COMPUTATIONAL CONVEX ANALYSIS AND ITS APPLICATIONS TO NUMERICAL SOLUTION OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Computational Convex Analysis focuses on the numerical computation of fundamental transforms of Convex Analysis. The objective of this paper is the application of the algorithms of Computational Convex Analysis to numerical computation of the solutions and their gradients of some nonlinear partial differential equations, in particular, Hamilton–Jacobi equations, Scalar Conservation Laws and Monge–Ampere Equations.

Key words and phrases: Semiconvex and Semiconcave functions, the Legendre–Fenchel transform, convex envelope, Hamilton–Jacobi equation, Burger's equation, Monge–Ampere equation.

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1 Introduction

The present article is concerned with the application of fundamental transforms of Convex Analysis like the Legendre–Fenchel transform, the Moreau envelope and the convex envelope of the given function to the numerical computation of solutions of some nonlinear PDE.

The Legendre–Fenchel transform, also named Fenchel conjugate or convex conjugate of a given real valued function $f(x)$, $x \in \mathbb{R}^n$, is defined in the following manner

$$f^*(x) = \sup_{y \in \mathbb{R}^n} \big((x, y) - f(y)\big), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where (x, y) denotes the usual scalar product of two vectors $x, y \in \mathbb{R}^n$.

The Moreau envelope of a real-valued function $f(x)$, $x \in \mathbb{R}^n$, is defined as follows

$$M_\lambda f(x) = \inf_{y \in \mathbb{R}^n} \Big(f(y) + \frac{|x - y|^2}{2 \cdot \lambda}\Big), \quad x \in \mathbb{R}^n, \tag{1.2}$$

and $\lambda > 0$ is a strictly positive real number.

The Fenchel conjugate has long been studied in a wide range of fields for its duality properties. What concerns the Moreau envelope it has been studied extensively both theoretically and algorithmically for its regularization properties. Its origin goes back to the work of Yosida [12] on maximal monotone operators and its behavior is well known in the field of convex analysis (Rockafellar [7]) and variational analysis (Rockafellar, Wets [8]).

We note that the following simple relation exists between the above fundamental transforms

$$M_\lambda f(x) = \frac{|x|^2}{2 \cdot \lambda} - \frac{1}{\lambda} \cdot \Big(\frac{|y|^2}{2} + \lambda \cdot f(y) \Big)^*(x), \tag{1.3}$$

$$f^*(x) = \frac{|x|^2}{2} - \lambda \cdot M_\lambda \Big(\frac{1}{\lambda} \cdot f(y) - \frac{|y|^2}{2 \cdot \lambda} \Big)(x), \tag{1.4}$$

where $f(x)$, $x \in \mathbb{R}^n$, is a real-valued function and $\lambda > 0$.

From the definition (1.1) and the relation (1.3) the important fact follows that the Legendre–Fenchel transform of any function is always a convex function and the Moreau envelope is a semiconcave function with the semiconcavity constant equal to $\frac{1}{\lambda}$. From relations (1.3) and (1.4) we note that the computation of the Moreau envelope is equivalent to the computation of the Legendre–Fechel conjugate, so algorithms for computing one transform are trivially extended to compute the other.

The third fundamental transform of convex analysis we are interested in is the convex hull or convex envelope of a function $f(x)$. By definition the convex envelope $\Gamma(f)$ of a real-valued continuous function $f(x)$, $x \in \mathbb{R}^n$, is the greatest convex function majorized by f (see Rockafellar [7, Section 5]).

It is a remarkable fact that an important relation exists between the Legendre–Fechel transform and the convex envelope of f, namely

$$\Gamma(f) = (f^*)^* = f^{**}, \tag{1.5}$$

that is the convex envelope of the continuous function $f(x)$, $x \in \mathbb{R}^n$, is the second Legendre–Fechel conjugate of f (Rockafellar, Wets [8,Chapter 11]).

Computational Convex Analysis focuses on the numerical computation of fundamental transforms of Convex Analysis. Motivated by the study of some Hamilton–Jacobi partial differential equations, computational algorithms have been developed to compute the Legendre–Fechel transform on grids. A log-linear algorithm named the Fast Legendre transform (FLT for short) was first introduced by Brenier [3] to be subsequently improved by a linear-time algorithm: The Linear-time Legendre transform (LLT) (Lucet [6]).

Another linear-time algorithm, motivated by applications in image processing, was obtained by computing the Moreau envelope (Deniau [4,Ph.D.]). The fast algorithm to compute the convex envelope on grids was developed by Barber, Dobkin and Huhdanpaa [1] and is called "The Quickhull Algorithm for Convex Hulls". As it is well known the convex hull of a finite set of points in $\mathbb{R}^n$ is the smallest convex set that contains the points. Computing the convex hull is the fundamental problem of computational geometry. The above authors succeeded in constructing the quickhull algorithm (QHULL for short) which works for any number n of Euclidean $\mathbb{R}^n$ space and is really fast for $n \leq 4$.

The outline of the paper is as follows.

In Section 2 we review the basic theoretical results on Hamilton–Jacobi equations, Scalar Conservation Laws and the Monge–Ampere equations. Section 3 is dedicated to our basic inequalities in Convex Analysis which constitute the theoretical foundation for the numerical computation of the gradients of the solutions of the above mentioned equations, while in Section 4 several particular examples of nonlinear partial differential equations are considered and subsequently solved numerically. The errors in approximating the analytically known solutions are given.

2 Hopf–Lax Formulas as Solutions to Hamilton–Jacobi Equations. Links with Scalar Conservation Laws

We start this section by considering the initial-value problem for the Hamilton–Jacobi Equation

$$\begin{cases} u_t + H(\operatorname{grad} u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where the Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ and the initial function $g : \mathbb{R}^n \to \mathbb{R}$ are given and the function $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is the unknown,

$$\operatorname{grad} u(x,t) = \Big(\frac{\partial u(x,t)}{\partial x_1}, \dots, \frac{\partial u(x,t)}{\partial x_n}\Big)$$

is the vector of partial derivatives of u with respect to argument x.

