LARGE TIME BEHAVIOR OF SOLUTION AND SEMI-DISCRETE SCHEME FOR ONE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION WITH SOURCE TERM

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Abstract

Initial-boundary value problem with mixed boundary conditions for one nonlinear integro-differential equation with source term is considered. The model arises at describing penetration of a magnetic field into a substance. Large time asymptotic as $t \to \infty$ is given. Corresponding semi-discrete difference scheme is studied as well. More general cases of nonlinearity are studied than one has been studied earlier.

Key words and phrases: Nonlinear partial integro-differential equation, asymptotic behavior, semi-discrete scheme.

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1 Introduction

In mathematical modeling nonlinear partial integro-differential models are received very often (see, for example, [1]-[5], [11], [12], [16], [22]). In the present work one such kind of equation, which describe the process of a magnetic field penetration into a substance is considered. The investigated model is obtained by reduction of the well-known Maxwell's system [17] to the following integro-differential form [10]

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2\,d\tau\right) rotH\right], \tag{1.1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

Our aim is to study asymptotic behavior as $t \to \infty$ and semi-discrete finite difference scheme for numerical solution of initial-boundary value problem with mixed boundary conditions for the one-dimensional case of equation (1.1) with nonlinear source term. Attention is paid to the investigation of a wider case of nonlinearity than already were studied.

Note that the investigation and numerical approximation of integro-differential parabolic models of (1.1) type are complex and still yields to the investigation only for special cases (see, for example, [6] - [10], [13] - [15], [18] - [20] and references therein).

2 Large Time Behavior of Solution

If the magnetic field has the form $H = (0, 0, U)$, $U = U(x,t)$, then, adding the source term $f(U)$, from (1.1) we obtain the following nonlinear integro-differential equation

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right] + f(U) = 0, \tag{2.1}$$

where

$$S(x,t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau. \tag{2.2}$$

In the cylinder $[0,1] \times [0,\infty)$ let us consider the following boundary and initial conditions:

$$U(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = 0, \tag{2.3}$$

$$U(x,0) = U_0(x), \tag{2.4}$$

where U_0 is a given function.

We use usual $L_2(0,1)$ and Sobolev spaces $H^k(0,1)$ and the corresponding norms. The symbol C below in this section denotes positive constant independent of t.

The following statement takes place.

Theorem 1. *If $a(S) = (1+S)^p$, $0 < p \le 1$, $f(U) = |U|^{q-2}U$, $q \ge 2$ and $U_0 \in H^3(0,1)$, $U_0(0) = \left.\frac{dU_0(x)}{dx}\right|_{x=1} = 0$, then problem (2.1) - (2.4) has not more than one solution and the following estimate holds as $t \to \infty$*

$$\left\|\frac{\partial U(x,t)}{\partial x}\right\| \le C\exp\left(-\frac{t}{2}\right).$$

Let us note that the same result as in Theorem 1 is true for a problem with first type homogeneous conditions on the whole boundary (see, for example, [14], [15]).

3 Convergence of the Semi-discrete Scheme

Let us consider the following problem:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[\left(1+\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)^p \frac{\partial U}{\partial x}\right] + f(U) = 0, \tag{3.1}$$

$$U(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = 0, \tag{3.2}$$

$$U(x,0) = U_0(x), \tag{3.3}$$

where $0 < p \le 1$ and f is an increasing function.

On [0,1] let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0,1,\dots,M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. In this section the semi-discrete approximation at (x_i, t) is designed by $u_i = u_i(t)$. The exact solution to the problem at (x_i, t) is denoted by $U_i = U_i(t)$. At points $i = 1, 2, \dots, M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences. We will use the following known notations, inner products and norms [21]:

$$(u,v) = h\sum_{i=1}^{M-1} u_i v_i, \quad (u,v] = h\sum_{i=1}^{M} u_i v_i,$$

$$\|u\| = (u,u)^{1/2}, \quad \|u]| = (u,u]^{1/2}.$$

$$u_{x,i}(t) = \frac{u_{i+1}(t) - u_i(t)}{h}, \quad u_{\bar{x},i}(t) = \frac{u_i(t) - u_{i-1}(t)}{h}.$$

Let us correspond to problem (3.1) - (3.3) the following semi-discrete scheme:

$$\frac{du_i}{dt} - \left\{\left(1+\int_0^t (u_{\bar{x},i})^2\, d\tau\right)^p u_{\bar{x},i}\right\}_x + f(u_i) = 0, \quad i = 1,2,\dots,M-1, \tag{3.4}$$

$$u_0(t) = u_{\bar{x},M}(t) = 0, \tag{3.5}$$

$$u_i(0) = U_{0,i}, \quad i = 0, 1, \ldots, M. \tag{3.6}$$

So, we obtained Cauchy problem (3.4) - (3.6) for the nonlinear system of ordinary integro-differential equations.

Multiplying equations (3.4) scalarly by $u(t) = (u_1(t), u_2(t), \ldots, u_{M-1}(t))$, after simple transformations we get

$$\frac{d}{dt}\|u(t)\|^2 + h\sum_{i=1}^{M}\left(1 + \int_0^t (u_{\bar{x},i})^2\, d\tau\right)^p (u_{\bar{x},i})^2 \leq 0.$$

From this we obtain the inequality

$$\|u(t)\|^2 + \int_0^t \|u_{\bar{x}}]|^2 d\tau \leq C, \tag{3.7}$$

where, here and below in this section, C denotes a positive constant which does not depend on h.

