ON THE REFINED THEORIES OF ELASTIC PLATES

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Abstract

In the present paper the solutions of Kirsch's type problems are considered by means of different theories (E. Reissner, A. Lurie, I. Vekua). The obtained results are compared with each other.

Key words and phrases: The elastic plate, Hooke's law, the displacement vector, the stress.

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1 Introduction

In the paper we consider Kirsch type problems by means of different refined theories of plates (E. Reissner, A. Lurie, I. Vekua).

The complex form of the $3-D$ system of the equation for the elastic plate, we have

1) the system of the equilibrium equations for the stress components:

$$\begin{cases} \partial_z(\sigma_{11}+\sigma_{22}-2i\sigma_{12})+\partial_{\overline{z}}(\sigma_{11}+\sigma_{22})+\partial_3\sigma_+ +\phi_+=0, \\ \partial_z\sigma_+ +\partial_{\overline{z}}\overline{\sigma}_+ +\partial_3\sigma_{33}+\phi_3=0; \end{cases} \tag{1.1}$$

2) Hooke's law

$$\begin{cases} \sigma_{11}-\sigma_{22}+2i\sigma_{12}=4\mu\partial_{\overline{z}}u_+, \\ \sigma_{11}+\sigma_{22}=2(\lambda+\mu)\theta+2\lambda\partial_3u_3, \\ \sigma_+=\sigma_{13}+i\sigma_{23}=\mu\big[2\partial_{\overline{z}}u_3+\partial_3u_+\big], \\ \sigma_{33}=\lambda\theta+(\lambda+2\mu)\partial_3u_3, \end{cases} \tag{1.2}$$

where

$$\theta=\partial_\gamma u_\gamma=\partial_z u_+ +\partial_{\overline{z}}\overline{u}_+ \ \ (\gamma=1,2), \tag{1.3}$$
$$u_+=u_1+iu_2, \quad \phi_+=\phi_1+i\phi_2$$

and λ and μ are Lame's constants;

3) the system of the equilibrium equations in the components of the displacement vector:

$$\begin{cases}\mu(\Delta u_+ + \partial_3^2 u_+) + 2(\lambda+\mu)\partial_{\overline{z}}(\theta + \partial_3 u_3) + \phi_+ = 0,\\ \mu(\Delta u_3 + \partial_3^2 u_3) + (\lambda+\mu)\partial_3(\theta + \partial_3 u_3) + \phi_3 = 0;\end{cases} \tag{1.4}$$

4) the boundary conditions on the border of the circle $|z| = R$:

$$\begin{cases}\sigma_{(rr)} + i\sigma_{(r\vartheta)} = \dfrac{1}{2}\Big[\sigma_{11} + \sigma_{22} + (\sigma_{11} - \sigma_{22} + 2i\sigma_{12})e^{-2i\vartheta}\Big],\\ \sigma_{(r3)} = \mathrm{Re}[\sigma_+ e^{-i\vartheta}] = -\,\mathrm{Im}\Big[\sigma_+ \dfrac{d\overline{z}}{ds}\Big],\end{cases} \tag{1.5}$$

where

$$\begin{cases}\sigma_{(rr)} = \dfrac{1}{2}\Big[\sigma_{11} + \sigma_{22} + \mathrm{Re}(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})e^{-2i\vartheta}\Big],\\ \sigma_{(\vartheta\vartheta)} = \dfrac{1}{2}\Big[\sigma_{11} + \sigma_{22} - \mathrm{Re}(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})e^{-2i\vartheta}\Big],\\ \sigma_{(r\vartheta)} = \dfrac{1}{2}\,\mathrm{Im}\Big[(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})e^{-2i\vartheta}\Big].\end{cases} \tag{1.6}$$

2 The Kirsch Type Problems

Now consider the Kirsch type problems. These problems were solved by N. Muskhelishvili's classical theory as well as A. Lurie and I. Vekua's approximation $N = 0$. In this case the condition $\sigma_{33}^{\infty} = 0$ isn't satisfied.

Let us see Lurie's approximation $N = 1$ (stress-strain).