Throughout this paper we will assume that $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous, i.e. there exists a nonnegative constant $c \geq 0$, such that

$$\big|g(x) - g(y)\big| \leq c \cdot |x-y|, \quad x, y \in \mathbb{R}^n. \tag{2.2}$$

What concerns the assumptions on Hamiltonian we will inevitably assume that

$$\begin{cases} H \text{ is } C^2 \text{ (i.e. twice continuously differentiable)}, \\ H \text{ is convex and } \lim\limits_{|x|\to\infty} \dfrac{H(x)}{|x|} = +\infty. \end{cases} \tag{2.3}$$

Definition 2.1. A C^2 convex function $H : \mathbb{R}^n \to \mathbb{R}$ is called uniformly convex (with constant $\gamma > 0$) if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 H(x)}{\partial x_i \partial x_j} \cdot y_i \cdot y_j \geq \gamma \cdot |y|^2 \text{ for all } x, y \in \mathbb{R}^n. \tag{2.4}$$

Definition 2.2. We say that the function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is semiconcave with constant $c \geq 0$ if for all $x, z \in \mathbb{R}^n$ the following one-sided inequality holds true

$$f(x+z) - 2 \cdot f(x) + f(x-z) \leq c \cdot |z|^2. \tag{2.5}$$

Definition 2.3. Consider the following infimal-convolution

$$u(x,t) = \min_{y\in\mathbb{R}^n} \Big\{ g(y) + t \cdot H^*\Big(\frac{x-y}{t}\Big)\Big\}, \tag{2.6}$$

where $t > 0$ and $x \in \mathbb{R}^n$. We call this expression the Hopf–Lax formula. (Here $H^*(x)$ is the Legendre–Fenchel conjugate of the Hamiltonian $H(x)$.)

The following lemma demonstrates that the function $u(x,t)$ inherits the semiconcavity property for any time instant $t > 0$ from the initial function $g(x)$.

Lemma 2.1. *Suppose that $g(x)$, $x \in \mathbb{R}^n$, is semiconcave with constant $c \geq 0$, then for any $t \geq 0$ the function $u(x,t)$ defined by formula* (2.6) *is also semiconcave with the same constant $c \geq 0$.*

Surprisingly enough, even if $g(x)$, $x \in \mathbb{R}^n$, is not semiconcave but if we require from convex Hamiltonian stronger condition of uniform convexity (2.4) it turns out that the Hopf–Lax function (2.6) will become semiconcave for times $t > 0$. Indeed, the following lemma holds true

Lemma 2.2. *Suppose that $H(x)$, $x \in \mathbb{R}^n$, is uniformly convex with constant $\gamma > 0$ and $u(x,t)$ is defined by the Hopf–Lax formula* (2.6). *Then for any strictly positive $t > 0$, the function $u(x,t)$ is semiconcave with constant $c = \frac{1}{\gamma \cdot t}$, i.e.*

$$u(x+z,t) - 2 \cdot u(x,t) + u(x-z,t) \leq \frac{1}{\gamma \cdot t} \cdot |z|^2 \text{ for all } x, z \in \mathbb{R}^n. \tag{2.7}$$

Definition 2.4. We say that a Lipschitz continuous function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a weak solution of the initial-value problem (2.1) provided

(a) $u(x,0) = g(x)$, $x \in \mathbb{R}^n$,

(b) $u_t(x,t) + H\big(\operatorname{grad} u(x,t)\big) = 0$ for a.e. $(x,t) \in \mathbb{R}^n \times (0,\infty)$, (2.8)

(c) there exists some constant $c \geq 0$, such that for all $t > 0$, $x, z \in \mathbb{R}^n$ it holds

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq c\Big(1 + \frac{1}{t}\Big) \cdot |z|^2. \tag{2.9}$$

Theorem 2.1 *Suppose the Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ is C^2 and satisfies* (2.3) *and the initial function $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous (i.e. satisfies* (2.2)*). If either g is semiconcave or H is uniformly convex, then*

$$u(x,t) = \min_{y \in \mathbb{R}^n} \Big\{ g(y) + t \cdot H^*\Big(\frac{x-y}{t}\Big)\Big\} \tag{2.10}$$

is the unique weak solution of the initial-value problem (2.1) *for the Hamilton–Jacobi equation.*

Now we turn to the initial-value problem for scalar conservation laws in one space dimension:

$$\begin{cases} u_t + \big(H(u)\big)_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{2.11}$$

where the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ and the initial function $g : \mathbb{R} \to \mathbb{R}$ are given and the function $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is the unknown, $(H(u))_x$ means the partial derivative of the function $H(u(x,t))$ with respect to x.

Equation (2.11) is a conservation law and our interest in it is due to the well-known connection between scalar conservation laws and Hamilton–Jacobi equations in one dimension.

Indeed, if $u(x,t)$ is an entropy solution (in Kruzkov's sense to be defined below) of the initial-value problem (2.11), then

$$v(x,t) = \int_0^x u(y,t)\,dy \tag{2.12}$$

is a weak solution of the initial-value problem for the following Hamilton–Jacobi equation

$$\begin{cases} v_t + H(v_x) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ v(x,0) = h(x), & x \in \mathbb{R}, \end{cases} \tag{2.13}$$

where

$$h(x) = \int\limits_0^x g(y)\,dy. \tag{2.14}$$

hence the solution $u(x,t)$ for the scalar conservation laws (2.11) can be obtained from the solution $v(x,t)$ for the Hamilton–Jacobi equation (2.13) by the following rule

$$u(x,t) = \frac{\partial}{\partial x}\, v(x,t), \quad t > 0. \tag{2.15}$$

Taking into account Theorem 2.1 (i.e. the formula (2.10)), the equality (2.15) can be written in the following manner

$$u(x,t) = \frac{\partial}{\partial x} \min_{y\in\mathbb{R}} \Big\{ h(y) + t \cdot H^*\Big(\frac{x-y}{t}\Big)\Big\}. \tag{2.16}$$

Next we define an entropy solution of the initial-value problem (2.11) for scalar conservation laws.

