_{A4} = \eta_{B4} = -\frac{\tilde{n}}{p}$, $\bar{\eta}_{A1} = \frac{(pz-1)\tilde{m}}{2\mu p^2}$,$\bar{\eta}_{B1} = \frac{(pz-3)\tilde{m}}{2\mu p^2}$, $\bar{\eta}_{A2} = \bar{\eta}_{B2} = \frac{\tilde{n}}{\mu p^2}$, $\bar{\eta}_{A3} = \frac{(pz-7)\tilde{m}}{2p^2}$, $\bar{\eta}_{B3} = \frac{(pz-9)\tilde{m}}{2p^2}$, $\bar{\eta}_{A4} = \bar{\eta}_{B4} = -\frac{\tilde{m}}{p}$; $\tilde{\eta}_1 = 1$, $\tilde{\eta}_2 = \tilde{\eta}_3 = \tilde{\eta}_4 = 0$;$\tilde{m} = \frac{\pi m}{x_1}$, $\tilde{n} = \frac{\pi n}{y_1}$.

According to Formulas (23) and (31), omitting the related explanations and comments, we can write (all together for convenience) formulas expressing displacements u, v and w.

$$\begin{aligned} a)\ \mathrm{u} &= \sum_{l=1}^{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{lmn}\bar{\eta}_{Al}e^{-pz} + B_{lmn}\bar{\eta}_{Bl}e^{-p(z-z_1)}\right] \\ &\qquad \times \cos(\tilde{m}x)\sin(\tilde{n}y) + \tilde{\eta}_l \int \tfrac{\partial \Psi_*}{\partial x} dz, \\ b)\ v &= \sum_{l=1}^{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{lmn}\eta_{Al}e^{-pz} + B_{lmn}\eta_{Bl}e^{-p(z-z_1)}\right] \\ &\qquad \times \cos\sin(\tilde{m}x)(\tilde{n}y) + \tilde{\eta}_l \int \tfrac{\partial \Psi_*}{\partial y} dz, \\ c)\ w &= \sum_{l=1}^{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\eta_l\left[A_{lmn}e^{-pz} + B_{lmn}e^{-p(z-z_1)}\right] \\ &\qquad \times \sin(\tilde{m}x)\sin(\tilde{n}y) + \tilde{\eta}_l\,\Psi_*. \end{aligned} \tag{32}$$

If we consider that the problem related to the partial solution of Equation (21) has been solved, then Formulas (32) give the solution of boundary value problem (9), (10), (15*b*), (16*b*), (17*a*).

Obviously, formulas of any of boundary value problems (9), (10), (15), (16), (17) can be written in an absolutely similar way to Formulas (32).

Furthermore, before we start looking for a partial solution of Equation (21), we should state some interesting, in our opinion, conclusions. These conclusions are given below in the form of the two following remarks.

5 Some Remarks Arising in the Process of Consideration of Boundary Value Problems of Thermoelasticity

Remark 2. Examining boundary value problems (9), (10), (15), (16), (17), we come to the following conclusion.

The thermoelastic stress/strain condition of the rectangular parallelepiped Ω with boundary conditions (15), (16) and (17) can be uniquely defined if at any point of the domain Ω we know

$$T, \quad div\overrightarrow{U}, \quad rot_z\overrightarrow{U} \quad \text{and} \quad w,$$

moreover, in this case the displacement vector $\overrightarrow{U}$ can be uniquely defined too.

The validity of the above-stated can be proved by Formulas (32) in which the components of the displacement vector u, v and w are expressed through harmonic functions D, K_z, T, Ψ_0 and the function Ψ_*

Remark 3. Let (15)* and (16)* denote non-homogeneous boundary conditions (15) and (16), respectively,. In other words, notation (15)* and (16)* will indicate the fact that on the lateral facets $x = x_j$ and $y = y_j$ of the parallelepiped Ω non-homogeneous boundary conditions of symmetry or antisymmetry are satisfied. Further, let (17)* denote boundary conditions (17a), for $\tau_0 = 0$ and $\tau_1 = 1$, and conditions (17b) - for $\tau_0 = 1$ and $\tau_1 = 0$. Taking into account the above-mentioned, we can assert the following.

By superposition of three boundary value problems, which are quite similar to boundary value problem (9), (10), (15), (16), (17)*, boundary value problem (9), (10), (15)*, (16)*, (17)* can be proved for a thermoelastic isotropic parallelepiped with non-homogeneity of the type $\mu = const$, $\lambda = \lambda(x, y, z)$.

Now we should explain what the three boundary value problems the sum of which gives the solution of the general boundary value problem (9), (10), (15)*, (16)*, (17)* imply.