Now (1.4) has the form

$$\begin{cases}\mu\Delta \overset{(0)}{u}_+ + 2(\lambda+\mu)\partial_{\overline{z}}\Big(\overset{(0)}{\theta} + \dfrac{1}{3}\overset{(1)}{u}_3\Big) = 0,\\ \Delta \overset{(1)}{u}_3 = 0.\end{cases} \tag{2.1}$$

The general solution of this system has the form

$$\begin{cases}\overset{(0)}{u}_+ = \dfrac{\lambda+3\mu}{\lambda+\mu}\,\varphi(z) - z\,\overline{\varphi'(z)} - \overline{\psi(z)} - \big[f(z) + z\,\overline{f'(z)}\,\big],\\ \overset{(1)}{u}_3 = 2\,\dfrac{\lambda+2\mu}{\lambda+\mu}\,h\big[f'(z) + z\,\overline{f'(z)}\,\big],\end{cases} \tag{2.2}$$

where

$$\overset{(0)}{\theta} = \frac{2\mu}{\lambda+\mu}\Big[\varphi'(z)+\overline{\varphi'(z)}\Big] - 2\Big[f'(z)+z\,\overline{f'(z)}\Big], \tag{2.3}$$

$\varphi(z)$, $\psi(z)$, $f(z)$ are the analytic functions of the complex variable $z = x_1 + ix_2$.

Now Hooke's law has the form:

$$\begin{cases} \overset{(0)}{\sigma}_{11} - \overset{(0)}{\sigma}_{22} + 2i\overset{(0)}{\sigma}_{12} = 4\mu\partial_{\overline{z}}\overset{(0)}{u}_{+} = -4\mu\Big[z\,\overline{\varphi''(z)} + \overline{\psi'(z)} + z\,\overline{f''(z)}\Big], \\ \overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{22} = 2(\lambda+\mu)\overset{(0)}{\theta} + \dfrac{2\lambda}{h}\overset{(1)}{u}_{3} = \\ \qquad = 4\mu\Big[\varphi'(z)+\overline{\varphi'(z)} - \dfrac{\mu}{\lambda+\mu}\big(f'(z)+\overline{f'(z)}\big)\Big], \\ \overset{(1)}{\sigma}_{+} = 2\mu\partial_{\overline{z}}\overset{(1)}{u}_{3} = 4\mu\,\dfrac{\lambda+2\mu}{\lambda+\mu}\,h\overline{f''(z)}, \\ \overset{(0)}{\sigma}_{33} = \lambda\overset{(0)}{\theta} + \dfrac{\lambda+2\mu}{h}\overset{(1)}{u}_{3} = \\ \quad = 2\mu\Big[\dfrac{\lambda}{\lambda+\mu}\big((\varphi'(z)+\overline{\varphi'(z)}\big) + \dfrac{3\lambda+4\mu}{\lambda+\mu}\big(f'(z)+\overline{f'(z)}\big)\Big]. \end{cases} \tag{2.4}$$

The boundary conditions are now written as

$$\begin{cases} \overset{(0)}{\sigma}_{(rr)} + i\overset{(0)}{\sigma}_{r\vartheta} = 2\mu\Big[\varphi'(z)+\overline{\varphi'(z)} - \dfrac{\mu}{\lambda+\mu}\big(f'(z)+\overline{f'(z)}\big) - \\ \qquad -\big(z\,\overline{\varphi''(z)} + \overline{\psi'(z)} + z\,\overline{f''(z)}\big)e^{-2i\vartheta}\Big], \\ \overset{(1)}{\sigma}_{(r3)} = 4\mu\,\dfrac{\lambda+2\mu}{\lambda+\mu}\,h\,\mathrm{Re}\big[\,\overline{f''(z)}e^{-i\vartheta}\big]. \end{cases} \tag{2.5}$$

3 Problem I. *Uni-Directional Tension of a Plate, Weakened by a Circular Hole.*

Let the edges of the hole be free from external stresses and let the tension in the direction Ox be equal to $\boldsymbol{p}$ at infinity, where $\boldsymbol{p}$ is a constant.

$$(\mathrm{I}_1)\ \ \overset{(0)}{\sigma}{}^{\infty}_{11} = \boldsymbol{p} = const,\ (\overset{(0)}{\sigma}_{12} = \overset{(0)}{\sigma}_{22} = \overset{(1)}{\sigma}_{+} = \overset{(0)}{\sigma}_{33})^{\infty} = 0,$$

$$\begin{cases} \big(\overset{(0)}{\sigma}_{11} - \overset{(0)}{\sigma}_{22} + 2i\overset{(0)}{\sigma}_{12}\big)^{\infty} = \boldsymbol{p} \\ \big(\overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{22}\big) = \boldsymbol{p} \\ \big(\overset{(0)}{\sigma}_{+}\big)^{\infty} = 0 \\ \big(\overset{(0)}{\sigma}_{33}\big)^{\infty} = 0 \end{cases} \qquad \Longrightarrow$$

$$\Longrightarrow \begin{cases} b_0 = -\dfrac{\boldsymbol{p}}{4\mu} \\ a_0 = \dfrac{\boldsymbol{p}}{8\mu} \cdot \dfrac{(\lambda+\mu)(3\lambda+2\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\ c_0 = -\dfrac{\boldsymbol{p}}{8\mu} \cdot \dfrac{\lambda(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \end{cases} ; \qquad (3.1)$$

