HIERARCHICAL MODELS OF THE SECOND TYPE FOR SPHERICAL SHELLS

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Abstract

In the present paper by means of the I. Vekua method the system of differential equations for shallow spherical shells is obtained, when on upper and lower face surface displacement are assumed to be known. Using the method of the small parameter approximate solutions of I. Vekua's equations for approximations $N = 0$ is constructed. The small parameter $\varepsilon = 2h/R$, where $2h$ is the thickness of the shell, R is the radius of the sphere.

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1 Equations of Equilibrium of an Elastic Medium

Let Ω denote a shell and the domain of the space occupied by this shell. Inside the shell, we consider a smooth surface S with respect to which the shell Ω lies symmetrically. The surface S is called the midsurface of the shell Ω. To construct the theory of shells, we use the more convenient coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\boldsymbol{R}$ of any point of the domain Ω can be represented in the form

$$\boldsymbol{R}(x^1, x^2, x^3) = \boldsymbol{r}(x^1, x^2) + x^3\boldsymbol{n}(x^1, x^2),$$

where $\boldsymbol{r}$ and $\boldsymbol{n}$ are the radius-vector and the unit vector of the normal of the surface S $(x^3 = 0)$, respectively, (x^1, x^2) are the Gaussian parameters of the midsurfaces [1].

Making use of tensor notation, we can write the equilibrium equation of the continuous medium and stress-strain relations (Hooke's law) in the form [1]

$$\nabla_i \sigma^{ij} + \Phi^j = 0 \quad (j = 1, 2, 3), \tag{1}$$

$$\sigma^{ij} = \lambda\theta g^{ij} + 2\mu e^{ij} \quad (i, j = 1, 2, 3), \tag{2}$$

where ∇_i are covariant derivatives with respect to the space coordinates x^i, σ^{ij} are contravariant components of the stress tensor, Φ^j are contravariant components of the volume force $\mathbf{\Phi}$, e^{ij} are contravariant components of the strain tensor, θ is the cubical dilatation

$$\theta = e_i^i = g_{ik}e^{ki},$$

g_{ij} and g^{ij} are covariant and contravariant components of the discriminant g of the metric quadratic form of the space, λ and μ are Lamé's constants

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)},$$

where E and σ are Young's modulus and Poisson's ratio, respectively.

We shall hereafter confine ourselves to the consideration of infinitesimal deformations of a shell. Therefore for components of the strain tensor we consider only linear relations, expressing them by the displacement vector. For the covariant components of the strain tensor these relations have the form

$$e_{ij} = \frac{1}{2}(\nabla_i U_j + \nabla_j U_i),$$

where U_i are covariant components of the displacement vector.

For thin or shallow shells we can write [1]

$$\boldsymbol{R}_\alpha \cong \boldsymbol{r}_\alpha, \quad \boldsymbol{R}^\alpha \cong \boldsymbol{r}^\alpha, \quad \boldsymbol{R}^3 = \boldsymbol{R}_3 = \boldsymbol{n}, \quad g \cong a,$$

where $\boldsymbol{R}_i$ and $\boldsymbol{R}^i$ are covariant and contravariant base vectors of the space, $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}^\alpha$ are covariant and contravariant base vectors of the midsurface, a is the discriminant of the metric tensor of the midsurface.

2 I. Vekua's reduction method

Multiplying both sides of equations (1) and (2) by Legendre polynomials $P_m\left(\frac{x^3}{h}\right)$ and then integrating with respect to x^3 from $-h$ to h we obtain the equivalent infinite system of 2-D equations [1-2]