Definition 2.5. We say that a function $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is an entropy solution of the initial-value problem (2.11) provided

$$\int\limits_0^\infty \int\limits_{-\infty}^\infty \big(u \cdot v_t + H(u)\cdot v_x\big)\,dx\,dt + \int\limits_{-\infty}^\infty (g\cdot v)\,dx\Big|_{t=0} = 0 \tag{2.17}$$

for all test functions $v : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ with compact support, and there exists some constant $c \geq 0$, such that

$$u(x+z,t) - u(x,t) \leq c\Big(1 + \frac{1}{t}\Big)\cdot z \tag{2.18}$$

for a.e. $x, z \in \mathbb{R}$, $t > 0$, with $z > 0$.

The following theorem is the classical result which asserts the existence and uniqueness of the entropy solution.

Theorem 2.2 *Assume $H : \mathbb{R} \to \mathbb{R}$ is C^2, uniformly convex ($H''(x) \geq \beta > 0$) and $g \in L^\infty(\mathbb{R})$. Then for each time $t > 0$ the function*

$$u(x,t) = \frac{\partial}{\partial x} \min_{y\in\mathbb{R}} \big\{ h(y) + t \cdot H^*\Big(\frac{x-y}{t}\Big)\Big\} \tag{2.19}$$

is defined for a.e x. This function $u(x,t)$ as a function of two variables in (x,t) is the unique entropy solution of the initial-value problem (2.11) for scalar conservation laws.

Let us consider the case of quadratic Hamiltonian

$$H(x) = \frac{1}{2}\cdot |x|^2, \quad x \in \mathbb{R}^n. \tag{2.20}$$

It is well-known and easy to calculate that

$$H^*(x) = H(x) = \frac{1}{2} \cdot |x|^2, \quad x \in \mathbb{R}^n. \tag{2.21}$$

In this particular case the Hopf–Lax formula (2.6) is the same as the Moreau envelope

$$M_t g(x) = \inf_{y \in \mathbb{R}^n} \Big(g(y) + \frac{|x-y|^2}{2 \cdot t} \Big), \quad t > 0, \quad x \in \mathbb{R}^n, \tag{2.22}$$

but we recall that by the relation (1.3) the Moreau envelope can be written in terms of the Legendre–Fenchel conjugate, hence we get

$$M_t g(x) = \frac{|x|^2}{2 \cdot t} - \frac{1}{t} \cdot \Big(\frac{|y|^2}{2} + t \cdot g(y) \Big)^*(x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

otherwise we obtain the following relation

$$u(x,t) = \frac{|x|^2}{2 \cdot t} - \frac{1}{t} \cdot \Big(\frac{|y|^2}{2} + t \cdot g(y) \Big)^*(x), \quad t > 0, \quad x \in \mathbb{R}^n, \tag{2.23}$$

where $u(x,t)$ is the weak solution of the following initial-value problem for Hamilton–Jacobi equation with quadratic Hamiltonian

$$\begin{cases} u_t + \frac{1}{2} \cdot |\operatorname{grad} u|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.24}$$

As the Legendre–Fenchel transform of any function is a convex function, we deduce from the relation (2.23) that for any $t > 0$ the function $u(x,t)$ is semiconcave even if the initial function $g(x)$ was not.

We shall consider now the important particular case of scalar conservation, laws the so called Burger's equation:

$$\begin{cases} u_t + \Big(\frac{u^2}{2} \Big)_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \tag{2.25}$$

This equation corresponds to the case of quadratic Hamiltonian $H(x) = \frac{1}{2}\, x^2$, $x \in \mathbb{R}$. We apply Theorem 2.2 (Lax–Oleinik formula) to this case and get that the unique entropy solution of the initial-value problem (2.25) for Burger's equation is given by the formula

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial x} \min_{y \in \mathbb{R}} \Big\{ h(y) + \frac{1}{t} \cdot \frac{|x-y|^2}{2} \Big\} = \\ &= \frac{\partial}{\partial x} \Big\{ \frac{x^2}{2 \cdot t} - \frac{1}{t} \cdot \Big(\frac{|y|^2}{2} + t \cdot h(y) \Big)^*(x) \Big\} = \\ &= \frac{x}{t} - \frac{1}{t} \cdot \frac{\partial}{\partial x} \Big(\frac{y^2}{2} + t \cdot h(y) \Big)^*(x). \end{aligned}$$

Thus we get the solution of the initial-value problem for Burger's equation (2.25) in case of $g \in L^\infty(\mathbb{R})$ as follows

$$u(x,t) = \frac{x}{t} - \frac{1}{t} \cdot \frac{\partial}{\partial x} \Big(\frac{y^2}{2} + t \cdot h(y) \Big)^*(x), \quad t > 0, \quad x \in \mathbb{R}, \tag{2.26}$$

where

$$h(y) = \int_0^y g(z)\, dz. \tag{2.27}$$

We discuss next the Monge–Ampere equation. The Monge-Ampere equation is a fully nonlinear elliptic PDE. Applications of the Monge-Ampere equation appear in the classical problem of prescribed Gauss curvature and in the problem of optimal mass transportation (with quadratic cost).

We shall present a simple (nine point stencil) finite difference method (see Benamou, Froese and Oberman [2]) which performs well for smooth as well as for singular solutions. The Monge-Ampere PDE in a planar domain $D \subset \mathbb{R}^2$ is the following

$$\det\big(\text{Hessian}\, u(x)\big) = f(x), \quad f(x) \geq 0, \tag{2.28}$$

or equivalently

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} \cdot \dfrac{\partial^2 u}{\partial y^2} - \Big(\dfrac{\partial^2 u}{\partial x \partial y}\Big)^2 = f \;\text{ in }\; D \subset \mathbb{R}^2, \\ \text{with Dirichlet boundary conditions } \; u = g \;\text{ on } \partial D \end{cases} \tag{2.29}$$

and the additional convexity constraint

$$u(x,y) \;\text{ is convex in }\; D, \tag{2.30}$$

which is required for the equation to be elliptic. Without the convexity constraint this equation does not have a unique solution. For example, taking the boundary function $g = 0$, if u is a solution, then $-u$ is also a solution.

