SOME ISSUES OF CONDUCTING FLUID UNSTEADY FLOWS IN A CIRCULAR TUBE

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Abstract

In this article the unsteady flow of a viscous incompressible electrically conducting fluid in annular pipe under external radial magnetic field is considered. An exact solution of the problem in general and specific forms is obtained.

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1 Introduction

The developed flows of conducting fluid in an annular pipe are recently rather detailed studied [1-9], and the possibility of obtain of new exact analytical exact solutions seems quite limited. However, such opportunities do exist and it is possible, as will be shown below, to find even simple new solutions that however, have a rather interesting qualitative features.

2 Basic Part

Let's consider the axial flow of an viscous incompressible conducting fluid in a pipe, whose cross-section is limited by circumferences $r = a$ and $r = b$ $(a < b)$ under the impact of external radial magnetic field $H_r = H_0 \frac{a}{r}$.

If we assume that along the axis of cylinder (oz) longitudinal differential of pressure $P = const$ is applied, the equations of magnets hydrodynamics would be reduced to the following system

$$\left.\begin{aligned} &\rho \frac{\partial \nu}{\partial t} = \eta \Delta \nu + \frac{H_r}{4\pi} \frac{\partial H}{\partial r} + P, \\ &\frac{\partial H}{\partial t} = \frac{c^2}{4\pi\sigma} \Delta H + H_r \frac{\partial \nu}{\partial r}, \end{aligned}\right\} \qquad (1)$$

where as the unknown functions are the velocity $\nu \equiv \nu_z(r, t)$ and the induced magnetic field $H \equiv H_z(r, t)$, ρ is the density, η is the viscosity, σ is the conductivity, c is the speed of light.

Herewith electric field has only one component on the axis φ

$$E \equiv E_\varphi = -\left[\frac{\nu}{c}H_r + \frac{c}{4\pi\sigma}\frac{\partial H}{\partial r}\right] \tag{2}$$

and is a transverse pressure drop $-\dfrac{\partial p}{\partial r} = \dfrac{H}{4\pi}\dfrac{\partial H}{\partial r}$.

Due to the subject of system (1) to the Laplace transform [9] with the parameter p at initial conditions

$$\nu|_{t=0} = \nu_0(r), \;\; H|_{t=0} = 0 \tag{3}$$

and introducing the notation (ν_* is the characteristic velocity)

$$\left.\begin{aligned}
&f = \mu \cdot p\frac{\bar{\nu}}{\nu_*}, \;\; \varphi = \lambda_p\frac{\bar{H}}{H_0}, \;\; \lambda^2 = \frac{H_0^2 a}{4\pi\eta\nu_*}, \;\; \mu^2 = \frac{4\pi\sigma a\nu_*}{c^2},\\
&\alpha^2 = \frac{ap}{\nu_*}R, \;\; \beta^2 = \frac{ap}{\nu_*}R_m, \;\; R = \frac{\rho a\nu_*}{\eta}, \;\; R_m = \frac{4\pi\sigma a\nu_*}{c^2},\\
&M = \lambda\mu = \frac{H_0\sigma}{c}\sqrt{\frac{\sigma}{\eta}}, \;\; N = -\frac{\mu a^2}{\eta\nu_*}(P + \rho p\nu_0),
\end{aligned}\right\} \tag{4}$$

we obtain the following simultaneous equations

$$f'' + \frac{1}{x}(f' + M\varphi') - \alpha^2 f = N, \quad \varphi'' + \frac{1}{x}(\varphi' + Mf') - \beta^2\varphi = 0, \tag{5}$$

where the dimensionless independent variable $x = \dfrac{r}{a}$ is introduced.

For the solvability of task to the system (5) is necessary to add the four boundary conditions, two of which are defined by specifying the velocities $\nu_a(t)$ and $\nu_b(t)$ and boundaries $r = a$ and $r = b$

$$\bar{\nu}|_{r=a} = \bar{\nu}_a(p), \;\; \bar{\nu}|_{r=b} = \bar{\nu}_b(p). \tag{6}$$

The other two conditions associated with continuity of tangent components of electric and magnetic fields would be obtained by considering the Maxwell equations in the conductors $r < a$ and $r < b$ (with conductivities σ_a and σ_b accordingly)

$$\left.\begin{aligned}
&\frac{c}{4\pi\sigma}\bar{H}'(a) - \frac{\sqrt{p}}{2\sqrt{\pi\sigma_a}}\frac{I_1\left(\dfrac{2a}{c}\sqrt{\pi p\sigma_a}\right)}{I_0\left(\dfrac{2a}{c}\sqrt{\pi p\sigma_a}\right)}\bar{H}(a) = -\frac{H_0}{c}\bar{\nu}_a,\\
&\frac{c}{4\pi\sigma}\bar{H}'(b) + \frac{\sqrt{p}}{2\sqrt{\pi\sigma_b}}\frac{K_1\left(\dfrac{2b}{c}\sqrt{\pi p\sigma_b}\right)}{K_0\left(\dfrac{2b}{c}\sqrt{\pi p\sigma_b}\right)}\bar{H}(b) = -\frac{H_0}{c}\frac{a}{b}\bar{\nu}_b(b).
\end{aligned}\right\} \tag{7}$$