$$\begin{aligned}
&\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha\beta} - b_\alpha^\beta \overset{(m)}{\sigma}{}^{\alpha 3} + \frac{2m+1}{h}\left(\overset{(m+1)}{\sigma}{}^{\beta 3} + \overset{(m+3)}{\sigma}{}^{\beta 3} + \cdots\right) + \overset{(m)}{\Phi}{}^{\beta} = 0,\\
&\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha 3} + b_\alpha^\beta \overset{(m)}{\sigma}{}^{\alpha}_{\beta} + \frac{2m+1}{h}\left(\overset{(m+1)}{\sigma}{}^{33} + \overset{(m+3)}{\sigma}{}^{33} + \cdots\right) + \overset{(m)}{\Phi}{}^{3} = 0,
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
&\overset{(m)}{\sigma}{}^{\alpha\beta} = \lambda\left[\nabla_\gamma \overset{(m)}{U}{}^\gamma - 2H\overset{(m)}{U}{}_3 - \frac{2m+1}{h}\left(\overset{(m-1)}{U}{}_3 + \overset{(m-3)}{U}{}_3 + \cdots\right)\right]a^{\alpha\beta}\\
&+\mu\left(\nabla^\beta \overset{(m)}{U}{}^\alpha + \nabla^\alpha \overset{(m)}{U}{}^\beta - 2b^{\alpha\beta}\overset{(m)}{U}{}_3\right)\\
&+\lambda\frac{2m+1}{h}\left(\overset{(+)}{U}{}_3 - (-1)^m\overset{(-)}{U}{}_3\right)a^{\alpha\beta},\\
&\overset{(m)}{\sigma}{}^{\alpha 3} = \mu\left[\nabla^\alpha \overset{(m)}{U}{}^3 + b^\alpha_\beta \overset{(m)}{U}{}^\beta - \frac{2m+1}{h}\left(\overset{(m-1)}{U}{}^\alpha + \overset{(m-3)}{U}{}^\alpha + \cdots\right)\right]\\
&+\mu\frac{2m+1}{h}\left(\overset{(+)}{U}{}^\alpha - (-1)^m\overset{(-)}{U}{}^\alpha\right),\\
&\overset{(m)}{\sigma}{}^{33} = \lambda\left(\nabla_\gamma \overset{(m)}{U}{}^\gamma - 2H\overset{(m)}{U}{}_3\right)\\
&-(\lambda+2\mu)\frac{2m+1}{h}\left(\overset{(m-1)}{U}{}_3 + \overset{(m-3)}{U}{}_3 + \cdots\right)\\
&+(\lambda+2\mu)\frac{2m+1}{h}\left(\overset{(+)}{U}{}_3 - (-1)^m\overset{(-)}{U}{}_3\right),
\end{aligned}\tag{4}$$

$$\overset{(\pm)}{U}{}^i = U^i(x^1, x^2, \pm h),$$

where ∇_α are covariant derivatives on the midsurface S $(x_3 = 0)$ of the shell, $a^{\alpha\beta}$ and $b^{\alpha\beta}$ are the contravariant components of the metric tensor and curvature tensor of the midsurface S, H denote the middle curvature of the midsurface S.

If we substitute expressions (3) into (4) equations we arrive at an infinite system of second-order equations with respect to components of vectors $\overset{(m)}{\boldsymbol{u}}$, when on upper and lower face surfaces displacements are assumed to be known.

An infinite system of equations (3) has the advantage that it contains two independent variables - Gaussian coordinates x^1, x^2 of the surface S. But the decrease in the number of independent variables one is achieved by increasing the number of equations to infinity, which, naturally, has an obvious practical inconvenience. Therefore it is necessary to make the next step at further simplification of the problem.

3 Approximation of Order $N = 0$ for Spherical Shells

Then we consider $N = 0$ approximation. In other words, in the previous equations it is assumed that

$$\overset{(m)}{U}{}^i = 0, \quad \overset{(m)}{\sigma}{}^{ij} = 0, \quad if \;\; m > 0.$$

For spherical shell of radius R we have

$$b_1^1 = b_2^2 = -\frac{1}{R}, \quad b_1^2 = b_2^1 = 0, \quad H = -\frac{1}{R}.$$

The system of equations (3-4) becomes

$$\begin{aligned} &\mu\nabla_\alpha\nabla^\alpha \overset{(0)}{U}{}^\beta + \mu\nabla_\alpha\nabla^\beta \overset{(0)}{U}{}^\alpha + \lambda\nabla^\beta\nabla_\alpha \overset{(0)}{U}{}^\alpha - \frac{\mu}{R^2}\overset{(0)}{U}{}^\beta \\ &+\frac{2\lambda+3\mu}{R}\nabla^\beta \overset{(0)}{U}{}^3 + \overset{(0)}{F}{}^\beta = 0, \\ &\mu\nabla_\alpha\nabla^\alpha \overset{(0)}{U}{}^3 - \frac{4(\lambda+\mu)}{R^2}\overset{(0)}{U}{}^3 - \frac{2\lambda+3\mu}{R}\nabla_\alpha \overset{(0)}{U}{}^\alpha + \overset{(0)}{F}{}^3 = 0, \end{aligned} \tag{5}$$