The numerical method involves simply discretizing the second derivatives using standard central differences on a uniform Cartesian grid. The result is

$$(D^2_{xx} u_{ij}) \cdot (D^2_{yy} u_{ij}) - (D^2_{xy} u_{ij})^2 = f_{ij}, \tag{2.31}$$

where

$$\begin{cases} D^2_{xx} u_{ij} = \dfrac{u_{i+1,j} + u_{i-1,j} - 2u_{ij}}{h^2}, \\ D^2_{yy} u_{ij} = \dfrac{u_{i,j+1} + u_{i,j-1} - 2u_{ij}}{h^2}, \\ D^2_{xy} u_{ij} = \dfrac{u_{i+1,j+1} + u_{i,j-1} - u_{i-1,j+1} - u_{i+1,j-1}}{4h^2}. \end{cases} \tag{2.32}$$

Introduce the notation

$$\begin{cases} a_1 = \dfrac{u_{i+1,j} + u_{i-1,j}}{2} & a_2 = \dfrac{u_{i,j+1} + u_{i,j-1}}{2}, \\ a_3 = \dfrac{u_{i+1,j+1} + u_{i-1,j-1}}{2} & a_4 = \dfrac{u_{i-1,j+1} + u_{i+1,j-1}}{2} \end{cases} \tag{2.33}$$

and rewrite (2.31) as a quadratic equation for u_{ij}:

$$4(a_1 - u_{ij})(a_2 - u_{ij}) - \frac{1}{4}(a_3 - a_4)^2 = h^4 f_{ij}. \tag{2.34}$$

Now solving for u_{ij} and selecting the smaller one (in order to select the locally convex solution), we obtain

$$u_{ij} = \frac{1}{2}(a_1 + a_2) - \frac{1}{2}\sqrt{(a_1 - a_2)^2 + \frac{1}{4}(a_3 - a_4)^2 + h^4 f_{ij}}\,. \tag{2.35}$$

We can now use Gauss-Seidel iteration to find the fixed point of (2.35).

The Dirichlet boundary conditions are enforced at boundary grid points. The convexity constraint (2.30) is not enforced (beyond the selection of the positive root in (2.35)).

3 Quantitative Estimates for the Gradients of the Semiconvex Functions

Consider a sequence $u_n(x)$, $x \in \mathbb{R}^n$, of differentiable convex functions, defined on an open convex domain D of the Euclidean $\mathbb{R}^n$ space which converges pointwise to the differentiable convex function $u(x)$, $x \in D$. It is a classical result (see Rockaffellar [7, Theorem 25.7, Section 25]) that on any compact subset K of D the sequence of gradients $\operatorname{grad} u_n(x)$ converges uniformly to the gradient $\operatorname{grad} u(x)$ of the limit function $u(x)$. But until recently there has been no quantitative estimate of the latter convergence in terms of the convergence of the initial convex functions $u_n(x)$ to its limit function $u(x)$. The two authors of this paper for one-dimensional case in 2005 [10] and for multidimensional case in 2014 [11] succecded to establish such an estimate and this type of estimates are the theoretical basis for the numerical approximation of the gradients of the solutions of nonlinear differential equations considered in this article.

We recall below these estimates for the multidimensional case [11]. Let D be a bounded open convex subset of Euclidean space $\mathbb{R}^n$ and $d_{\partial D}(x)$ be the distance function from an arbitrary point $x \in D$ to its boundary ∂D.

Theorem 3.1 *Let u and v be two bounded convex functions in D. Then the following weighted energy inequality is valid for the difference $u - v$*

$$\int\limits_D |\operatorname{grad} u - \operatorname{grad} v|^2 \cdot \frac{d_{\partial D}^2}{n}\, dx \le$$
$$\le 5 \cdot \operatorname{meas} D \cdot \|u - v\|_{L^\infty(D)} \cdot \big(\|u\|_{L^\infty(D)} + \|v\|_{L^\infty(D)}\big). \quad (3.1)$$

Proposition 3.1. *Let u and v be two bounded semiconcave functions in D with the semiconcavity constants c_u and c_v, respectively. Then the following energy inequality holds for the difference $u-v$ of two semiconcave functions*

$$\int\limits_D |\operatorname{grad} u - \operatorname{grad} v|^2 \cdot \frac{d_{\partial D}^2}{n}\, dx \le 5 \cdot \operatorname{meas} D \cdot \|u - v\|_{L^\infty(D)} \times$$
$$\times \Big(\|u\|_{L^\infty(D)} + \|v\|_{L^\infty(D)} + 2 \max(c_u, c_v) \cdot \|v_0\|_{L^\infty(D)}\Big), \quad (3.2)$$

where

$$v_0(x) = \frac{1}{2} \cdot |x|^2. \quad (3.3)$$

Next we remind that the convex envelope of a bounded continuous function u in D is defined as the supremum of all convex functions which are majorized by the function u:

$$\Gamma(u) = \sup\Big\{v(x) : \ v(x) \text{ convex in } D,\ v(x) \le u(x) \text{ for all } x \in D\Big\}. \quad (3.4)$$

Proposition 3.2. *On the space $C(D) \cap L^\infty(D)$ the mapping $u \to \Gamma(u)$ possesses the following important property*

$$\int\limits_D \big|\operatorname{grad} \Gamma(u) - \operatorname{grad} \Gamma(v)\big|^2 \cdot \frac{d_{\partial D}^2}{n}\, dx \le$$
$$\le 5 \cdot \operatorname{meas} D \cdot \|u - v\|_{L^\infty(D)} \cdot \big(\|u\|_{L^\infty(D)} + \|v\|_{L^\infty(D)}\big). \quad (3.5)$$

The consequence of the Theorem 3.1 is that if we have a sequence of arbitrary convex functions $u_n(x)$ (not necessarily differentiable) which converges pointwise to the limit convex function $u(x)$ on the open convex set D, then on the arbitrary compact subset K of D the sequence of gradients $\operatorname{grad} u_n(x)$ converges in $L^2(K)$ to the $\operatorname{grad} u(x)$.