Boundary value problem (9), (10), (15), (16), (17)* will be considered to be the first one. Boundary value problem (9), (10), (15)*, (16), $(17)^0$ will be considered to be the second one and boundary value problem (9), (10), (15), (16)*, $(17)^0$ will be considered the third one, number $(17)^0$ denoting homogeneous (zero) boundary conditions in (17)*, i.e. the case when in conditions (17)* we have $f_{j1} = 0$, $f_{j2} = 0$, $f_{j3} = 0$, $f_{j4} = 0$, and in the case $(17a)^*$ we have $\tau_0 = 0$, $\tau_1 = 1$, while in the case $(17b)^*$ $\tau_0 = 1$, $\tau_1 = 0$.

6 Partial Solution Construction for Poisson's Equation (21)

Now we will find a partial solution of Equation (21).

First of all we assume that the function $g(x,\, y,\, z)$can be expressed by a uniformly converging infinite trigonometric threefold Fourier series. From this infinite series one can always obtain a threefold finite trigonometric series which represents the function $g(x,\, y,\, z)$ with any a priori defined degree of precision. By virtue of the just stated when any application problem of thermoelasticity (including thermoelasticity problems arising in precision engineering) is being solved use of the corresponding finite series (" corresponding finite series implies a sufficiently high approximation precision of the function g by a finite series) for the function $g(x,\, y,\, z)$ practically in no way limits either precision or effectiveness of the solution.

Taking into consideration the above-stated and Conditions (18) we can represent the function $g(x,\, y,\, z)$ in the following form

$$g(x,\, y,\, z) = \sum_{h=0}^{h_0} \sum_{m=0}^{m_0} \sum_{n=0}^{n_0} \left[g_{mnh}^{(c)} \cos(\tilde{h}z) + g_{mnh}^{(s)} \sin(\tilde{h}z) \right] \times \cos(\tilde{m}x) \cos(\tilde{n}y), \tag{33}$$

where $\tilde{h} = \tilde{h}(h)$, $\tilde{m} = \frac{\pi m}{x_1}$, $\tilde{n} = \frac{\pi n}{y_1}$; h_0, m_0, n_0 are natural numbers and $h_0 \geq 1$, $m_0 \geq 1$, $n_0 \geq 1$; $g_{mnh}^{(c)}$ $g_{mnh}^{(s)}$ are Fourier's functions $g(x,\, y,\, z)$.

Examining Formula (33) one can easily conclude that when constructing a partial solution of Equation (21) i.e. when constructing the function Ψ_*, it is sufficient to take just one summand instead of the whole series (33). Indeed, the members of series (33) differ from each other only by their constants $g_{mnh}^{(c)}$, $g_{mnh}^{(s)}$, $\tilde{h}$, $\tilde{m}$, $\tilde{n}$, so having a partial solution for a single member of the series, if we sum the partial solutions we will obtain the desired full partial solution.

The mentioned single member out of the set of members of series (33) can be written in the following form :

$$G(x,\, y,\, z) = \left[\bar{G}_{mnh} \cos(\tilde{h}z) + \tilde{G}_{mnh} \sin(\tilde{h}z) \right] \times \times \cos\left(\frac{\pi m_1}{x_1} x \right) \cos\left(\frac{\pi n_1}{y_1} y \right), \tag{34}$$

where $\bar{G}_{mnh}$,$\tilde{G}_{mnh}$ are constants depending on m, n and h; $\tilde{h} = \tilde{h}(h)$; m_1 and n_1 are some particular natural numbers.

Actually, one can believe that this completes the construction of the function Ψ_*, if we assume that a partial solution of Equation (21) has been constructed when in Equation (21) the function $g(x,\, y,\, z)$ is substituted by the function $G(x,\, y,\, z)$. We should indicate that the function $G(x,\, y,\, z)$ will appear when we examine one example at the end of the present paper. It is just this example that will complete the final understanding of the algorithm of the search for a partial solution of Equation (21).

Further we will consider non-homogeneity of the type:

$$\mu = const, \qquad \lambda = \lambda(z). \tag{35}$$

This non-homogeneity is particularly important due to the fact that when conditions (35) are satisfied the class of analytically solvable boundary value and boundary value contact problems of thermoelasticity can be extended. Let us examine this in more detail.

7 Problem Solution in the Case When $g = g(z)$

In the case when $g = g(z)$, boundary conditions(15) and (16) remain unchanged while the number of boundary conditions on the facets $z = z_j$ can markedly increase. Specifically, boundary conditions on the indicated facets can be considered arbitrary although in the forthcoming we assume that on each of the facets $z = z_j$ one of the four basic boundary conditions is defined.