(I_2) The boundary conditions on the border of the circle $|z| = R$:

$$\begin{cases} \overset{(0)}{\sigma}_{(rr)} + i\overset{(0)}{\sigma}_{(r\vartheta)} = 0 \\ \overset{(0)}{\sigma}_{(r3)}\big|_R = 0 \end{cases} \Longrightarrow \begin{cases} a_2 = -\dfrac{\boldsymbol{p}R^2}{4\mu} \\ b_2 = \dfrac{\boldsymbol{p}R^2}{4\mu}, \quad b_4 = -\dfrac{3\boldsymbol{p}R^4}{4\mu} \\ c_n = 0, \quad n \geq 1 \end{cases} . \qquad (3.2)$$

(3.1) and (3.2) $\Longrightarrow$

$$\begin{cases} \varphi'(z) = a_0 + \dfrac{a_2}{z^2}, \\ \psi'(z) = b_0 + \dfrac{b_2}{z^2} + \dfrac{b_4}{z^4}, \\ f'(z) = c_0. \end{cases} \qquad (3.3)$$

(1.6) and (3.3) $\Longrightarrow$

$$\begin{cases} \overset{(0)}{\sigma}_{(rr)} = \dfrac{\boldsymbol{p}}{2}\left[1 - \dfrac{R^2}{r^2}\right] + \dfrac{\boldsymbol{p}}{2}\left[1 - \dfrac{4R^2}{r^2} + \dfrac{3R^4}{r^4}\right]\cos 2\vartheta, \\ \overset{(0)}{\sigma}_{(\vartheta\vartheta)} = \dfrac{\boldsymbol{p}}{2}\left[1 + \dfrac{R^2}{r^2}\right] - \dfrac{\boldsymbol{p}}{2}\left[1 + \dfrac{3R^4}{r^4}\right]\cos 2\vartheta, \\ \overset{(0)}{\sigma}_{(r\vartheta)} = \dfrac{\boldsymbol{p}}{2}\left[1 + \dfrac{2R^2}{r^2} - \dfrac{3R^4}{r^4}\right]\sin 2\vartheta. \end{cases} \qquad (3.4)$$

At the internal boundary (i.e. for $r = R$), as was to be expected, one has

$$\overset{(0)}{\sigma}_{(rr)} = \overset{(0)}{\sigma}_{(r\vartheta)} = 0,$$

while the value of $\overset{(0)}{\sigma}_{(\vartheta\vartheta)}$ is given by

$$\overset{(0)}{\sigma}_{(\vartheta\vartheta)} = \boldsymbol{p}[1 - 2\cos 2\vartheta] \ \text{ on } L. \qquad (3.5)$$

The maximum value of $\overset{(0)}{\sigma}_{(\vartheta\vartheta)}$ thus occurs for $\cos 2\vartheta = -1$, i.e. for

$$\vartheta = \pm\frac{\pi}{2},$$

where

$$\max \overset{(0)}{\sigma}_{(\vartheta\vartheta)} = 3\boldsymbol{p} \tag{3.6}$$

so that the value of the tensile is increased.

The coefficient of concentration

$$K = \frac{\max \overset{(0)}{\sigma}_{(\vartheta\vartheta)}}{\boldsymbol{p}} = 3. \tag{3.7}$$

The displacements have the following forms:

$$u_r + iu_\vartheta + u_+ e^{-i\vartheta} \Longrightarrow$$

$$\Longrightarrow \begin{cases} u_r = \dfrac{\boldsymbol{p}}{4\mu r}\left\{\dfrac{\lambda+2\mu}{3\lambda+2\mu}\, r^2 + R^2 + \Big[\dfrac{2(\lambda+2\mu)}{\lambda+\mu}\, R^2 + r^2 - \dfrac{R^4}{r^2}\Big]\cos 2\vartheta\right\} \\ u_\vartheta = -\dfrac{\boldsymbol{p}}{4\mu r}\left\{\dfrac{2}{\lambda+\mu}\, R^2 + r^2 + \dfrac{R^4}{r^2}\right\}\sin 2\vartheta \end{cases},$$

$$u_3 = -\frac{\boldsymbol{p}}{2\mu}\cdot\frac{\lambda}{3\lambda+2\mu}\, h. \tag{3.8}$$

4 Problem II. *Bi-Axial Tension.*

The problem of bi-axial tension of a plate with a circular hole is solved even more easily.

