ANALYTICAL SOLUTION OF CLASSICAL AND NON-CLASSICAL BOUNDARY VALUE CONTACT PROBLEMS OF THERMOELASTICITY FOR CYLINDRICAL BODIES CONSISTING OF COMPRESSIBLE AND INCOMPRESSIBLE ELASTIC LAYERS

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Abstract

Static thermoelastic equilibrium is considered for an N-layer along the radial coordinate body bounded by coordinate surfaces of a circular cylindrical system of coordinates. Each layer is isotropic and homogeneous and some of the layers may be composed of an incompressible elastic material. On the flat boundaries of the cylindrical body boundary conditions of either symmetrical or anti-symmetrical continuous extension of the solution are imposed. Between the layers contact conditions of rigid, sliding or other type of contact may be defined. The stated problems are solved using the method of separation of variables,the general solution being represented by means of harmonic functions. The solution of the problems is reduced to the solution of systems of algebraic equations with block diagonal matrices. At the end of the paper an application example is given which illustrates the applied approach for an analytical solution of problems.

Key words and phrases: Thermoelastic equilibrium of multilayer cylindrical body. Method of separation variables. Classical and non-classical boundary value contact problems.

AMS subject classification: 74B05, 74F05.

1 Introduction

The paper deals with a solution of some application problems of thermoelasticity for a multilayer cylindrical body and was supported by National Scientific Foundation for applied studies AR/91/5-109/11 (Agreement N 10/17). There are a number of papers [1-4] devoted to the solution of three-dimensional problems of elastic equilibrium of cylindrical bodies. Reviews of related studies are given in monographs [1], [2]. In contrast to the above-mentioned works we study a whole class of boundary value and boundary value contact problems of thermoelasticity for multilayer cylindrical bodies bodies the vast majority of which have been solved for the first time [5-9]. It should be noted that some layers may consist of an incompressible elastic material. Analytical solutions of the studied problems are obtained by means of the method of separation of variables. A computer program is also given that enables one to perform numerical procedures and give a visual representation of the solutions.

For an N-layer along the radial coordinate body bounded by coordinate surfaces of a circular cylindrical system of coordinates static thermoelastic equilibrium [5],[6] is considered. On the flat boundaries of the cylindrical body symmetry or anti-symmetry conditions of a continuous extension of the solution are imposed [8]. Contact conditions of a rigid, sliding or other type of contact may be imposed between the layers. Arbitrary boundary conditions are defined on cylindrical boundary surfaces. The problems are solved analytically using the method of separation of variables. The general solution is expressed by means of harmonic functions. Problem solution is reduced to the solution of systems of linear algebraic equations with block diagonal matrices.

2 Equilibrium equations. Statement of the problem

Consider a multilayer cylindrical body which in the circular cylindrical system of coordinates $r,\ \alpha,\ z$ occupies the domain $\Omega = \Omega_1 \bigcup \Omega_2 \bigcup \cdots \bigcup \Omega_\mathrm{k} \bigcup \cdots \bigcup \Omega_N$, $\Omega_k = \{r_{k-1} < r < r_k,\ 0 < \alpha < \alpha_1,\ 0 < z < z_1\}$, $k = \overline{1, N}$, where r_j, α_1, z_1, $j = \overline{0, N}$ are positive constants. Hence the domain Ω represents a cylindrical body which consists of N layers (see Fig. 1). This multilayer elastic body is called the curvilinear coordinate parallelepiped. It is assumed in the paper that $N = \overline{1,\ 12}$, i.e. we can have a one-layer, two-layer, etc. twelve-layer body. Note that twelve - the maximal number of layers- is taken in order to make software definite. But if the number of layers exceeds twelve the program can be easily adjusted.

Assume that every domain Ω_k is filled by an isotropic homogeneous elastic material. As we know, homogeneous equilibrium equations have the following form [5] :

$$
\begin{aligned}
& r\partial_r\sigma_{rr}^{(k)} + \partial_\alpha\sigma_{r\alpha}^{(k)} + r\partial_z\sigma_{rz}^{(k)} + \sigma_{rr}^{(k)} - \sigma_{\alpha\alpha}^{(k)} = 0, \\
& \partial_\alpha\sigma_{\alpha\alpha}^{(k)} + r\partial_z\sigma_{\alpha z}^{(k)} + \frac{1}{r}\partial_r(r^2\sigma_{\alpha r}^{(k)}) = 0, \qquad (1) \\
& r\partial_z\sigma_{zz}^{(k)} + \partial_r(r\sigma_{zr}^{(k)}) + \partial_\alpha\sigma_{z\alpha}^{(k)} = 0,
\end{aligned}
$$

where $\partial_r \equiv \frac{\partial}{\partial r}$, $\partial_\alpha \equiv \frac{\partial}{\partial \alpha}$, $\partial_z \equiv \frac{\partial}{\partial z}$; $\sigma_{rr}^{(k)}$, $\sigma_{\alpha\alpha}^{(k)}$, $\sigma_{zz}^{(k)}$are normal stresses,

$\sigma_{\alpha r}^{(k)} = \sigma_{r\alpha}^{(k)}$, $\sigma_{zr}^{(k)} = \sigma_{rz}^{(k)}$, $\sigma_{\alpha z}^{(k)} = \sigma_{z\alpha}^{(k)}$ are tangential stresses. The Duhamel-Neumann relations are expressed as [10]

$$\sigma_{rr}^{(k)} = \frac{E^{(k)}}{1+\nu^{(k)}} \left\{ \frac{1-\nu^{(k)}}{1-2\nu^{(k)}} \frac{1}{r} \left[\partial_r \left(r u^{(k)} \right) + \partial_\alpha v^{(k)} \right] - \right.$$
$$\left. -\frac{1}{r} \left(\partial_\alpha v^{(k)} + u^{(k)} \right) + \frac{\nu^{(k)}}{1-2\nu^{(k)}} \partial_z w^{(k)} \right\} - \beta^{(k)} T^{(k)},$$