Let us state the indicated boundary conditions.

For $z = z_j \quad (j = 0, 1; \ z_0 = 0)$:

$$a) \quad Z_z = f_{j1}\,, Z_x = f_{j2}, Z_y = f_{j3}\,, \tau_0 T + \tau_1 \tfrac{\partial T}{\partial z} = f_{j4} \ \Rightarrow \Rightarrow \frac{\partial}{\partial z} K_z = \frac{\partial f_{j3}}{\partial x} - \frac{\partial f_{j2}}{\partial y}$$

or

$$b) \quad w = f_{j1}\,, u = f_{j2}, v = f_{j3}, \tau_0 T + \tau_1 \tfrac{\partial T}{\partial z} = f_{j4} \ \Rightarrow \Rightarrow K_z = \frac{\partial f_{j3}}{\partial x} - \frac{\partial f_{j2}}{\partial y}$$

or

$$c) \quad w = f_{j1}\,, Z_x = f_{j2}\,, \ Z_y = f_{j3}\,, \ \tau_0 T + \tau_1 \tfrac{\partial T}{\partial z} = f_{j4}\,; \tag{36}$$

from the first three conditions the functions

$$\frac{\partial}{\partial z} D, \quad \frac{\partial}{\partial z} K_z, \quad \frac{\partial u}{\partial z}, \quad \frac{\partial v}{\partial z}, \quad K_x, \quad K_y$$

are defined on the boundary$z = z_j$ or

$$d) \quad Z_z = f_{j1}\,, \ u = f_{j2}, \ v = f_{j3}, \ \tau_0 T + \tau_1 \frac{\partial T}{\partial z} = f_{j4}\,;$$

from the first three conditions the functions

$$D, \quad K_z, \quad \frac{\partial w}{\partial z}, \quad \frac{\partial}{\partial z} K_x, \quad \frac{\partial}{\partial z} K_y.$$

are defined on the boundary$z = z_j$.

Naturally, boundary conditions (36) markedly expand compared to Conditions (17), the class of boundary value problems of thermoelasticity studied in the given paper. Moreover, non-homogeneity of type (35) allows us to find a partial solution of Equation (21) easier than in the case $g = g(x,\, y,\, z)$, and complete the solution of the corresponding boundary value problem of thermoelasticity.

We will find a partial solution of Equation (21) in this section. As for problems we have often mentioned above dealing with a multilayer piecewise non-homogeneous rectangular parallelepiped their statement and solution methods are given in the next section.

We will search for a partial solution of Equation (21), similar to Series (33), as a form of the following single finite series

$$g(z) = \sum_{h=0}^{h_0} \left[\bar{g}_h \cos\left(\tilde{h}z\right) + \tilde{g}_h \sin\left(\tilde{h}z\right) \right] \ , \tag{37}$$

where $\tilde{h} = \tilde{h}(h)$, h_0 is a particular natural number and $\bar{g}_h$ and $\tilde{g}_h$ are Fourier coefficients of the function $g(z)$.

Due to the same reason as in the case of Decomposition (33), we take just one member of series (37), which can be written in the following way:

$$G^*(z) = \bar{G}_h \cos\left(\tilde{h}z\right) + \tilde{G}_h \sin\left(\tilde{h}z\right) \ , \tag{38}$$

where $\bar{G}_h$ and $\tilde{G}_h$ are constants.

We consider the construction of Equation (21) in the case when

$$D,\ T,\ \Psi_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{lmn} e^{-pz} + B_{lmn} e^{p(z-z_1)} \right] \sin(\tilde{m}x) \sin(\tilde{n}y) \,, \tag{39}$$

where $l = 1,\ 3,\, 4$; $\tilde{m} = \frac{\pi m}{x_1}$, $\tilde{n} = \frac{\pi n}{y_1}$. Note that solution construction in other cases will not be very different from the case under consideration.

If we search now for the partial solution of Equation (21) in the form of $\Psi_* = \sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} F_{mn}(z) \sin(\tilde{m}x) \sin(\tilde{n}y)$, we will have

$$\frac{d^2 F_{mn}(z)}{dz^2} - p^2 F_{mn}(z) = \left[\bar{G}_h \cos\left(\tilde{h}z\right) + \tilde{G}_h \sin\left(\tilde{h}z\right) \right] \ . \tag{40}$$