The a priori estimate (3.7) guarantees the global solvability of problem (3.1) - (3.3).

The principal aim of the present section is the proof of the following statement.

Theorem 2. *If $0 < p \leq 1$, f is an increasing function and problem (3.1) - (3.3) has a sufficiently smooth solution $U = U(x,t)$, then solution $u = u(t) = (u_1(t), u_2(t), \ldots, u_{M-1}(t))$ of the problem (3.1) - (3.3) tends to $U = U(t) = (U_1(t), U_2(t), \ldots, U_{M-1}(t))$ as $h \to 0$ and the following estimate is true*

$$\|u(t) - U(t)\| \leq Ch. \tag{3.8}$$

Proof. For $U = U(x,t)$ we have:

$$\frac{dU_i}{dt} - \left\{\left(1 + \int_0^t (U_{\bar{x},i})^2 d\tau\right)^p U_{\bar{x},i}\right\}_x + f(U_i) = \psi_i(t), \tag{3.9}$$
$$i = 1, 2, \ldots, M-1,$$

$$U_0(t) = U_{\bar{x},M}(t) = 0, \tag{3.10}$$

$$U_i(0) = U_{0,i}, \;\; i = 0, 1, \ldots, M, \tag{3.11}$$

where

$$\psi_i(t) = O(h).$$

Let $z_i(t) = u_i(t) - U_i(t)$. From (3.1) - (3.3) and (3.9) - (3.11) we have:

$$\frac{dz_i}{dt} - \left\{ \left(1 + \int_0^t (u_{\bar{x},i})^2 d\tau \right)^p u_{\bar{x},i} - \left(1 + \int_0^t (U_{\bar{x},i})^2 d\tau \right)^p U_{\bar{x},i} \right\}_x + f(u_i) - f(U_i) = -\psi_i(t), \tag{3.12}$$

$$z_0(t) = z_{\bar{x},M}(t) = 0,$$

$$z_i(0) = 0.$$

Multiplying equation (3.12) scalarly by $z(t) = (z_1(t), z_2(t), \ldots, z_{M-1}(t))$, using the discrete analogue of the formula of integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \sum_{i=1}^{M} \left\{ \left(1 + \int_0^t (u_{\bar{x},i})^2 d\tau \right)^p u_{\bar{x},i} - \left(1 + \int_0^t (U_{\bar{x},i})^2 d\tau \right)^p U_{\bar{x},i} \right\} z_{\bar{x},i} h$$
$$+h \sum_{i=1}^{M-1} \left(f(u_i) - f(U_i) \right) (u_i - U_i) = -h \sum_{i=1}^{M-1} \psi_i z_i. \tag{3.13}$$

Note that,

$$\left\{ \left(1 + \int_0^t (u_{\bar{x},i})^2 d\tau \right)^p u_{\bar{x},i} - \left(1 + \int_0^t (U_{\bar{x},i})^2 d\tau \right)^p U_{\bar{x},i} \right\} (u_{\bar{x},i} - U_{\bar{x},i})$$
$$= p \int_0^1 \left(1 + \int_0^t \left[U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i}) \right]^2 d\tau \right)^{p-1}$$
$$\times \frac{d}{dt} \left(\int_0^t \left[U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i}) \right] (u_{\bar{x},i} - U_{\bar{x},i}) d\tau \right)^2 d\xi$$
$$+ \int_0^1 \left(1 + \int_0^t \left[U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i}) \right]^2 d\tau \right)^p d\xi \, (u_{\bar{x},i} - U_{\bar{x},i})^2 .$$

After substituting this equality in (3.13), taking into account monotonicity of f, integrating received equality on $(0,t)$ and using formula of integrating by parts we get

$$\|z\|^2+2h\sum_{i=1}^{M}\int_0^t\int_0^1\left(1+\int_0^{t'}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2d\tau'\right)^p(u_{\bar{x},i}-U_{\bar{x},i})^2\,d\xi d\tau$$

$$+ph\sum_{i=1}^{M}\int_0^1\left(1+\int_0^{t}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2d\tau\right)^{p-1}$$

$$\times\left(\int_0^t\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right](u_{\bar{x},i}-U_{\bar{x},i})d\tau\right)^2d\xi$$

$$-p(p-1)h\sum_{i=1}^{M}\int_0^1\int_0^t\left(1+\int_0^{t'}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2d\tau'\right)^{p-2}$$

$$\times\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2$$

$$\times\left(\int_0^{t'}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right](u_{\bar{x},i}-U_{\bar{x},i})d\tau'\right)^2d\xi d\tau=-2h\sum_{i=1}^{M-1}\psi_i z_i.$$

Taking into account relation $0<p\le 1$ we have from the last equality

$$\|z(t)\|^2\le\int_0^t\|z(\tau)\|^2d\tau+\int_0^t\|\psi_i\|^2d\tau. \tag{3.14}$$

From (3.14) we get (3.8), and Theorem 2 thus is proved.

Various numerical experiments for the studied schemes are carried out. The results of these numerical experiments agree with theoretical researches.

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