$$\sigma_{\alpha\alpha}^{(k)} = \frac{E^{(k)}}{1+\nu^{(k)}} \left\{ \frac{1-\nu^{(k)}}{1-2\nu^{(k)}} \frac{1}{r} \left[\partial_r \left(r u^{(k)} \right) + \partial_\alpha v^{(k)} \right] - \right.$$
$$\left. -\frac{1}{r} \partial_r u^{(k)} + \frac{\nu^{(k)}}{1-2\nu^{(k)}} \partial_z w^{(k)} \right\} - \beta^{(k)} T^{(k)},$$

$$\sigma_{zz}^{(k)} = \frac{E^{(k)}}{1+\nu^{(k)}} \left\{ \frac{\nu^{(k)}}{1-2\nu^{(k)}} \frac{1}{r} \left[\partial_r \left(r u^{(k)} \right) + \partial_\alpha v^{(k)} \right] + \right. \tag{2}$$
$$\left. + \frac{1-\nu^{(k)}}{1-2\nu^{(k)}} \partial_z w^{(k)} \right\} - \beta^{(k)} T^{(k)},$$

$$\sigma_{rz}^{(k)} = \frac{E^{(k)}}{2(1+\nu^{(k)})} \left[\partial_z u^{(k)} + \partial_r w^{(k)} \right],$$

$$\sigma_{z\alpha}^{(k)} = \frac{E^{(k)}}{2(1+\nu^{(k)})} \left[\frac{1}{r} \partial_\alpha w^{(k)} + \partial_z v^{(k)} \right],$$

$$\sigma_{r\alpha}^{(k)} = \frac{E^{(k)}}{2(1+\nu^{(k)})} \left[r \partial_r \left(\frac{v^{(k)}}{r} \right) + \frac{1}{r} \partial_\alpha u^{(k)} \right],$$

where $\nu^{(k)}$ is Poisson's ratio for the k-th layer, $E^{(k)}$is Young's modulus for the k-th layer; $\vec{U}^{(k)} = \left(u^{(k)},\ v^{(k)},\ w^{(k)} \right)$ is the displacement vector for the k–th layer; $\beta^{(k)}$is a coefficient depending on the thermal characteristics of the k–th layer; $T^{(k)}$ is the temperature change in the k–th layer satisfying Laplace's equation

$$\left(\frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial_{\alpha\alpha} + \partial_{zz} \right) T^{(k)} = 0. \tag{3}$$

We also consider cases when some of the layers of the body under study are made of incompressible thermoelastic materials. In this case the incompressible layers are also assumed to be isotropic and homogeneous. Material incompressibility implies a property such that under any strain and at any point the volume remains unchanged. Besides, studying the static thermoelastic equilibrium, we assume that the change in the temperatures in the incompressible layers, similar to the classical case, satisfies Laplace's equation (3).

Let the j–th layer of the curvilinear coordinate parallelepiped under study consist of an incompressible thermoelastic material. Then equilibrium equations written in the invariant form will be the following [11]

$$\begin{cases} \text{grad } \left(s^{(j)} + 4\nu^{(j)}T^{(j)}\right) - \text{rot rot } \overrightarrow{U}^{(j)} = 0, \\ \text{div } \overrightarrow{U}^{(j)} = 3\nu^{(j)}T^{(j)}, \end{cases} \quad \text{in } \Omega_j, \tag{4}$$

where $s^{(j)}$is the so called hydrostatic pressure of the j–th layer.

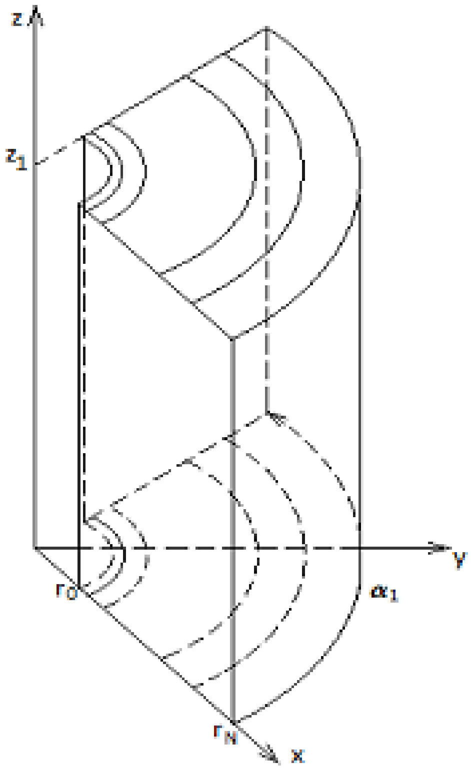

Fig.1. Multilayer cylindrical body

On the flat boundaries of each layer the following homogeneous boundary conditions are defined:

$$
\text{For } \alpha = \alpha_j : \quad a)\ \ \sigma_{\alpha\alpha}^{(k)} = 0,\ \ u^{(k)} = 0,\ \ w^{(k)} = 0,\ \ T^{(k)} = 0
$$
or
$$
b)\ \ v^{(k)} = 0,\ \ \sigma_{\alpha r}^{(k)} = 0,\ \ = 0,\ \ \sigma_{\alpha z}^{(k)} = 0,\ \ \partial_\alpha T^{(k)} = 0\,, \qquad (5)
$$
$$
r_{k-1} < r < r_k, \quad k = \overline{1,\, N}, \quad 0 < z < z_1;
$$

$$
\text{For } z = z_j : \quad a)\ \ \sigma_{zz}^{(k)} = 0,\ \ u^{(k)} = 0,\ \ v^{(k)} = 0,\ \ T^{(k)} = 0
$$
or
$$
b)\ \ w^{(k)} = 0,\ \ \sigma_{zr}^{(k)} = 0,\ \ \sigma_{z\alpha}^{(k)} = 0,\ \ \partial_z T^{(k)} = 0\,, \qquad (6)
$$
$$
r_{k-1} < r < r_k, k = \overline{1,\, N}, 0 < \alpha < \alpha_1\,,
$$

where $j = 0,\ 1$, and $\alpha_0 = 0,\ z_0 = 0$.

Note that boundary conditions (5a) and (6a) are conditions of an anti-symmetric continuous extension of the solution while boundary conditions (5b) and (6b) are conditions of a symmetric continuous extension of the solution [8].

Thus on the flat boundaries of the domain Ω we will have nine different combinations of symmetry and anti-symmetry conditions (see Fig. 2).