Since it is not a problem to write a partial solution of Equation (40) we can consider the function Ψ_* to be known. And if this is true, we will know the form of all basic function through which displacements are expressed in the case of any of boundary value (9), (10), (15*b*), (16*b*), (36). Indeed, these are functions D, $T$and Ψ_0, defined by Formulas (39), the function K_z, defined by formulas of Example 1 and, finally, the function Ψ_*. that we have defined. Using the listed functions we compose series for the boundary conditions selected from (36) (e.g., we assume that these are Conditions (36 a) both for $z = 0$ and for $z = z_1$) and compare them with their corresponding trigonometric series representing the functions $f_{j1}, f_{j2}, f_{j3}, f_{j4}$. As a result we have an infinite system of linear algebraic equations with respect to unknown $A_{lmn}, B_{lmn}, l = 1, 2, 3, 4$ with a block diagonal matrix. Each block of the infinite matrix is of the eighths order and always breaks into two blocks of the second order and one block of the fourth order. None of the listed block matrices is degenerate.

Hence the solution of boundary value problems (9), (10), (15b), (16b), (36) can be considered complete.

8 Multilayer Piecewise Non-Homogeneous Rectangular Parallelepiped

Consider a piecewise non-homogeneous rectangular parallelepiped which is multilayer along the coordinate z and occupies the domain Ω_z. Here Ω_z is a union of the domains $\Omega_{z_1} = \{0 < x < x_1, \ 0 < y < y_1, \ 0 < z < z_1\}$, $\Omega_{z_2} = \{0 < x < x_1, \ 0 < y < y_1, \ z_1 < z < z_2\}$, . . ., $\Omega_{z_q} = \{0 < x < x_1, \ 0 < y < y_1, \ z_{q-1} < z < z_q\}$, which contact with each other along the planes $z = z_j$, where $j = 1, \ 2, \ \ldots \ , q-1$, and q is the number of layers (Fig. 2). Each layer has its own elastic and thermal characteristics. For $x = x_j$ (here $j = 0, \ 1$) for all layers some of Conditions (15) are simultaneously satisfied, while for $y = y_j \, (j = 0, \ 1)$ some of conditions (16) are simultaneously satisfied.

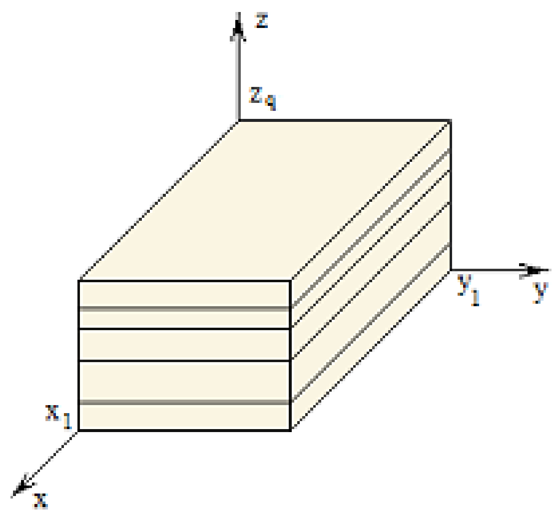

Fig. 2. The studied piecewise non-homogeneous rectangular parallelepiped multilayer along the coordinate z

If a body occupies the domain Ω_z, Conditions (15), (16) and (36) are satisfied on the boundaries of this domain, with z_1 in (36) being substituted by z_q. On the contact planes $z = z_j$ ($j = 1,\ 2,\ \dots,\ q-1\,;\ z = z_j$ is the contact plane of the layer number j, which contacts with the layer number $j+1$), various contact conditions can be defined. We will give some of them here.

1. Contact conditions with a rigid contact between the layers.

$$\begin{aligned} &T_j - T_{j+1} = 0,\ k_j^*\frac{\partial T_j}{\partial z} - k_{j+1}^*\frac{\partial T_{j+1}}{\partial z} = 0,\\ &w_j - w_{j+1} = 0, Z_{z\,j} - Z_{z\,j+1} = 0, u_j - u_{j+1} = 0,\\ &Z_{x\,j} - Z_{x\,j+1} = 0,\ v_j - v_{j+1} = 0,\ Z_{y\,j} - Z_{y\,j+1} = 0, \end{aligned} \tag{41}$$

   where k^* is a coefficient of heat conductivity.