Now we return to the Hamilton–Jacobi equation with quadratic Hamiltonian (2.24) and to the Burger's equation (2.25) (which is the particular case of scalar conservation laws). Observe that the solution of the first one (2.23) is the Legendre–Fenchel transform of the given function and the solution of the second one (2.26) is the derivative of the Legendre–Fenchel transform and by the definition the Legendre–Fenchel transform of any function (not necessarily convex) is inevitably a convex function. Therefore applying any algorithm of the computational convex analysis to compute the Legendre–Fenchel transform which preserves the convexity property we conclude that we automatically get the convergence in L^2 of the computed numerical gradients to the gradient of the solution of the Hamilton–Jacobi equation (2.24). In a similar manner we obtain L^2-approximation of the solution of Burger;s equation (2.25) via the formula (2.26).

Next we consider the Monge–Ampere PDE in a planar domain (2.29). Numerical experiments show that the finite difference method (2.31)–(2.35) performs well even for singular solutions, though the numerical solution does not preserve the convexity property. Nevertheless, provided that the finite difference numerical solution gives the uniform approximation in the domain D to the unique convex solution of the planar Monge–Ampere PDE (2.29), we next apply our basic idea: we construct the convex envelope of the computed numerical solution by the "Quickhull algorithm for Convex Hulls" (Barber, Dobkin and Huhdanpaa [1]) and we replace the computed numerical solution by its convex envelope. The first advantage is that the obtained numerical approximation is a convex function and hence it better imitates the exact convex solution then the previous one and the second essential advantage comes from our L^2-estimate (3.5), which tells us that the gradient of the computed convex envelope approximates in L^2 the gradient of the exact solution (we note that the exact solution u of the Monge–Ampere PDE (2.29)–(2.30) is convex and hence $\Gamma(u) = u$).

4 Examples

In this section we consider several particular examples of the above mentioned nonlinear PDE and solve them numerically. We shall compare the numerical solutions with the exact solutions and calculate the errors of approximation.

For Hamilton–Jacobi equation (2.24) we consider two particular initial conditions

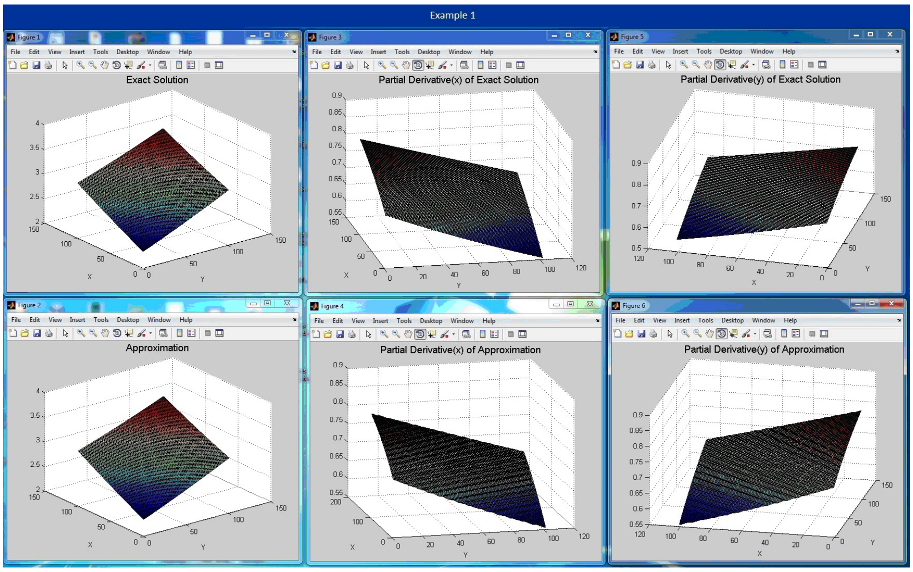

Figure 1.

Example 1.

$$g(x) = |x|, \quad x \in \mathbb{R}^2. \tag{4.1}$$

Example 2.

$$g(x) = -|x|, \quad x \in \mathbb{R}^2. \tag{4.2}$$

The explicit solutions are known (see Evans [5, Section 3.3, pp. 135–136]):

in case of $g(x) = |x|$ we have

$$u(x,t) = \begin{cases} |x| - \dfrac{t}{2}\,, & \text{if } \ |x| \geq t, \\ \dfrac{|x|^2}{2 \cdot t}\,, & \text{if } \ |x| \leq t, \quad t > 0, \end{cases} \tag{4.3}$$