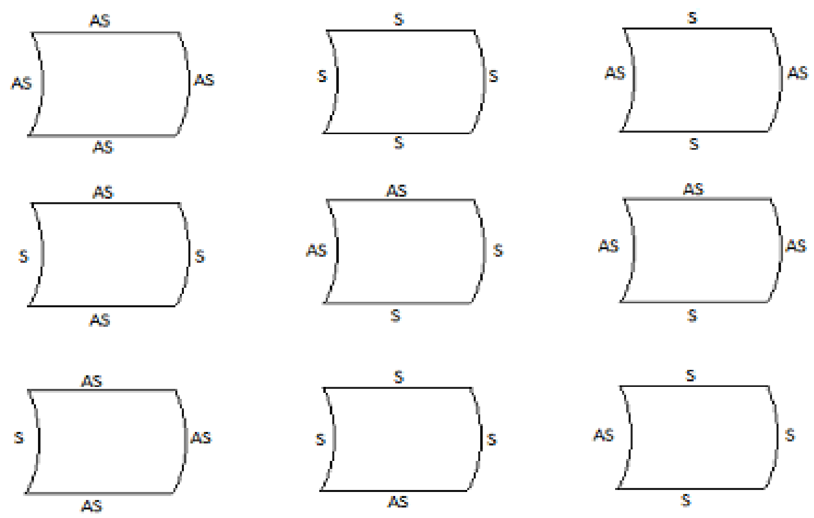

Fig. 2. Boundary conditions on flat boundaries

The following boundary conditions are defined on cylindrical boundary surfaces.

$$\text{For } r = r_{\delta_j}: \quad \begin{aligned} &\sigma_{rr}^{(j)} = F_1^{(j)}(\alpha,\ z) \quad \text{or} \quad u^{(j)} = f_1^{(j)}(\alpha,\ z), \\ &\sigma_{r\alpha}^{(j)} = F_2^{(j)}(\alpha,\ z) \quad \text{or} \quad v^{(j)} = f_2^{(j)}(\alpha,\ z), \\ &\sigma_{rz}^{(j)} = F_3^{(j)}(\alpha,\ z) \quad \text{or} \quad w^{(j)} = f_3^{(j)}(\alpha,\ z), \\ &\partial_r T^{(j)} = t^{(j)}(\alpha,\ z) \quad \text{or} \quad T^{(j)} = \tau^{(j)}(\alpha,\ z), \end{aligned} \tag{7}$$

where $j = 1,\ N,\quad \delta_j = \begin{cases} 0, & j = 1, \\ N, & j = N, \end{cases}$ and $r_0 = 0$; $F_i^{(j)}(\alpha,\ z)$, $f_i^{(j)}(\alpha,\ z)$, $t^{(j)}(\alpha,\ z)$ and $\tau^{(j)}(\alpha,\ z)$, $i = 1, 2, 3$ are defined functions which in the domain $\omega = \{0 < \alpha < \alpha_1,\ 0 < z < z_1\}$ are decomposed into uniformly converging Fourier series.

Hence arbitrary boundary conditions may be defined on the cylindrical boundary surfaces of the domain Ω.

As for contact conditions, the article shows cases when contact conditions of rigid or sliding contact may be defined between neighboring layers.

Rigid contact conditions have the following form:

$$r = r_j: \quad \begin{aligned} &u^{(j)} = u^{(j+1)}, \quad v^{(j)} = v^{(j+1)}, \quad w^{(j)} = w^{(j+1)}; \\ &\sigma_{r\alpha}^{(j)} = \sigma_{r\alpha}^{(j+1)}, \quad \sigma_{rz}^{(j)} = \sigma_{rz}^{(j+1)}, \quad \sigma_{rr}^{(j)} = \sigma_{rr}^{(j+1)}; \\ &T^{(j)} = T^{(j+1)}, \quad \mathrm{k}^{(\mathrm{j})}\partial_{\mathrm{r}} T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)}, \\ &\qquad j = \overline{1, N-1}. \end{aligned} \tag{8}$$

Sliding contact conditions have the following form:

$$r = r_j: \quad \begin{aligned} &\sigma_{rr}^{(j)} = \sigma_{rr}^{(j+1)}, \quad u^{(j)} = u^{(j+1)}, \quad \sigma_{r\alpha}^{(j)} = 0, \\ &\sigma_{r\alpha}^{(j+1)} = 0, \quad \sigma_{rz}^{(j)} = 0, \quad \sigma_{rz}^{(j+1)} = 0; \\ &T^{(j)} = T^{(j+1)}, \quad \mathrm{k}^{(\mathrm{j})}\partial_{\mathrm{r}} T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)}, \\ &\qquad j = \overline{1, N-1}. \end{aligned} \tag{9}$$

In formulas (8) and (9) $k^{(j)}$is thermal conductivity of the j-th layer.

Thus between layers $\Omega_j$and Ω_{j+1} either (8) type conditions (rigid contact) can be defined or (9) type contact conditions (sliding contact). Although the paper deals with (8) or (9) type contact conditions, we can analytically solve boundary value contact problems in the case when other types of conditions are defined between the neighboring layers, e.g. we can consider contact conditions of the following form:

$$r = r_j : \quad w^{(j)} = w^{(j+1)}, \quad \sigma_{rz}^{(j)} = \sigma_{rz}^{(j+1)}, \quad \sigma_{r\alpha}^{(j)} = 0,$$

$$\sigma_{r\alpha}^{(j+1)} = 0, \quad \sigma_{rr}^{(j)} = 0, \quad \sigma_{rr}^{(j+1)} = 0;$$

$$T^{(j)} = T^{(j+1)}, \quad \mathrm{k}^{(\mathrm{j})}\partial_{\mathrm{r}} T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)}$$
$$j = \overline{1, N-1}.$$

3 Solution of the stated boundary value contact problems

It is proved that the general solution of equation system (1), (2) can be represented by means of four harmonic functions [5]. Just for simplicity we write out general solutions expressed by means of four harmonic functions for the case when on one of the flat boundaries $\alpha = const$ anti-symmetry conditions are defined while on the other – those of symmetry. For the general case when symmetry and anti-symmetry conditions are arbitrarily mixed up the solutions are given in paper [5]. The above-mentioned refers to boundary conditions on flat boundaries. As for boundary conditions on cylindrical surfaces they are arbitrary in all of the above-mentioned cases.