2. The contact between the layers takes place only in a normal direction, i.e. along the coordinate z, while in the tangential direction (along x and y) the contact planes of the layers $z = z_j$ are free and do not contact. This is the so called sliding contact.

$$\begin{array}{l} T_j - T_{j+1} = 0,\ k_j^*\frac{\partial T_j}{\partial z} - k_{j+1}^*\frac{\partial T_{j+1}}{\partial z} = 0,\ w_j - w_{j+1} = 0,\\ Z_{z\,j} - Z_{z\,j+1} = 0,\ Z_{x\,j} = 0,\ Z_{x\,j+1} = 0,\ Z_{y\,j} = 0,\ Z_{y\,j+1} = 0. \end{array} \tag{42}$$

3. The contact between the layers is along the coordinate z, while in the tangential direction the contact planes of the layers $z = z_j$ are fixed (do not allow displacements)

$$\begin{array}{l} T_j - T_{j+1} = 0,\ k_j^*\frac{\partial T_j}{\partial z} - k_{j+1}^*\frac{\partial T_{j+1}}{\partial z} = 0,\ w_j - w_{j+1} = 0,\\ Z_{z\,j} - Z_{z\,j+1} = 0,\ u_j = 0,\ u_{j+1} = 0,\ v_j = 0,\quad v_{j+1} = 0. \end{array} \tag{43}$$

When we state the problem of establishing the thermoelastic equilibrium of a piecewise non-homogeneous multilayer parallelepiped we do the following. Bearing in mind boundary conditions (15) and (16) (for a complete definiteness we assume that both for $z = 0$ and $z = z_j$ stresses and temperature are defined) we write the expressions for the functions $D^{(j)}$, $K_z^{(j)}$, $T^{(j)}$, $\Psi_0^{(j)}$ and $\Psi_*^{(j)}$ for each of the layers and following the methods of the previous section obtain an infinite system of linear algebraic equations with a block diagonal matrix of the $8q$-th order. The latter matrix can be decomposed into two matrices of the $2q$-th order and one matrix of the $4q$-th order. Respectively, the solution of the whole problem becomes much simpler. Solvability of the infinite systems of equations, convergence of the corresponding infinite series and uniqueness of the obtained solutions can be easily shown.

Besides the listed contact conditions we can also consider a number of other contact conditions under which the solutions can be effectively constructed.

9 Solution of a Boundary Value Problem of Thermoelasticity in the Case When $G = A_0 + A\cos\left(\tilde{h}z\right)\cos\left(\frac{\pi m_1}{x_1}x\right)\cos\left(\frac{\pi n_1}{y_1}y\right)$

First we should note that after the problem with non-homogeneity indicated in the heading (A_0 and A that appear in the heading are constant) of this section has been solved it will be quite easy to solve the problem with non-homogeneity defined by Formula (34) as well.

In this section we construct the solution of the boundary value problem (9), (10), (15b), (16b), (17b)^0 for $z = 0$, (17b)* for $z = z_1$ (explanations related to Formulas (17b)^{0}and (17b)*are given in Section 4). We assume that under Conditions (17b)* for $z = z_1$ $f_{12} = 0$, $f_{13} = 0$.

It follows from Equation (19b) and the boundary conditions for the function K_z that at any point of the closed domain $\bar{\Omega}$ the function $K_z = 0$.

From equations (19a), (19c)and (20b)and boundary conditions for the functions D, T and Ψ_0 ,using the method of separation of variables we have

$$
\begin{aligned}
&a)\ D = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{1mn}\sinh(pz)\sin\left(\frac{\pi m}{x_1}x\right)\sin\left(\frac{\pi n}{y_1}y\right),\\
&b)\ T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{3mn}\sinh(pz)\sin\left(\frac{\pi m}{x_1}x\right)\sin\left(\frac{\pi n}{y_1}y\right),\\
&c)\ \Psi_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{4mn}\cosh(pz)\sin\left(\frac{\pi m}{x_1}x\right)\sin\left(\frac{\pi n}{y_1}y\right).
\end{aligned}
\tag{44}
$$

From boundary conditions (17b)* for $z = z_1$, for functions $D$and T we have

$$
D = f_{11}(x, y) + 4\mu k f_{14}(x, y), \quad T = f_{14}(x, y).
$$

Bearing in mind Conditions (15b), (16b), and expanding the functions f_{11}

and f_{14} , into trigonometric Fourier series we have

$$a)\ \ D|_{z=z_1} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(f_{11mn} + 4\mu k f_{14mn})\sin\left(\frac{\pi m}{x_1}x\right)\sin\left(\frac{\pi n}{y_1}y\right), \tag{45}$$

$$b)\ \ T|_{z=z_1} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{14mn}\sin\left(\frac{\pi m}{x_1}x\right)\sin\left(\frac{\pi n}{y_1}y\right),$$

where both f_{11mn} and f_{14mn} - are Fourier's coefficients of the functions f_{11} and f_{14} respectively.