Further the particular case is investigated when the Reynolds viscous (R) and magnetic (R_m) numbers are equal to each other. At this $\beta = \alpha$ and the system (5) would be accurately integrated in cylindrical functions. In the case of $P = \nu_0 = 0$ the general integral has the form

$$\left.\begin{aligned} f &= \xi^{-m}[AI_{-m}(\xi) + BK_m(\xi)] + \xi^m[CI_m(\xi) + DK_m(\xi)], \\ \varphi &= \xi^{-m}[AI_{-m}(\xi) + BK_m(\xi)] - \xi^m[CI_m(\xi) + DK_m(\xi)]. \end{aligned}\right\} \quad (8)$$

At that it is designated $\xi = ax, \;\; m = \dfrac{M}{2}$.

The solution of problem would be obtained when you find the values A, B, C, D from the boundary conditions (6)-(7) and apply the inversion formula of Riemann-Mellin.

Let's carry out the calculations for the case of liquid flow of one cylinder of radius $a(b \to \infty)$ that is considered as perfectly conducting ($\sigma_a \to \infty$). In this case, from the conditions of infinity it would be assumed $A = C = 0$, and B and D would be found from condition ($\nu_* = \nu_a = cost$)

$$\bar{\nu}(a) = \frac{\nu_a}{P}, \;\; \frac{d\bar{H}}{d\xi}\bigg|_{z=1} = -\frac{H_0\mu^2}{pa}\,, \quad (9)$$

after that the transformed solution has the following form

$$\left.\begin{aligned} \bar{\nu} &= \frac{\nu_a}{p}\left[\frac{chm\ln\dfrac{a}{r}}{K_m(a)} - \frac{m}{a}\frac{shm\;\ln\dfrac{a}{r}}{K'_m(a)}\right]K_m(\xi), \\ \bar{H} &= \frac{\nu}{\lambda}\frac{H_0}{p}\left[\frac{shm\ln\dfrac{a}{r}}{K_m(a)} - \frac{m}{a}\frac{chm\;\ln\dfrac{a}{r}}{K'_m(a)}\right]K_m(\xi). \end{aligned}\right\} \quad (10)$$

Due to the calculations carried out with the Riemann-Mellin by way of integrating on section along the negative part of real line of the complex variable p plane, would be obtained the final solution of problem in the following form

$$\left.\begin{aligned} \nu &= \nu_{st} + \frac{2\nu_0}{\pi}\Big[\Phi_1\; chm \cdot \ln\frac{a}{r} - m\Phi_2\; shm \cdot \ln\frac{a}{r}\Big], \\ H &= H_{st} + \frac{2\mu H_0}{\pi\lambda}\Big[\Phi_1\; shm \cdot \ln\frac{a}{r} - m\Phi_2\; chm \cdot \ln\frac{a}{r}\Big], \end{aligned}\right\} \quad (11)$$

where the corresponding steady mode is expressed by the formulae

$$\nu_{st} = \nu_0\left(\frac{a}{r}\right)^M, \quad H_{st} = \frac{\mu H_0}{\lambda}\left(\frac{a}{r}\right)^M \quad (12)$$

and the designations are introduced $\left(\tau = \dfrac{\eta t}{a^2 \rho}\right)$,

$$\left.\begin{aligned} \Phi_1 &= \int_0^\infty \frac{J_m(ux)Y_m(u) - Y_m(ux)J_m(u)}{I_m^2(u) + Y_m^2(u)} e^{-m^2} \frac{du}{u}, \\ \Phi_2 &= \int_0^\infty \frac{J_m(ux)Y_m'(u) - Y_m(ux)J_m'(u)}{I_m'^2(u) + Y_m'^2(u)} e^{-m^2} \frac{du}{u^2}. \end{aligned}\right\} \tag{13}$$

Thus, the exact solution of problem in this case is presented by well convergent quadratures.

Let's mention one particular case where the solution is expressed through a cylindrical function.