Components of the displacement vector are expressed by means of the following formulas:

$$\begin{aligned}
2\mu_k u^{(k)} &= \partial_r\left(r\partial_r\Phi^{(k)} + G^{(k)}\right) + \frac{1}{r}\partial_\alpha\Psi^{(k)} - \\
&\quad - \frac{\mu_k - \left(1+\nu^{(k)}\right)}{1-\nu^{(k)}}\gamma^{(k)}\partial_r\left(r\partial_r\tilde{T}^{(k)}\right), \\
2\mu_k v^{(k)} &= \partial_\alpha\left(\partial_r\Phi^{(k)} + \frac{1}{r}G^{(k)}\right) - \partial_r\Psi^{(k)} - \\
&\quad - \frac{\mu_k\left(1+\nu^{(k)}\right)}{1-\nu^{(k)}}\gamma^{(k)}\partial_{r\alpha}\tilde{T}^{(k)}, \\
2\mu_k w^{(k)} &= \partial_z\left(r\partial_r\Phi^{(k)} + G^{(k)}\right) + 4\left(1-\nu^{(k)}\right)\partial_z\Phi^{(k)} - \\
&\quad - \frac{\mu_k\left(1+\nu^{(k)}\right)}{1-\nu^{(k)}}\gamma^{(k)}\partial_z\left(r\partial_r\tilde{T}^{(k)}\right), \quad k = \overline{1, N}.
\end{aligned} \tag{10}$$

where $\Phi^{(k)}$, $\Psi^{(k)}$, $G^{(k)}$ are arbitrary harmonic functions in the domain Ω_k; $\gamma^{(k)}$is a linear heat expansion coefficient of the k-th layer, which is expressed through $\beta^{(k)}$ by means of the formula

$$\gamma^{(k)} = \frac{1-2\nu^{(k)}}{2\mu_k\left(1+\nu^{(k)}\right)}\beta^{(k)};$$

$\tilde{T}^{(k)}$ is also a harmonic function, which is related to the function $T^{(k)}$ by means of

$$T^{(k)} = \partial_{zz}\tilde{T}^{(k)}.$$

If we substitute formulas (10) in Duhamel-Neumann relations (2), then we shall have stresses expressed by means of the introduced harmonic functions. In particular, for the stresses $\sigma_{rr}^{(k)}$, $\sigma_{r\alpha}^{(k)}$ and $\sigma_{rz}^{(k)}$ we will have the following formulas:

$$\begin{aligned}
\sigma_{rr}^{(k)} &= \partial_{rr}\left(r\partial_r\Phi^{(k)} + G^{(k)}\right) + 2\nu^{(k)}\partial_{zz}\Phi^{(k)} + \partial_{\alpha r}\left(\frac{1}{r}\Psi^{(k)}\right) - \\
&\quad -\frac{\mu_k\left(1+\nu^{(k)}\right)}{1-\nu^{(k)}}\gamma_k\left[\partial_{zz}\tilde{T}^{(k)} + \partial_{rr}\left(r\partial_r\tilde{T}^{(k)}\right)\right],\\
\sigma_{r\alpha}^{(k)} &= \partial_{r\alpha}\left(\partial_r\Phi^{(k)} + \frac{1}{r}G^{(k)}\right) - \frac{1}{2}\partial_{zz}\Psi^{(k)} - \partial_{rr}\Psi^{(k)} - \\
&\quad -\frac{\mu_k\left(1+\nu^{(k)}\right)}{1-\nu^{(k)}}\gamma_k\partial_{rr\alpha}\tilde{T}^{(k)},\\
\sigma_{rz}^{(k)} &= \partial_{zr}\left(r\partial_r\Phi^{(k)} + G^{(k)}\right) + 2(1-\nu^{(k)})\partial_{zr}\Phi^{(k)} + \frac{1}{2r}\partial_{z\alpha}\Psi^{(k)} - \\
&\quad -\frac{\mu_k\left(1+\nu^{(k)}\right)}{1-\nu^{(k)}}\gamma_k\partial_{zr}\left(r\partial_r\tilde{T}^{(k)}\right).
\end{aligned} \tag{11}$$

It is proved that the general solution of a system of equilibrium equations for an incompressible material (3) can be obtained from formulas (9) and (10), if instead of $\nu^{(k)}$ we take the value $\frac{1}{2}$. As for hydrostatic pressure, its value is defined by formula

$$\mu^{(k)}s^{(k)} = \partial_z\Phi^{(k)} - \mu^{(k)}\nu^{(k)}T^{(k)}.$$

It should be noted that in the case of an incompressible material as well boundary conditions (4)–(6) are imposed. The same can be also said about contact conditions, i.e. in the case when one of the two neighboring layers is incompressible, contact conditions (7) or (8) still hold.

Furthermore, using the method of separation of variables and bearing in mind boundary conditions defined on the flat boundaries of the cylindrical body, we establish the form of harmonic functions $\Phi^{(k)}$, $\Psi^{(k)}$, $G^{(k)}$ and $\tilde{T}^{(k)}$.

In particular, here we consider the following three cases of boundary conditions on flat surfaces of the body under study.

Problem 1. On all of the four flat boundary surfaces anti-symmetry conditions are defined while non-homogeneous conditions on the cylindrical boundary surfaces are symmetrical with respect to the planes $\alpha = \dfrac{\alpha_1}{2}$, $z = \dfrac{z_1}{2}$.

Problem 2. On all of the four flat boundary surfaces symmetry conditions are defined while non-homogeneous conditions on the cylindrical boundary surfaces are antisymmetrical with respect to the planes $\alpha = \dfrac{\alpha_1}{2}$, $z = \dfrac{z_1}{2}$.

Problem 3. On the boundaries $\alpha = 0$ and $z = 0$ anti-symmetry conditions are defined, while on the boundaries $\alpha = \alpha_1$ and $z = z_1$symmetry conditions are given (Fig. 2, 5).