(II_1) Conditions at infinity

$$(\overset{(0)}{\sigma}_{11})^\infty = (\overset{(0)}{\sigma}_{22})^\infty = \boldsymbol{p} = const, \quad (\overset{(0)}{\sigma}_{12} = \overset{(1)}{\sigma}_{+} = \overset{(0)}{\sigma}_{33})^\infty = 0,$$

$$\begin{cases} \big(\overset{(0)}{\sigma}_{11} - \overset{(0)}{\sigma}_{22} + 2i\overset{(0)}{\sigma}_{12}\big)^\infty = 0 \\ \big(\overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{22}\big) = 2\boldsymbol{p} \\ \big(\overset{(1)}{\sigma}_{+}\big)^\infty = 0 \\ \big(\overset{(0)}{\sigma}_{33}\big)^\infty = 0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} b_0 = 0 \\ a_0 = \dfrac{\boldsymbol{p}}{4\mu}\cdot\dfrac{(\lambda+\mu)(3\lambda+4\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\ c_0 = -\dfrac{\boldsymbol{p}}{4\mu}\cdot\dfrac{\lambda(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \end{cases}; \tag{4.1}$$

(II_2) The boundary conditions on the border of the circle

$$\begin{cases} \overset{(0)}{\sigma}_{(rr)} + i\overset{(0)}{\sigma}_{(r\vartheta)}\big|_R = 0 \\ \overset{(1)}{\sigma}_{(r3)}\big|_R = 0 \end{cases} \Longrightarrow \begin{cases} a_2 = 0 \\ b_2 = \dfrac{\boldsymbol{p}R^2}{2\mu}, \quad b_4 = 0 \\ c_n = 0, \quad n \geq 1 \end{cases}. \tag{4.2}$$

(4.1) and (4.2) $\Longrightarrow$

$$\begin{cases} \varphi'(z) = a_0 \\ \psi'(z) = \dfrac{b_2}{z^2} \\ f'(z) = c_0 \end{cases}. \tag{4.3}$$

(1.6) and (4.3) $\Longrightarrow$

$$\begin{cases} \overset{(0)}{\sigma}_{(rr)} = \boldsymbol{p}\Big[1 - \dfrac{R^2}{r^2}\Big] \\ \overset{(0)}{\sigma}_{(\vartheta\vartheta)} = \boldsymbol{p}\Big[1 + \dfrac{R^2}{r^2}\Big] \\ \overset{(0)}{\sigma}_{(r\vartheta)} = 0 \end{cases}. \tag{4.4}$$

At the internal boundary $\overset{(0)}{\sigma}_{(rr)} = \overset{(0)}{\sigma}_{(r\vartheta)} = 0$ and $\overset{(0)}{\sigma}_{(\vartheta\vartheta)} = 2\boldsymbol{p}$. The displacements have the form:

$$u_r + iu_\vartheta = u_+ e^{-i\vartheta} \Longrightarrow \begin{cases} u_r = \dfrac{\boldsymbol{p}}{2\mu r}\left\{\dfrac{\lambda + 2\mu}{3\lambda + 2\mu} r^2 + R^2\right\} \\ u_\vartheta = 0 \end{cases}, \tag{4.5}$$

$$\overset{(1)}{u}_3 = -\frac{\boldsymbol{p}}{\mu} \cdot \frac{\lambda}{3\lambda + 2\mu} h.$$

5 Problem III. *Uniform Normal Pressure, Applied to the Edge of a Circular Hole.*

Consider now the case when the edge of the hole is subject to uniform normal pressure $\boldsymbol{p}$ and when the stresses vanish ar infinity.