Comparing Series $(45a)$with Series $(44a)$ for $z = z_1$, we find

$$A_{1mn} = \frac{f_{11mn} + 4\mu k f_{14mn}}{\sinh(pz_1)}. \tag{46}$$

In quite a similar way we obtain

$$A_{14mn} = \frac{f_{14mn}}{\sinh(pz_1)}. \tag{47}$$

In order to find a partial solution of Equation (21), we substitute the function $G = A_0 + A\cos\left(\tilde{h}z\right)\cos\left(\frac{\pi m_1}{x_1}x\right)\cos\left(\frac{\pi n_1}{y_1}y\right)$ and Formula $(44a)$. in its right-hand side. As a result we obtain the following equation

$$\begin{aligned}\Delta\Psi = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_0(z)\sin\left(\tfrac{\pi m}{x_1}x\right)\sin\left(\tfrac{\pi n}{y_1}y\right) + \\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_1(z)\sin\left(\tfrac{\pi(m+m_1)}{x_1}x\right)\sin\left(\tfrac{\pi(n+n_1)}{y_1}y\right) \\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_1(z)\sin\left(\tfrac{\pi(m-m_1)}{x_1}x\right)\sin\left(\tfrac{\pi(n-n_1)}{y_1}y\right) \\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_1(z)\sin\left(\tfrac{\pi(m+m_1)}{x_1}x\right)\sin\left(\tfrac{\pi(n-n_1)}{y_1}y\right) \\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_1(z)\sin\left(\tfrac{\pi(m-m_1)}{x_1}x\right)\sin\left(\tfrac{\pi(n+n_1)}{y_1}y\right),\end{aligned} \tag{48}$$

where

$$F_0(z) = pAA_{1mn}\cosh(pz), \quad F_1(z) = \tfrac{AA_{1mn}}{4}\Big[p\cos(\tilde{h}z)\cosh(pz) - \tilde{h}\sin(\tilde{h}z)\sinh(pz)\Big].$$

We will search for a partial solution of Equation 48 separately for each type of the series in the right-hand side. Omitting four summands in the right-hand side of Equation (48), we can preserve, for example, the second summand. It is evident that if we find a partial solution in this case, we will easily find it in the remaining four cases in a quite similar way. Summing the five partial solutions we obtain the desired partial solution Ψ_*.

If we search for a partial solution of the equation

$$\Delta\tilde{\Psi} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_1(z)\sin\left(\frac{\pi(m+m_1)}{x_1}x\right)\sin\left(\frac{\pi(n+n_1)}{y_1}y\right)$$

in the form of $\tilde{\Psi}_* = \sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}\tilde{F}_{1mn}(z)\sin\left(\frac{\pi(m+m_1)}{x_1}x\right)\sin\left(\frac{\pi(n+n_1)}{y_1}y\right)$, we will have

$$\frac{d^2\tilde{F}_{1mn}}{dz^2} - \tilde{p}^2\tilde{F}_{1mn} = F_1(z), \tag{49}$$

where $\tilde{p} = \sqrt{\left[\frac{\pi(m+m_1)}{x_1}\right]^2 + \left[\frac{\pi(n+n_1)}{y_1}\right]^2}$.

It will not be a problem to find a partial solution of Equation (49); therefore we believe that a partial solution Ψ_* has been found.. If this is true, we can consider that the function Ψ_0

has been found and this completes the solution of the problem stated in the given section.

10 Solution of a Particular Problem for a Rectangular Parallelepiped

Let an isotropic rectangular parallelepiped be defined the dimensions of which, for simplicity, are: $x_1 = y_1 = 4\pi,\ z = \pi$. This parallelepiped occupying the domain $\Omega = \{0 < x < 4\pi,\ 0 < y < 4\pi,\ 0 < z < \pi\}$,has non-homogeneity of the type: $\mu = const,\ \ g = c_0 + c_1\cos(2z)$, where $c_0$and c_1-are constant and $c_0 > 0,\ c_0 > |c_1|$so that for any $z \in [0;\pi]$, we have $g > 0$. On the facets $x = 0,\ x = 4\pi,\ y = 0,\ y = 4\pi\ \ z = 0$ boundary conditions of antisymmetry are defined and for $z = \pi$

$$w = \sin\left(\frac{x}{4}\right)\sin\left(\frac{y}{4}\right),\ \ Z_x = 0,\ \ Z_y = 0,\ \ \frac{\partial T}{\partial z} = 0. \tag{50}$$

A problem is stated: to find all values of the constants $c_0$and c_1, for which $w\,(2\pi,\,2\pi,\,0) = 0$. Naturally, the problem can be solved only after boundary problem (9), (10), (15b), (16b), $(17)^0$ for $z = 0$, (50). has been solved.