It can be easily seen that taking into account conditions of continuous extension of the solution the first two problems can be reduced to the third one. In the case of Problem 3 the functions $\Phi^{(k)}$, $\Psi^{(k)}$, $G^{(k)}$ and $\tilde{T}^{(k)}$ have the following form:

$$
\begin{aligned}
\Phi^{(k)} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}^{(k)}\frac{I_p(qr)}{I_p(qr_1)} + B_{mn}^{(k)}\frac{K_p(qr)}{K_p(qr_0)}\right)\times \\
&\quad \times \sin\frac{(2m-1)\,\pi\alpha}{2\alpha_1}\sin\frac{(2n-1)\,\pi z}{2z_1}, \\
\Psi^{(k)} &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(C_{mn}^{(k)}\frac{I_p(qr)}{I_p(qr_1)} + D_{mn}^{(k)}\frac{K_p(qr)}{K_p(qr_0)}\right)\times \\
&\quad \times \cos\frac{(2m-1)\,\pi\alpha}{2\alpha_1}\sin\frac{(2n-1)\,\pi z}{2z_1}, \\
G^{(k)} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(E_{mn}^{(k)}\frac{I_p(qr)}{I_p(qr_1)} + F_{mn}^{(k)}\frac{K_p(qr)}{K_p(qr_0)}\right)\times \\
&\quad \times \sin\frac{(2m-1)\,\pi x}{2x_1}\sin\frac{(2n-1)\,\pi ny}{2z_1},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\tilde{T}^{(k)} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\gamma_{mn}^2}\left(t_{mn}^{(k)}\frac{I_p(qr)}{I_p(qr_1)} + T_{mn}^{(k)}\frac{K_p(qr)}{K_p(qr_0)}\right)\times \\
&\quad \times \sin\frac{(2m-1)\,\pi\alpha}{2\alpha_1}\sin\frac{(2n-1)\,\pi z}{2z_1},
\end{aligned}
\tag{13}
$$

where $I_p(qr)$ and $K_p(qr)$are the first and second kind modified Bessel functions, correspondingly, $p = (2m-1)\,\pi/2\alpha_1$,$q = (2n-1)\,\pi/2z_1$.

In order to find the desired displacements and stresses the expressions for functions $\Phi^{(k)}$, $\Psi^{(k)}$, $G^{(k)}$ and $\tilde{T}^{(k)}$(formulas (12) and (13)) are substituted in the corresponding formulas (9) or (10). For example, if we substitute expressions (12) in formulas (10), displacements will be expressed in the following way

$$
\begin{aligned}
2\mu_k u^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Bigg\{ & \left(q\frac{I_p^{/}(qr)}{I_p(qr_1)} + q^2\frac{I_p^{//}(qr)}{I_p(qr_1)} \right) A_{mn}^{(k)} + \\
& + \left(q\frac{K_p^{/}(qr)}{K_p(qr_0)} + q^2\frac{K_p^{//}(qr)}{K_p(qr_0)} \right) B_{mn}^{(k)} - \\
& - p\frac{1}{r}\frac{I_p(qr)}{I_p(qr_1)} C_{mn}^{(k)} - p\frac{1}{r}\frac{K_p(qr)}{K_p(qr_0)} D_{mn}^{(k)} + \\
& + q\frac{I_p^{/}(qr)}{I_p(qr_1)} E_{mn}^{(k)} + q\frac{K_p^{/}(qr)}{K_p(qr_0)} F_{mn}^{(k)} - \\
& - \frac{\mu_k\left(1+\nu_k\right)}{1+\nu_k}\gamma^{(k)} \left[\left(q\frac{I_p^{/}(qr)}{I_p(qr_1)} + q^2\frac{I_p^{//}(qr)}{I_p(qr_1)} \right) t_{mn}^{(k)} + \right. \\
& \left. + \left(q\frac{K_p^{/}(qr)}{K_p(qr_0)} + q^2\frac{K_p^{//}(qr)}{K_p(qr_0)} \right) T_{mn}^{(k)} \right] \Bigg\} \times \\
& \times \sin\frac{(2m-1)\,\pi\alpha}{2\alpha_1} \sin\frac{(2n-1)\,\pi z}{2z_1},
\end{aligned}
$$

$$
\begin{aligned}
2\mu_k v^{(k)} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \Bigg\{ & pq\left(\frac{I_p^{/}(qr)}{I_p(qr_1)} A_{mn}^{(k)} + \frac{K_p^{/}(qr)}{K_p(qr_0)} B_{mn}^{(k)} \right) - \\
& - q\left(\frac{I_p^{/}(qr)}{I_p(qr_1)} C_{mn}^{(k)} + \frac{K_p^{/}(qr)}{K_p(qr_0)} D_{mn}^{(k)} \right) C_{mn}^{(k)} + \\
& + D_{mn}^{(k)} + p\frac{1}{r}\left(\frac{I_p(qr)}{I_p(qr_1)} E_{mn}^{(k)} + \frac{K_p(qr)}{K_p(qr_0)} F_{mn}^{(k)} \right) - \\
& - \frac{\mu_k\left(1+\nu_k\right)}{1+\nu_k} pq\gamma^{(k)} \left(\frac{I_p^{/}(qr)}{I_p(qr_1)} t_{mn}^{(k)} + \frac{K_p^{/}(qr)}{K_p(qr_0)} T_{mn}^{(k)} \right) \Bigg\} \times \\
& \times \cos\frac{(2m-1)\,\pi\alpha}{2\alpha_1} \sin\frac{(2n-1)\,\pi z}{2z_1},
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
2\mu_k w^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Bigg\{ & \left(q^2\frac{rI_p^{/}(qr)}{I_p(qr_1)} + 4(1-\nu^{(k)})q\frac{I_p^{//}(qr)}{I_p(qr_1)} \right) A_{mn}^{(k)} + \\
& + \left(q^2\frac{rK_p^{/}(qr)}{K_p(qr_0)} + 4(1-\nu^{(k)})q\frac{K_p^{//}(qr)}{K_p(qr_0)} \right) B_{mn}^{(k)} + \\
& + q\left(\frac{I_p(qr)}{I_p(qr_1)} E_{mn}^{(k)} + \frac{K_p(qr)}{K_p(qr_0)} F_{mn}^{(k)} \right) - \\
& - \frac{\mu_k\left(1+\nu_k\right)}{1+\nu_k} q^2\gamma^{(k)} \left(\frac{rI_p^{/}(qr)}{I_p(qr_1)} t_{mn}^{(k)} + \frac{rK_p^{/}(qr)}{K_p(qr_1)} T_{mn}^{(k)} \right) \Bigg\} \times \\
& \times \sin\frac{(2m-1)\pi\alpha}{2\alpha_1} \cos\frac{(2n-1)\pi z}{2z_1}\ .
\end{aligned}
$$