where

$$\begin{aligned} &\overset{(0)}{F}{}^\beta = \overset{(0)}{\Phi}{}^\beta + \frac{\lambda}{h}\nabla^\beta\left(\overset{(+)}{U}{}^3 + \overset{(-)}{U}{}^3\right) + \frac{\mu}{hR}\left(\overset{(+)}{U}{}^\beta + \overset{(-)}{U}{}^\beta\right), \\ &\overset{(0)}{F}{}^3 = \overset{(0)}{\Phi}{}^3 + \frac{\mu}{h}\nabla_\alpha\left(\overset{(+)}{U}{}^\alpha + \overset{(-)}{U}{}^\alpha\right) - \frac{2\lambda}{hR}\left(\overset{(+)}{U}{}^3 + \overset{(-)}{U}{}^3\right). \end{aligned} \tag{6}$$

Let us consider the isometric coordinates on the sphere

$$\xi = tg\frac{\vartheta}{2}\cos\varphi, \quad \eta = tg\frac{\vartheta}{2}\sin\varphi,$$

where ϑ and φ are geographical coordinates. For the shallow spherical shell the coordinate ϑ varies inside the small segment: $0 \le \vartheta \le \vartheta_0$. Therefore one can put

$$\xi = \frac{\vartheta}{2}\cos\varphi, \quad \eta = \frac{\vartheta}{2}\sin\varphi.$$

Further it will be more convenient to consider the following new coordinates [3]

$$x^1 = \frac{R}{2h}\vartheta\cos\varphi, \quad x^2 = \frac{R}{2h}\vartheta\sin\varphi.$$

Then for the metric quadratic form we obtain the formula

$$ds^2 = (2h)^2((dx^1)^2 + (dx^2)^2) = (2h)^2 dz d\bar{z},$$

where

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2.$$

Now the system of equations of the elastic spherical shell may be written in the following form

$$\begin{aligned}
&\mu\Delta \overset{(0)}{U}_1 + (\lambda+\mu)\frac{\partial \overset{(0)}{\Theta}}{\partial x^1} + \varepsilon(2\lambda+3\mu)\frac{\partial \overset{(0)}{U}_3}{\partial x^1} - \varepsilon^2\mu \overset{(0)}{U}_1 + \overset{(0)}{F}_1 = 0,\\
&\mu\Delta \overset{(0)}{U}_2 + (\lambda+\mu)\frac{\partial \overset{(0)}{\Theta}}{\partial x^2} + \varepsilon(2\lambda+3\mu)\frac{\partial \overset{(0)}{U}_3}{\partial x^2} - \varepsilon^2\mu \overset{(0)}{U}_2 + \overset{(0)}{F}_2 = 0,\\
&\mu\Delta \overset{(0)}{U}_3 - \varepsilon(2\lambda+3\mu)\overset{(0)}{\Theta} - \varepsilon^2 4(\lambda+\mu)\overset{(0)}{U}_3 + \overset{(0)}{F}_3 = 0,
\end{aligned} \tag{7}$$

$$\overset{(0)}{\Theta} = \frac{\partial \overset{(0)}{U}_1}{\partial x^1} + \frac{\partial \overset{(0)}{U}_2}{\partial x^2},$$

where ε is the small parameter

$$\varepsilon = \frac{2h}{R}.$$

Introduce the notation

$$\overset{(0)}{U}_i = u_i.$$

Let us try to construct the solutions of the form [3-5]

$$u_i = \sum_{k=0}^{\infty} \overset{(k)}{u}_i \varepsilon^k. \tag{8}$$

The formal substitution of (8) into (7) shows that series (8) may satisfy equations (7) if the following equations are fulfilled

$$\begin{aligned}
&\mu\Delta \overset{(k)}{u}_1 + (\lambda+\mu)\frac{\partial \overset{(k)}{\Theta}}{\partial x^1} = \overset{(k)}{X}_1,\\
&\mu\Delta \overset{(k)}{u}_2 + (\lambda+\mu)\frac{\partial \overset{(k)}{\Theta}}{\partial x^2} = \overset{(k)}{X}_2,
\end{aligned} \tag{9}$$