We start the solution of the problem bearing in mind that Equations (19b), (19c)and boundary conditions imply that throughout the domain $\Omega\ \ K_z = 0,\ \ T = 0$.

We search for the functions D and Ψ_0 in the form

$$a)\ D = A_1 \sinh\left(\frac{z}{\sqrt{8}}\right) \sin\left(\frac{x}{4}\right) \sin\left(\frac{y}{4}\right),$$
$$b)\ \Psi_0 = A_4 \cosh\left(\frac{z}{\sqrt{8}}\right) \sin\left(\frac{x}{4}\right) \sin\left(\frac{y}{4}\right), \tag{51}$$

where A_1 and A_4 are constant.

Somewhat further the establishment of the constant A_4 in ($51b$) will complete the solution of our problem, while it can be shown for A_1 from ($51a$) that $A_1 = \frac{\mu}{4}$.

Substitute the function $g = c_0 + c_1 \cos(2z)$ and Formula ($51a$) in the right-hand side of Equation (21). After a transformation of the obtained expression we have

$$\Delta\Psi = \frac{\mu}{8\sqrt{2}} \left[c_1 \cosh\left(\frac{z}{\sqrt{8}}\right) \cos(2z) - 4\sqrt{2} c_1 \sinh\left(\frac{z}{\sqrt{8}}\right) \sin(2z) \right.$$
$$\left. + c_0 \cosh\left(\frac{z}{\sqrt{8}}\right) \right] \sin\left(\frac{x}{4}\right) \sin\left(\frac{y}{4}\right). \tag{52}$$

If we search for a partial solution of Equation (52) in the form of $\Psi_* = F_2(z) \sin\left(\frac{x}{4}\right) \sin\left(\frac{y}{4}\right)$, we wll have the following equation

$$\frac{d^2 F_2}{dz^2} - \frac{1}{8} F_2 = \frac{\mu}{8\sqrt{2}} \left[c_1 \cosh\left(\frac{z}{\sqrt{8}}\right) \cos(2z) \right.$$
$$\left. - 4\sqrt{2} c_1 \sinh\left(\frac{z}{\sqrt{8}}\right) \sin(2z) + c_0 \cosh\left(\frac{z}{\sqrt{8}}\right) \right]. \tag{53}$$

With the help of [14] and the corresponding table of integrals a partial solution of Equation (53) can be easily written. But we will do it somewhat later, specifically, after we have obtained expressions for the displacement w.

In our case Formula (22) will take the form $w = \Psi_0 - \frac{z}{2\mu} D + \Psi_*$, and the first of Conditions (50) will be written in the following way

$$\left(\Psi_0 - \frac{z}{2\mu} D + \Psi_* \right)_{z=\pi} = \sin\left(\frac{x}{4}\right) \sin\left(\frac{y}{4}\right). \tag{54}$$

From Equality (54) and the expression for the function Ψ_*, we have

$$A_4 = \frac{\frac{\pi}{8} \sinh \frac{\pi}{\sqrt{8}} + 1 - F_2(\pi)}{\cosh \frac{\pi}{\sqrt{8}}}.$$

Applying the last equality, we will have

$$w = \left[\frac{\frac{\pi}{8}\sinh\frac{\pi}{\sqrt{8}} + 1 - F_2(\pi)}{\cosh\frac{\pi}{\sqrt{8}}} \cosh\frac{z}{\sqrt{8}} - \frac{z}{8}\sinh\frac{z}{\sqrt{8}} + F_2(z) \right] \times \sin\left(\frac{x}{4}\right)\sin\left(\frac{y}{4}\right) . \tag{55}$$

for the displacement w .

Overcoming some tiresome computing difficulties we have

$$F_2(\pi) = \frac{\sqrt{2}\mu\cosh\frac{\pi}{\sqrt{8}}}{48}\left[\left(10 - 12\tanh\frac{\pi}{\sqrt{8}}\right)c_1 + \left(3\pi\sqrt{2}\tanh\frac{\pi}{\sqrt{8}} - 6\right)c_0\right],$$

$$F_2(0) = \frac{\sqrt{2}\mu}{24}(5c_1 + 3c_0).$$

Using these formulas we can write

$$\frac{8\cosh\frac{\pi}{\sqrt{8}}}{8+\sinh\frac{\pi}{\sqrt{8}}}w(2\pi,\ 2\pi,\ 0) = 1 + \frac{2\sqrt{2}\mu\sinh\frac{\pi}{\sqrt{8}}}{8+\sinh\frac{\pi}{\sqrt{8}}}c_1 - \frac{\pi\mu\sinh\frac{\pi}{\sqrt{8}}}{8+\sinh\frac{\pi}{\sqrt{8}}}c_0,$$