The defined functions on boundary cylindrical surfaces of a multilayer cylindrical body are extended into corresponding trigonometric series. Expressions for functions defined on the boundary and contact surfaces and appropriate extensions into function series defined on boundary cylindrical surfaces of the body are substituted into corresponding conditions (7), (8) or (9) and expressions for identical trigonometric functions are equated. As a result, infinite systems of algebraic equations are obtained for the desired coefficients of harmonic functions, the main matrix of these equations for a fixed $m = \bar{m}$ and arbitrary values of n from 1 to infinity having a block diagonal form shown in Fig.3.

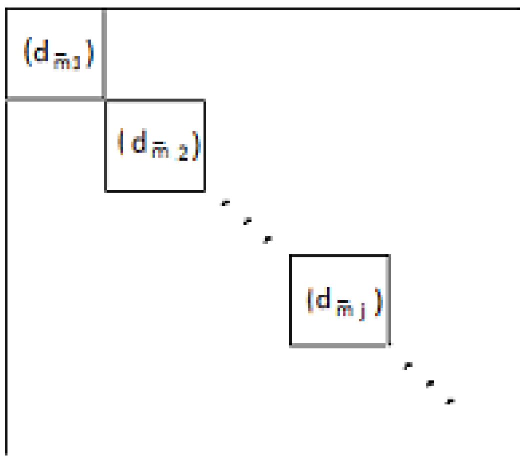

Fig.3. Form of the main matrix

In all of the investigated nine cases of boundary conditions on flat boundaries of the cylinder the main matrix corresponding to linear algebraic equations of any of the above-stated problems will have the form shown in Fig.3. Each of its (d_{ij}) blocks represents a matrix of the following form ($\times$ denotes non-zero elements of the matrix):

$$
\left(
\begin{array}{l}
\times\times\;\;\times\times\times\times \\
\times\;\;\times\times\times\times\times \\
\times\;\;\times\;\;\;\;\times\times\times \\
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Fig. 4. Form of the matrix $(d_{ij})(i = 1, 2, \ldots; \quad j = 1, 2, \ldots)$

The N-layer cylindrical body under consideration has $N-1$ contact cylindrical surfaces contact conditions on which generate $6\,(N-1)$ equations while boundary conditions on boundary cylindrical surfaces of the body give three more equations each. As a result, for each fixed m and n there is a system of $6N$ equations with $6N$ unknowns, i.e. the main matrix represents a square matrix of the dimension $6N \times 6N$.

As it was noted above, Problems 1 and 2 result from a continuous extension of the solutions. In particular, Problem 1 results from a continuous symmetrical extension of the solution through $\alpha = \alpha_1$ and $z = z_1$, while we have Problem 2 when the solution is continuously anti-symmetrically extended through $\alpha = 0$ and $z = 0$.

As for the change in temperature, this problem is reduced to integration of Laplace's equation for a multilayer cylindrical domain.

Convergence of the obtained series can be easily proved as well as the uniqueness of the obtained solutions.

For a numerical implementation and visual representation of the solutions of the problems studied above a computer program using MATLAB language has been developed [12]. The program has a simple form, is user-friendly and when necessary can be supplemented by a corresponding computation program module.

4 Numerical example of elastic equilibrium determination for a one-layer cylindrical body

We end the given paper with the following application problem. In particular, find the elastic equilibrium of a cylindrical body $\Omega = \{r_0 < r < r_1,\, 0 < \alpha < \dfrac{2\pi}{3},\, 0 < z < 2z_1\}$ when on flat lateral surfaces $\alpha = 0$, $\alpha = \dfrac{2\pi}{3}$, $z = 0$ and $z = 2z_1$ anti-symmetry conditions are satisfied, while the cylindrical boundary surface $r = r_0$ is free of stresses

$$\text{For } \ r = r_0: \quad \sigma_{r\alpha} = 0, \ \ \sigma_{rz} = 0, \ \ \sigma_{rr} = 0; \tag{15}$$

On the cylindrical boundary surface $r = r_1$ the following boundary conditions

$$\text{For } \ r = r_1: \quad \sigma_{r\alpha} = 0, \ \ \sigma_{rz} = 0, \ \ \sigma_{rr} = f(\alpha,\, z) = p \sin\frac{3\alpha}{2} \sin\frac{\pi z}{2z_1}, \tag{16}$$

are defined where $p = const$.

In addition to the determination of the elastic equilibrium of the given body another problem is also stated – to find how the displacement u at the point $M\left(\dfrac{r_1 + r_2}{2},\, \dfrac{\pi}{3},\, z_1\right)$ depends on the value z_1. The above-stated problem using symmetry of the function $f(\alpha,\, z)$ defined on the boundary with respect to the planes $\alpha = \dfrac{\pi}{3}$ and $z = z_1$, is reduced to the following boundary value problem: to find the elastic equilibrium of the cylindrical body $\tilde{\Omega} = \left\{r_0 < r < r_1, 0 < \alpha < \frac{\pi}{3},\, 0 < z < z_1\right\}$ when on the flat lateral boundary surfaces $\alpha = 0$ and $z = 0$ anti-symmetry conditions are satisfied while on the surfaces $\alpha = \frac{\pi}{3}$ and $z = z_1$ symmetry conditions hold. Boundary conditions on cylindrical surfaces remain unchanged as well as another aim of the given example - to determine how the value of the displacement u at the point $M\left(\dfrac{r_1 + r_2}{2},\, \dfrac{\pi}{3},\, z_1\right)$ depends on the value z_1.