and in case of $g(x) = -|x|$ we have

$$u(x,t) = -|x| - \frac{t}{2}\,, \quad t \geq 0. \tag{4.4}$$

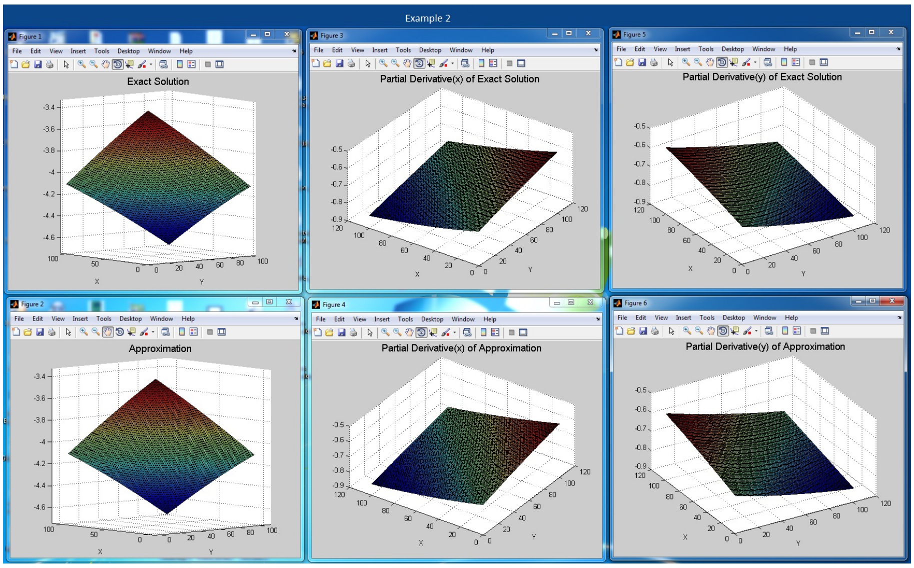

Figure 2.

For Burger's equation (2.25) we shall consider the following initial conditions

Example 3.

$$g(x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 - x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x \geq 1. \end{cases} \tag{4.5}$$

Example 4.

$$g(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x > 1. \end{cases} \tag{4.6}$$

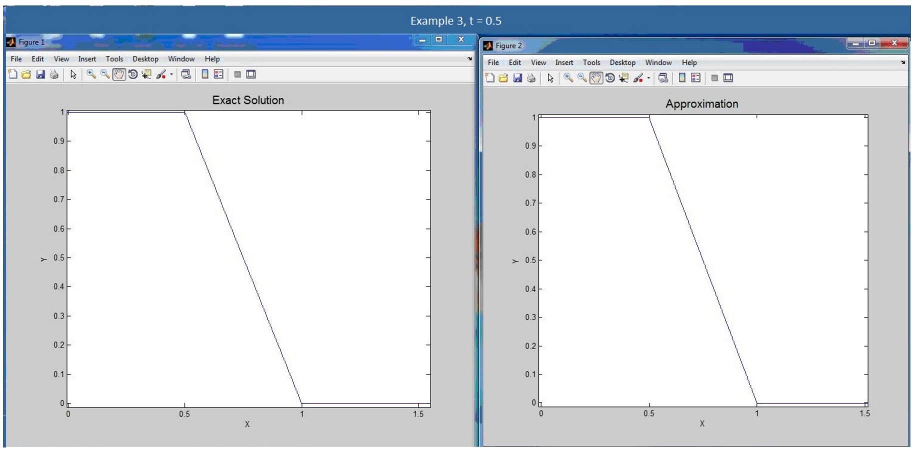

Figure 3.

A curve of the discontinuity $x = s(t)$ for the solution $u(x,t)$ of the Burgers' equation is called a shock curve (shock wave). The explicit solutions for these examples are known as well (see Evans [5, Section 3.4, pp.140–143]):

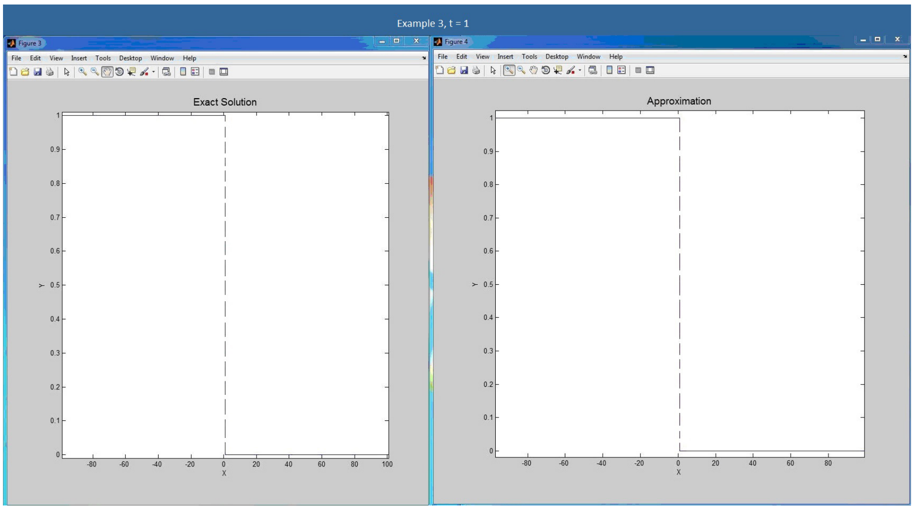

Figure 4.

In case of the initial condition in Example 3 we have

$$\begin{cases} u(x,t) = \begin{cases} 1, & \text{if } x \le t, \\ \dfrac{1-x}{1-t}, & \text{if } t \le x \le 1, \\ 0, & \text{if } x \ge 1, \end{cases} & 0 \le t < 1, \\ u(x,t) = \begin{cases} 1, & \text{if } x < \dfrac{1+t}{2}, \\ 0, & \text{if } x > \dfrac{1+t}{2}, \end{cases} & t \ge 1, \end{cases} \tag{4.7}$$

$$S(t) = \frac{1+t}{2} \text{ is a shock curve for } t \ge 1, \tag{4.8}$$

and in case of the Example 4 we have

$$\begin{cases} u(x,t) = \begin{cases} 0, & \text{if } \ x < 0, \\ \dfrac{x}{t}\,, & \text{if } \ 0 < x < t, \\ 1, & \text{if } \ t < x < 1 + \dfrac{t}{2}\,, \\ 0, & \text{if } \ x > 1 + \dfrac{t}{2}\,, \end{cases} & 0 < t < 2, \\ u(x,t) = \begin{cases} 0, & \text{if } \ x < 0, \\ \dfrac{x}{t}\,, & \text{if } \ 0 < x < (2 \cdot t)^{1/2}, \\ 0, & \text{if } \ x > (2 \cdot t)^{1/2}, \end{cases} & t \geq 2. \end{cases} \tag{4.9}$$

$$S(t) = (2 \cdot t)^{1/2} \ \text{ is a shock curve for } \ t \geq 2. \tag{4.10}$$