Let's suppose that a moving fluid would be considered as ideal, i.e. in the original equations (1) would be assumed $\eta = 0$. In this case, the converted velocity is expressed due to terms of transferred induced magnetic field by the formula.

$$\bar{\nu} = \frac{1}{\rho}\left[\frac{H_0 a}{4\pi\rho r}\bar{H}' + \nu_0\right], \tag{14}$$

and the unknown function $\bar{H}(r)$ satisfies the certain second order equation. If introduce a dimensionless quantity $z(x) = \dfrac{p\bar{H}}{H_0}$, then the mentioned equation in case $P = 0$ would make the following form

$$\left(1 + \frac{1}{pTx^2}\right) z'' + \frac{1}{x}\left(1 - \frac{1}{pTx^2}\right) z' - 4\gamma pTz = 0, \tag{15}$$

where $\gamma = \dfrac{\pi a^2 \sigma^2 H_0^2}{c^4 \rho}$, $\quad T = \dfrac{c^2 \rho}{H_0^2 \sigma}$.

The substitution $x^2 = \dfrac{1}{\rho T}\left(\dfrac{u}{\gamma} - 1\right)$ leads to the equation

$$\frac{d^2 z}{du^2} + \frac{1}{u}\frac{dz}{du} - \frac{z}{u} = 0, \tag{16}$$

integrable in the Bessel functions [9]

$$z = AI_0(2\sqrt{u}) + BK_0(2\sqrt{u}). \tag{17}$$

To determine the values A and B in the right parts of boundary conditions (7) would be introduced values of velocities at $r = a$ and $r = b$ taken from (14). As for the conditions (6), for the case of ideal fluid, they make no sense at all.

If we turn, as above, to the case $\sigma_a \to \infty$, $b \to \infty$, from the requirements boundedness of field at infinity, we have $A = 0$, and B would be found from the condition ($\nu_0 = const$)

$$\left.\frac{dz}{du}\right|_{z=1} = -\frac{2\nu_0 T}{(1+pT)a}\ , \tag{18}$$

after that the solution of problem takes the following form

$$\frac{H}{H_0} = \frac{2\nu_0 T\gamma}{a}\frac{1}{2\pi i}\int\limits_{\sigma_0 - i\infty}^{\sigma_0 + i\infty}\frac{K_0(2\sqrt{u})}{\sqrt{u_0}K_1(2\sqrt{u_a})}e^{pt}\frac{dp}{p}, \tag{19}$$

where

$$u = \gamma\left(1 + pT\frac{r^2}{a^2}\right), \quad u_a = U|_{r=a}. \tag{20}$$

3 Conclusion

Thus, through bypassing the branching points $p_1 = -\frac{a^2}{r^2 T}$ and $p_2 = -\frac{1}{T}$ the solution in terms of the real integrals can de expressed. The formula for velocity on the surface of flowed cylinder has a particularly simple form:

$$\nu|_{r=a} = \nu_0 e^{-\frac{1}{T}}. \tag{21}$$

References

1. Loitsiansky L.G. *Mechanic of a fluid and gas.* Moscow, Sciences, 1987, 840 p., (in Russian).

2. Vatazin A. B., Lubimov G. A., Regirer C. A. *Magnetohydrodynamic Flow in Channels.* (In Russian) Moscow, Nauka, 1970.

3. Tsutskiridze V. N. Heat transfer with the flow of conducting fluid in circular pipes with finite conductivity under uniform transverse magnetic fields. *Appl. Math. Inform. Mech.* **12** (2007), no. 2, 111 -114.

4. Sharikadze J. V. An approximate solution of three dimensional boundary layer with strong suction. *Transactions of the International Conference "Problems of Continuum Mechanics"*, Tbilisi, 2007, 78-82.

5. Tsutskiridze V. N. Heat exchange at laminar motion of incompressible liquid with variable physical properties in the pipe with porous walls. I*nternational Scientific Journal of IFToMM "Problems of Mechanics"*. Tbilisi, 2008, 3(32), 61-64.

6. Tsutskiridze V.N., Jikidze L. A. Heat transfer in tubes with porous walls under internal heat sources. *Proc. A. Razmadze Math. Inst.* **154** (2010), 127-133.

7. Sharikadze J. V., Tsutskiridze V.N., Jikidze L. A. The flow conducting liquid in magnetohydrodynamics tubes with heat transfer. *International Scientific Journal of IFToMM "Problems of Mechanics"*, Tbilisi, 2011, 3(44), 63–70.

8. Jikidze L. A., Tsutskiridze V.N. Approximate method for solving unsteady rotation problem porous plate in the conducting fluid with account heat transfer in case of variable electroconductivity. *Several Problems of Applied Mathematics and Mechanics. Series: Science and Technology ¡¡Mathematical Physics¿¿(ebook)*, New York, 2013, pp. 157-164, ISBN: 978-1-62081-603-5.

9. Sharikadze J. V., Tsutskiridze V. N. Pulsating flow of weakly conductive liquid with heat transfer.*Appl. Math. Inform. Mech.* **18** (2013), no. 2, 42-53.