It can be easily seen that in this case the harmonic functions Φ, Ψ and G will have the following form

$$\begin{aligned}
\Phi &= \left\{ A\, I_{3/2}\left(\frac{\pi r}{2z_1}\right) + B\, I_{-3/2}\left(\frac{\pi r}{2z_1}\right) \right\} \sin\frac{3\alpha}{2} \sin\frac{\pi z}{2z_1}, \\
\Psi &= \left\{ C\, I_{3/2}\left(\frac{\pi r}{2z_1}\right) + D\, I_{-3/2}\left(\frac{\pi r}{2z_1}\right) \right\} \cos\frac{3\alpha}{2} \sin\frac{\pi z}{2z_1}, \qquad (17) \\
G &= \left\{ E\, I_{3/2}\left(\frac{\pi r}{2z_1}\right) + F\, I_{-3/2}\left(\frac{\pi r}{2z_1}\right) \right\} \sin\frac{3\alpha}{2} \sin\frac{\pi z}{2z_1},
\end{aligned}$$

where $A,\ B,\ C,\ D,\ E,\ F$ are the desired constants.

Substitute formulas (17) in the relations (11) and then substituting the obtained expressions in boundary conditions (15) and (16) and comparing coefficients of identical trigonometric functions we obtain the following system of equations with respect to the coefficients $A,\ B,\ C,\ D,\ E,\ F$.

$$\begin{aligned}
&\tilde{A}(r_0)\, A + \tilde{B}(r_0)\, B + \tilde{C}(r_0)\, C + \tilde{D}(r_0)\, D + \frac{\pi^2}{4z_1^2} I_{3/2}^{//}\left(\frac{\pi r_0}{2z_1}\right) E + \\
&+ \frac{\pi^2}{4z_1^2} I_{-3/2}^{//}\left(\frac{\pi r_0}{2z_1}\right) F = 0, \\
&\frac{3}{2} I_{3/2}^{//}\left(\frac{\pi r_0}{2z_1}\right) A + \frac{3}{2} I_{-3/2}^{//}\left(\frac{\pi r_0}{2z_1}\right) B + \frac{\pi^2}{4z_1^2}\left[\frac{1}{2} I_{3/2}\left(\frac{\pi r_0}{2z_1}\right) - \right. \\
&\left. - I_{3/2}^{//}\left(\frac{\pi r_0}{2z_1}\right)\right] C + \frac{\pi^2}{4z_1^2}\left[\frac{1}{2} I_{3/2}\left(\frac{\pi r_0}{2z_1}\right) - I_{3/2}^{//}\left(\frac{\pi r_0}{2z_1}\right)\right] D - \\
&- \tilde{C}(r_0)\, E - \tilde{D}(r_0)\, F = 0, \qquad (18) \\
&\tilde{E}(r_0)\, A + \tilde{F}(r_0)\, B - \frac{3\pi}{8z_1}\frac{1}{r_0} I_{3/2}\left(\frac{\pi r_0}{2z_1}\right) C - \frac{3\pi}{8z_1}\frac{1}{r_0} I_{-3/2}\left(\frac{\pi r_0}{2z_1}\right) D + \\
&+ \frac{\pi^2}{4z_1^2} I_{3/2}^{/}\left(\frac{\pi r_0}{2z_1}\right) E + \frac{\pi^2}{4z_1^2} I_{-3/2}^{/}\left(\frac{\pi r_0}{2z_1}\right) F = 0, \\
&\tilde{A}(r_1)\, A + \tilde{B}(r_1)\, B + \tilde{C}(r_1)\, C + \tilde{D}(r_1)\, D + \frac{\pi^2}{4z_1^2} I_{3/2}^{//}\left(\frac{\pi r_1}{2z_1}\right) E + \\
&+ \frac{\pi^2}{4z_1^2} I_{-3/2}^{//}\left(\frac{\pi r_1}{2z_1}\right) F = p,
\end{aligned}$$

$$\frac{3}{2}I^{//}_{3/2}\left(\frac{\pi r_1}{2z_1}\right)A+\frac{3}{2}I^{//}_{-3/2}\left(\frac{\pi r_1}{2z_1}\right)B+\frac{\pi^2}{4z_1^2}\left[\frac{1}{2}I_{3/2}\left(\frac{\pi r_1}{2z_1}\right)-\right.$$
$$\left.-I^{//}_{3/2}\left(\frac{\pi r_1}{2z_1}\right)\right]C+\frac{\pi^2}{4z_1^2}\left[\frac{1}{2}I_{3/2}\left(\frac{\pi r_1}{2z_1}\right)-I^{//}_{3/2}\left(\frac{\pi r_1}{2z_1}\right)\right]D-$$
$$-\tilde{C}(r_1)\,E-\tilde{D}(r_1)\,F=0, \tag{18}$$
$$\tilde{E}(r_1)\,A+\tilde{F}(r_1)\,B-\frac{3\pi}{8z_1}\frac{1}{r_0}I_{3/2}\left(\frac{\pi r_1}{2z_1}\right)C-\frac{3\pi}{8z_1}\frac{1}{r_1}I_{-3/2}\left(\frac{\pi r_1}{2z_1}\right)D+$$
$$+\frac{\pi^2}{4z_1^2}I^{/}_{3/2}\left(\frac{\pi r_1}{2z_1}\right)E+\frac{\pi^2}{4z_1^2}I^{/}_{-3/2}\left(\frac{\pi r_1}{2z_1}\right)F=0,$$

where

$$\tilde{A}(r)=\partial_r\left[\frac{\pi}{2z_1}I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right)+\frac{\pi^2}{4z_1^2}I^{//}_{3/2}\left(\frac{\pi r}{2z_1}\right)\right]-\frac{\nu\pi^2}{2z_1^2}I_{3/2}\left(\frac{\pi r}{2z_1}\right),$$

$$\tilde{B}(r)=\partial_r\left[\frac{\pi}{2z_1}I^{/}_{-3/2}\left(\frac{\pi r}{2z_1}\right)+\frac{\pi^2}{4z_1^2}I^{//}_{-3/2}\left(\frac{\pi r}{2z_1}\right)\right]-\frac{\nu\pi^2}{2z_1^2}I_{-3/2}\left(\frac{\pi r}{2z_1}\right),$$

$$\tilde{C}(r)=\frac{3}{2}\left(\frac{1}{r^2}I_{3/2}\left(\frac{\pi r}{2z_1}\right)-\frac{\pi}{2rz_1}I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right)\right),$$

$$\tilde{D}(r)=\frac{3}{2}\left(\frac{1}{r^2}I_{-3/2}\left(\frac{\pi r}{2z_1}\right)-\frac{\pi}{2rz_1}I^{/}_{-3/2}\left(\frac{\pi r}{2z_1}\right)\right),$$

$$\tilde{E}\,(r)=\frac{\pi^2}{4z_1^2}\left[(3-2\nu)I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right)+\frac{\pi}{2z_1}I^{//}_{3/2}\left(\frac{\pi r}{2z_1}\right)\right],$$

$$\tilde{F}\,(r)=\frac{\pi^2}{4z_1^2}\left[(3-2\nu)I^{/}_{-3/2}\left(\frac{\pi r}{2z_1}\right)+\frac{\pi}{2z_1}I^{//}_{-3/2}\left(\frac{\pi r}{2z_1}\right)\right].$$