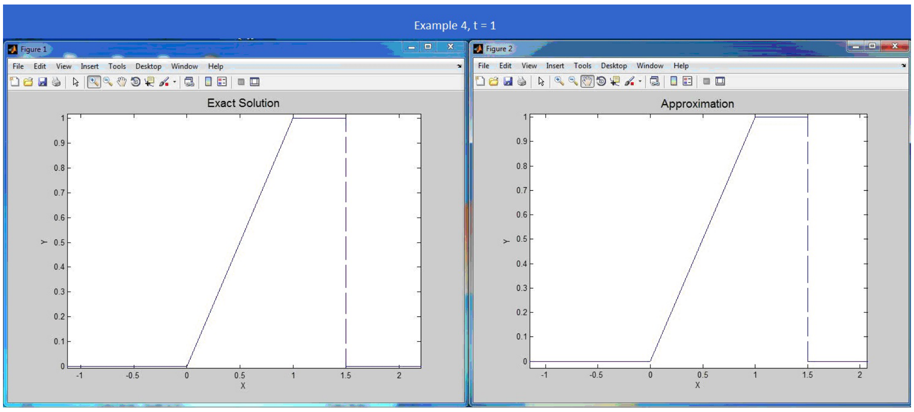

Figure 5.

Next we consider two exact solutions for the Monge–Ampere PDE (2.29)–(2.30) on the square $[0,1] \times [0,1]$.

Example 5.

$$\begin{cases} u(x,y) = \exp\Big(\dfrac{x^2 + y^2}{2}\Big), \\ f(x,y) = (1 + x^2 + y^2) \cdot \exp(x^2 + y^2). \end{cases} \tag{4.11}$$

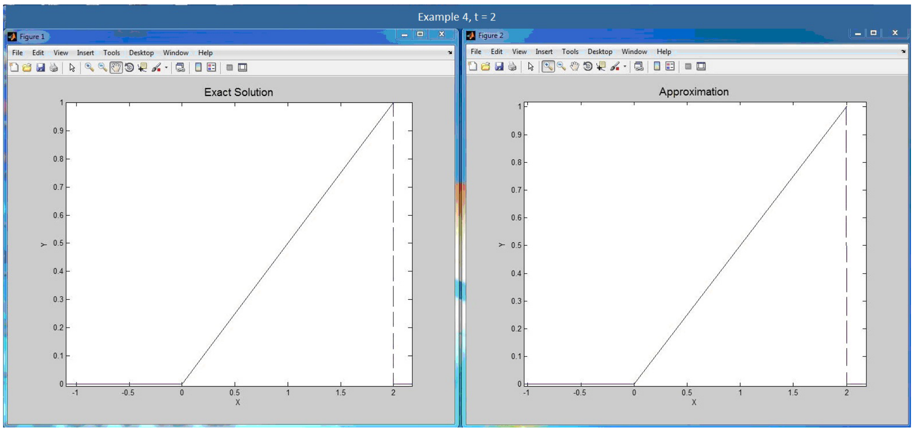

Figure 6.

Example 6.

$$\begin{cases} u(x, y) = \dfrac{2 \cdot \sqrt{2}}{3} \cdot (x^2 + y^2)^{3/4}, \\ f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}. \end{cases} \tag{4.12}$$

In this example the function f blows up at the boundary point $(0, 0)$.

We note that we use fast algorithm to accelerate computations in the finite difference method (2.31)–(2.35). For the Hamilton–Jacobi equation (Examples 1 and 2 above) we consider the numerical solutions $u(x, t)$ at the time instant $t = 1$ and the spatial argument x restricted to the square area $(2, 3) \times (2, 3)$.

Errors for the exact solution and its gradient for Example 1 on an $N \times N$ grid:

N	uniform error for the exact solution	uniform error for the exact gradient	L^2-error for the exact gradient
1001	3×10^{-7}	0.004	0.0012

Errors for the exact solution and its gradient for Example 2 on an $N \times N$ grid:

N	uniform error for the exact solution	uniform error for the exact gradient	L^2-error for the exact gradient
1001	2×10^{-7}	0.004	0.0011

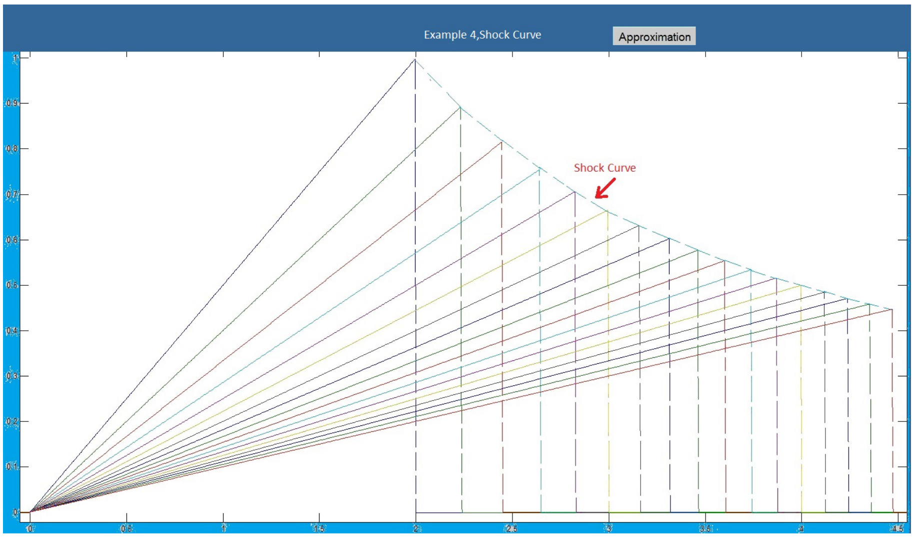

Figure 7.