or

$$-\frac{c_1}{Q_1} + \frac{c_0}{Q_2} + \frac{w(2\pi,\ 2\pi,\ 0)}{Q_3} = 1, \tag{56}$$

where

$$Q_1 = \frac{8+\sinh\frac{\pi}{\sqrt{8}}}{2\sqrt{2}\mu\sinh\frac{\pi}{\sqrt{8}}},\ Q_2 = \frac{8+\sinh\frac{\pi}{\sqrt{8}}}{\pi\mu\sinh\frac{\pi}{\sqrt{8}}},\ Q_3 = \frac{8+\sinh\frac{\pi}{\sqrt{8}}}{8\cosh\frac{\pi}{\sqrt{8}}}.$$

Formula (56) defines the plane equation in intervals in the rectangular system of coordinates $c_1,\ c_0,\ w(2\pi,\ 2\pi,\ 0)$. We can show this plane in the system of coordinates $c_1,\ c_0,\ w(2\pi,\ 2\pi,\ 0)$, assuming that c_1 is an abscissa, c_0 is an ordinate and $w\,(2\pi,\ 2\pi,\ 0)$ is an applicata.

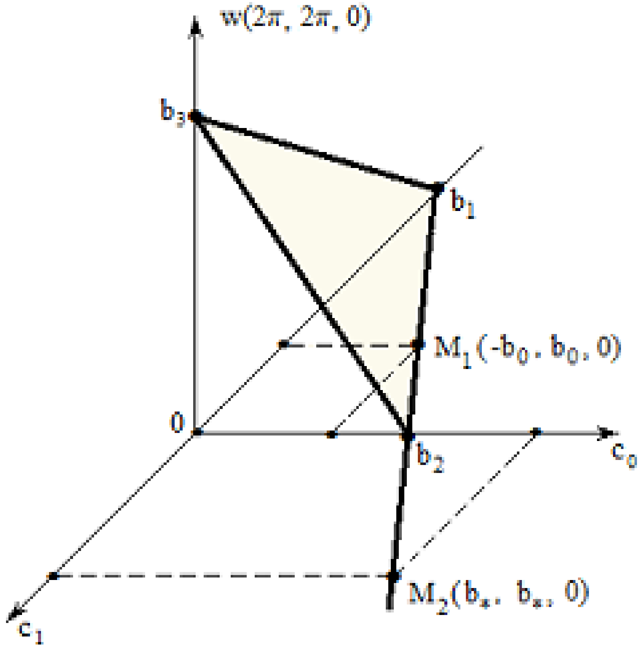

Fig. 3. Establishment of the desired constants $c_1$and c_0, bringing about the condition $w\,(2\pi,\,2\pi,\,0) = 0$

In Fig. 3 the lengths of the intervals ob_1, ob_2 and ob_3 are Q_1(abscissa), Q_2 (ordinate) and Q_3 (z-axis), respectively. Besides, any point of the line located on the interval b_1b_2, turns the applicata $w(2\pi,\,2\pi,\,0)$into zero, although the abscissa and ordinate will be the desired couple of the values $c_1$and c_0, defining the function $g$only for some points of this line.

It is obvious that both on the interval b_1b_2 and on the continuation of this interval (in the direction from the point b_1 to the point b_2), a point can be found with equal in the modulus abscissa and ordinate. Let $M_1\,(-b_0,\,b_0,\,0)$ and $M_2\,(b_*,\,b_*,\,0)$, be such points. Then it is evident that all values of c_1 will be located between $-b_0$ and b_*, i.e. $c_1 \in (-b_0,\,b_*)$.

The equation for our plane, for $w\,(2\pi,\,2\pi,\,0) = 0$, will take the form

$$-\frac{c_1}{Q_1} + \frac{c_0}{Q_2} = 1. \tag{57}$$

We note again that $Q_1 > Q_2,\;\; c_0 > 0,\;\; c_0 > |c_1|$.

With the help of (57) we will have $Q_2b_0+Q_1b_0 = Q_1Q_2\;\; -Q_2b_*+Q_1b_* = Q_1Q_2$, and these equalities imply that $b_0 = \dfrac{Q_1Q_2}{Q_1+Q_2}$ and $b_* = \dfrac{Q_1Q_2}{Q_1-Q_2}$.

Hence, the solution of our problem is

$$c_1 \in \left(-\frac{Q_1Q_2}{Q_1+Q_2},\; \frac{Q_1Q_2}{Q_1-Q_2}\right),$$

$$c_0 = \frac{Q_2}{Q_1} c_1 + Q_2 \,.$$

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