The values of modified Bessel functions entering system (18) are computed according to the well-known formulas

$$I_{3/2}\,(x)=\sqrt{\frac{2}{\pi x}}\left(\cosh x-\frac{\sinh x}{x}\right),I_{-3/2}\,(x)=\sqrt{\frac{2}{\pi x}}\left(\sinh x-\frac{\cosh x}{x}\right)$$

and the resulting relations

$$I^{/}_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\sinh x - \frac{3\cosh x}{2x} + \frac{3\sinh x}{2x^2} \right),$$

$$I^{/}_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\cosh x - \frac{3\sinh x}{2x} + \frac{3\cosh x}{2x^2} \right),$$

$$I^{//}_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\cosh x - \frac{2\sinh x}{x} + \frac{3,75\cosh x}{x^2} - \frac{3,75\sinh x}{x^3} \right),$$

$$I^{//}_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\sinh x - \frac{2\cosh x}{x} + \frac{3,75\sinh x}{x^2} - \frac{3,75\cosh x}{x^3} \right).$$

According to formulas (10) and (17) the displacements will be expressed in the following way

$$\begin{aligned} 2\mu u = &\left\{ \left[\frac{\pi}{2z_1} I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right) + \frac{\pi^2}{4z_1^2} I^{//}_{3/2}\left(\frac{\pi r}{2z_1}\right) \right] A + \left[\frac{\pi}{2z_1} I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right) + \right.\right. \\ &\left. + \frac{\pi^2}{4z_1^2} I^{//}_{3/2}\left(\frac{\pi r}{2z_1}\right) \right] B - \frac{3}{2}\frac{1}{r} I_{3/2}\left(\frac{\pi r}{2z_1}\right) C - \frac{3}{2}\frac{1}{r} I_{-3/2}\left(\frac{\pi r}{2z_1}\right) D + \\ &+ \frac{\pi}{2z_1} I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right) E + \frac{\pi}{2z_1} I^{/}_{-3/2}\left(\frac{\pi r}{2z_1}\right) F \\ &\left. + \frac{\pi}{2z_1} I^{/}_{-3/2}\left(\frac{\pi r}{2z_1}\right) F \right\} \sin\frac{3\alpha}{2} \sin\frac{\pi z}{2z_1}, \end{aligned}$$

$$\begin{aligned} 2\mu v = &\left\{ \frac{3\pi}{4z_1} I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right) A + \frac{3\pi}{4z_1} I^{/}_{-3/2}\left(\frac{\pi r}{2z_1}\right) B - \frac{\pi}{2z_1} I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right) C - \right. \\ &- \frac{\pi}{2z_1} I^{/}_{-3/2}\left(\frac{\pi r}{2z_1}\right) D + \frac{3}{2}\frac{1}{r} I_{3/2}\left(\frac{\pi r}{2z_1}\right) E + \\ &\left. + \frac{3}{2}\frac{1}{r} I_{-3/2}\left(\frac{\pi r}{2z_1}\right) F \right\} \cos\frac{3\alpha}{2} \sin\frac{\pi z}{2z_1}, \end{aligned}$$

$$\begin{aligned} 2\mu w = &\left\{ \left[\frac{\pi^2 r}{4z_1^2} I^{/}_{3/2}\left(\frac{\pi r}{2z_1}\right) + \frac{4(1-\nu)\pi}{2z_1} I_{3/2}\left(\frac{\pi r}{2z_1}\right) \right] A + \left[\frac{\pi^2 r}{4z_1^2} I^{/}_{-3/2}\left(\frac{\pi r}{2z_1}\right) + \right.\right. \\ &\left. + \frac{4(1-\nu)\pi}{2z_1} I_{-3/2}\left(\frac{\pi r}{2z_1}\right) \right] B + \frac{\pi}{2z_1} I_{3/2}\left(\frac{\pi r}{2z_1}\right) E + \\ &\left. + \frac{\pi}{2z_1} I_{-3/2}\left(\frac{\pi r}{2z_1}\right) F \right\} \sin\frac{3\alpha}{2} \cos\frac{\pi z}{2z_1}. \end{aligned}$$

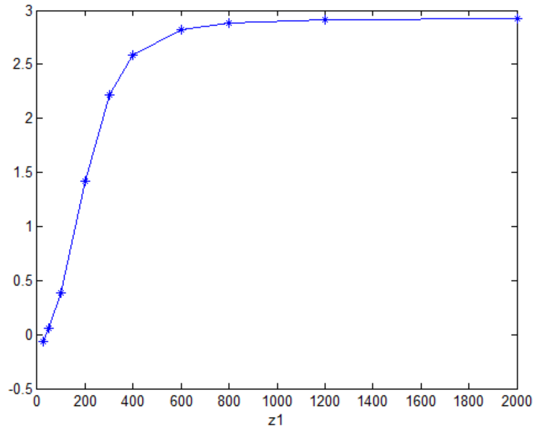

Fig.5. Graph showing how the displacement u depends on z_1 ($r_0 = 192$sm, $r_1 = 208$sm, $\nu = 0,3$, $E = 2 \cdot 10^6$kg /cm ^{2}).

The desired dependence of the displacement u on z_1 after the solution of the boundary value problem is shown in Fig.5.

Just as we expected , for the time being, the value u is also growing along with z_1 and then it stabilizes and tends to a certain definite value.

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