$$\mu\Delta \overset{(k)}{u}_3 = \overset{(k)}{X}_3, \tag{10}$$

where

$$\overset{(k)}{X}_1 = -\overset{(0)}{F}_1 - (2\lambda+3\mu)\frac{\partial \overset{(k-1)}{u}_3}{\partial x^1} + \mu \overset{(k-2)}{u}_1,$$

$$\overset{(k)}{X}_2 = -\overset{(0)}{F}_2 - (2\lambda+3\mu)\frac{\partial \overset{(k-1)}{u}_3}{\partial x^2} + \mu \overset{(k-2)}{u}_2,$$

$$\overset{(k)}{X}_3=-\overset{(0)}{F}_3+(2\lambda+3\mu)\overset{(k-1)}{\Theta}+4(\lambda+\mu)\overset{(k-2)}{u}_3.$$

For each fixed k equations (9) and (10) coincide with equations of plane theory of elasticity and Poisson. The right parts of equations (9-10) are well-known quantities, defined by functions $\overset{(0)}{u}_i$, $\overset{(1)}{u}_i$,... $\overset{(k-1)}{u}_i$.

The complex form of the system (9-10) is:

$$\begin{aligned}&\mu\Delta\overset{(k)}{u}_+ +2(\lambda+\mu)\partial_{\bar z}\overset{(k)}{\Theta}\,\partial x^1=\overset{(k)}{X}_+,\\&\mu\Delta\overset{(k)}{u}_3=\overset{(k)}{X}_3,\end{aligned}\tag{11}$$

where

$$\overset{(k)}{u}_+=\overset{(k)}{u}_1+i\overset{(k)}{u}_2,\quad \overset{(k)}{X}_+=\overset{(k)}{X}_1+i\overset{(k)}{X}_2,$$

$$\partial_z=\frac{\partial}{\partial z},\quad \partial_{\bar z}=\frac{\partial}{\partial\bar z},\quad \Delta=4\frac{\partial^2}{\partial z\partial\bar z},$$

$$\left[\frac{\partial}{\partial z}=\frac12\left(\frac{\partial}{\partial x^1}-i\frac{\partial}{\partial x^2}\right),\quad \frac{\partial}{\partial\bar z}=\frac12\left(\frac{\partial}{\partial x^1}+i\frac{\partial}{\partial x^2}\right)\right].$$

The general solutions of this system are written as follow:

$$\overset{(k)}{u}_+=\varkappa\overset{(k)}{f}(z)-z\overline{\overset{(k)}{f}{}'(z)}-\overline{\overset{(k)}{g}{}'(z)}+\overset{(k)}{u}_{+p},$$

$$\overset{(k)}{u}_3=\overset{(k)}{\psi}(z)+\overline{\overset{(k)}{\psi}(z)}+\overset{(k)}{u}_{3p},$$

where $\varkappa=\dfrac{\lambda+3\mu}{\lambda+\mu}$, $\overset{(k)}{f}(z)$, $\overset{(k)}{g}(z)$ and $\overset{(k)}{\psi}(z)$ are any analytic functions of complex variable z, $\overset{(k)}{u}_{+p}$ and $\overset{(k)}{u}_{+3}$ are particular solutions of the system (11):

$$\overset{(k)}{u}_{+p}=\frac1\pi\frac{\lambda+3\mu}{4\mu(\lambda+2\mu)}\int\int_S\overset{(k)}{X}_+(\xi,\eta)\ln|\zeta-z|d\xi d\eta$$

$$-\frac1\pi\frac{\lambda+\mu}{8\mu(\lambda+2\mu)}\int\int_S\overset{(k)}{X}_+(\xi,\eta)\frac{\zeta-z}{\bar\zeta-\bar z}d\xi d\eta,$$

$$\overset{(k)}{u}_{+3}=\frac{1}{4\mu\pi}\int\int_S\overset{(k)}{X}_3(\xi,\eta)\ln|\zeta-z|d\xi d\eta,$$

where $\zeta=\xi+i\eta$.

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