Convergence results for the Burgers' equation (Examples 3 and 4) for the times $t = 0.5$, $t = 1$ and $t = 2$ and the argument x restricted to the interval $(-100, +100)$.

L^2-errors for the exact solution for Example 3:

number of grid points N	time argument t	L^2-error for the exact gradient
2000001	0.5	0.0007
2000001	1	0.0007

L^2-errors for the exact solution for Example 4:

number of grid points N	time argument t	L^2-error for the exact solution
2000001	1	0.0007
2000001	2	0.0003

Below the numerical solution for the Example 4 at times $t = 2, 2.5, 3, 3.5, 4, \ldots, 10$ are plot on the same figure. Observe the movement of the shock curve on this figure

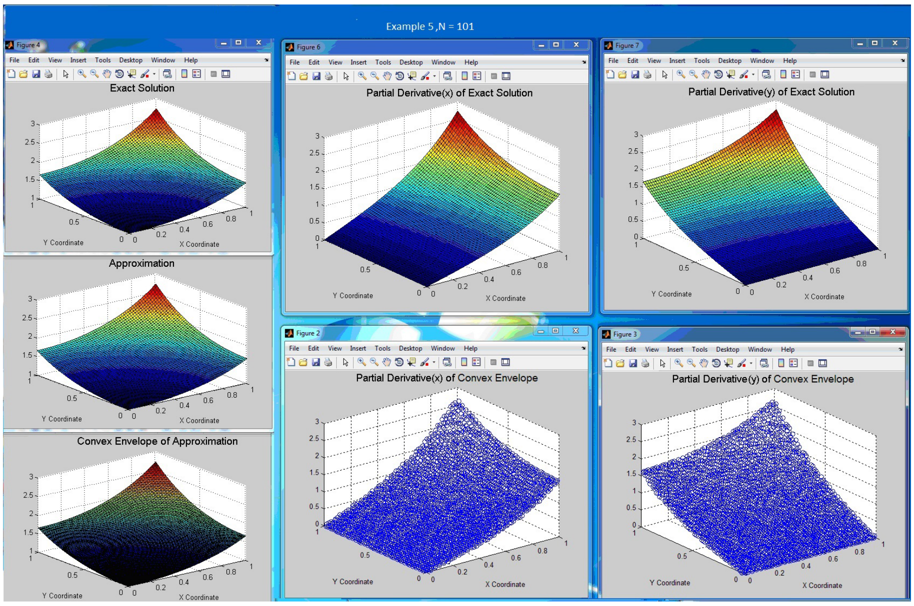

Figure 8.

The Monge–Ampere equations (the Examples 5 and 6) are considered on the square $[0, 1] \times [0, 1]$.

In the tables below for the different grid points we compute the number of iterations, the computation times, the errors of approximation of the exact solution and of the exact gradient.

Computation times and errors for the exact solution and its gradient for the Example 5 on an $N \times N$ grid:

N	Number of iterations	Computation times	uniform error for the exact solution	uniform error for the exact gradient	L^2-error for the exact gradient
21	1362	1 sec.	1.5×10^{-4}	0.1255	0.011
61	10840	10 sec.	1.8×10^{-5}	0.0441	0.0038
101	28764	60 sec.	6.7×10^{-6}	0.0267	0.0023
141	54802	300 sec.	3.4×10^{-6}	0.0192	0.0016

Computation times and errors for the exact solution and its gradient for the Example 6 on an $N \times N$ grid:

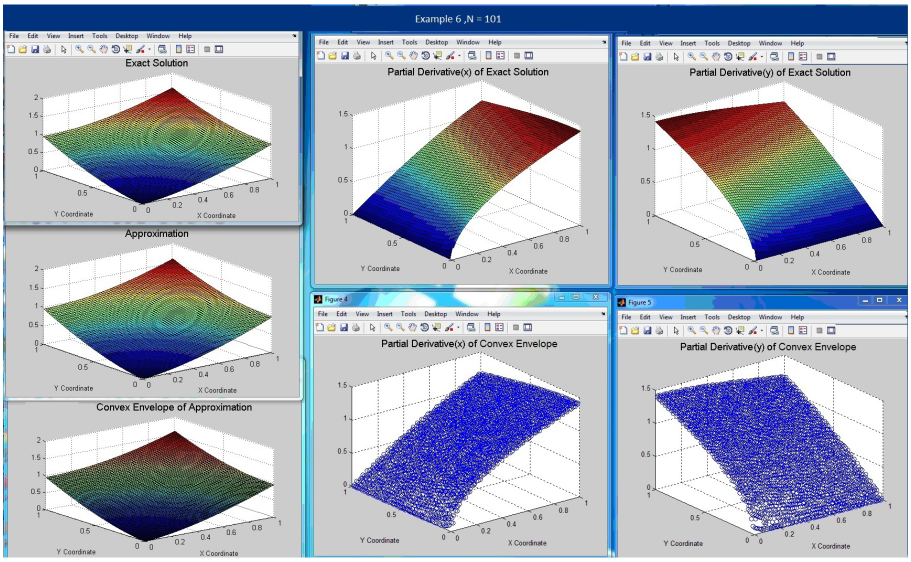

Figure 9.

N	Number of iterations	Computation times	uniform error for the exact solution	uniform error for the exact gradient	L^2-error for the exact gradient
21	1397	1 sec.	5×10^{-4}	0.1511	0.0077
61	11065	10 sec.	1×10^{-4}	0.0887	0.0027
101	29312	70 sec.	4.9×10^{-5}	0.0689	0.0016
141	55768	300 sec.	2.9×10^{-5}	0.0583	0.0011

We give the surface plots (for Examples 5 and 6) of the following functions:

(a) the exact solution,

(b) finite difference numerical approximation,

(c) the convex envelope of the numerical approximation,

(d) partial derivative w.r. to x of the exact solution,

(e) partial derivative w.r. to y of the exact solution,

(f) partial derivative w.r. to x of the convex envelope,

(g) partial derivative w.r. to y of the convex envelope.

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