We have

(III$_1$) the conditions at infinity

$$\sigma_{ij}^{\infty} = 0 \ \ (i,j=1,2,3),$$

$$\begin{cases} \big(\overset{(0)}{\sigma}_{11} - \overset{(0)}{\sigma}_{22} + 2i\overset{(0)}{\sigma}_{12}\big)^{\infty} = 0 \\ \big(\overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{22}\big)^{\infty} = 0 \\ \big(\overset{(1)}{\sigma}_{+}\big)^{\infty} = 0 \\ \big(\overset{(0)}{\sigma}_{33}\big)^{\infty} = 0 \end{cases} \Longrightarrow \begin{cases} b_0 = 0 \\ a_0 = 0 \\ c_0 = 0 \end{cases} ; \tag{5.1}$$

(III$_2$) the boundary conditions on the border of the circle

$$\overset{(0)}{\sigma}_{(rr)}\big|_R = -\boldsymbol{p}, \quad \overset{(0)}{\sigma}_{(r\vartheta)}\big|_R = 0, \quad \overset{(0)}{\sigma}_{(r3)}\big|_R = 0,$$

$$\begin{cases} \overset{(0)}{\sigma}_{(rr)} + \overset{(0)}{\sigma}_{(r\vartheta)} = -\boldsymbol{p} \\ \overset{(0)}{\sigma}_{(r3)} = 0 \end{cases} \Longrightarrow \begin{cases} a_2 = 0 \\ b_2 = \dfrac{\boldsymbol{p}R^2}{2\mu}, \ \ b_4 = 0 \\ c_n = 0, \ \ n \geq 1 \end{cases} . \tag{5.2}$$

(5.1) and (5.2) $\Longrightarrow$

$$\begin{cases} \varphi'(z) = 0 \\ \psi'(z) = \dfrac{b_2}{z^2} \\ f'(z) = 0 \end{cases} . \tag{5.3}$$

(1.6) and (5.3) $\Longrightarrow$

$$\begin{cases} \overset{(0)}{\sigma}_{(rr)} = -\dfrac{\boldsymbol{p}R^2}{r^2} \\ \overset{(0)}{\sigma}_{(\vartheta\vartheta)} = \dfrac{\boldsymbol{p}R^2}{r^2} \\ \overset{(0)}{\sigma}_{(r\vartheta)} = 0 \end{cases} . \tag{5.4}$$

The displacements have the following forms:

$$u_r + iu_\vartheta = u_+ e^{-i\vartheta} \Longrightarrow \begin{cases} u_r = \dfrac{\boldsymbol{p}R^2}{2\mu r} \\ u_\vartheta = 0 \end{cases} , \tag{5.5}$$

$$u_3 = 0.$$

6 Lurie's Method (N=1)

Consider now the same types of problems for the **bending** and solve them by means of A. Lurie's method.

Lurie's approximation $N = 1$ **(bending)**.

We have

1) the system of the equilibrium equations in the components of the displacement vector:

$$\begin{cases} \mu\Delta \overset{(1)}{u}_{+} + 2(\lambda+\mu)\partial_{\overline{z}} \overset{(1)}{\theta} = 0, \\ \mu\Delta \overset{(0)}{u}_{3} + \dfrac{\lambda+\mu}{h} \overset{(0)}{\theta} = 0, \end{cases} \tag{6.1}$$

where

$$\overset{(1)}{\theta} = \frac{2\mu}{\lambda+\mu}\left[\varphi'(z) + \overline{\varphi'(z)}\right]; \tag{6.2}$$

2) the general solution of this system has the form:

$$\begin{cases} \overset{(1)}{u}_{+} = \dfrac{\lambda+3\mu}{\lambda+\mu}\,\varphi(z) - z\,\overline{\varphi'(z)} - \overline{\psi(z)}, \\ \overset{(0)}{u}_{3} = -\dfrac{1}{2h}\left(\overline{z}\varphi(z) + z\,\overline{\varphi(z)}\right) + f(z) + \overline{f(z)}; \end{cases} \tag{6.3}$$

3) now Hooke's law has the form:

$$\begin{cases} \overset{(1)}{\sigma}_{11} - \overset{(1)}{\sigma}_{22} + 2i\overset{(1)}{\sigma}_{12} = 4\mu\partial_{\overline{z}}\overset{(1)}{u}_{+} = -4\mu\left[z\,\overline{\varphi''(z)} + \overline{\psi'(z)}\right], \\ \overset{(1)}{\sigma}_{11} + \overset{(1)}{\sigma}_{22} = 2(\lambda+\mu)\overset{(1)}{\theta} = 4\mu\left[\varphi'(z) + \overline{\varphi'(z)}\right], \\ \overset{(0)}{\sigma}_{+} = \mu\left[2\partial_{\overline{z}}\overset{(0)}{u}_{3} + \dfrac{1}{h}\overset{(1)}{u}_{+}\right] = \\ \qquad = \dfrac{2\mu}{h}\left[\dfrac{\mu}{\lambda+\mu}\,\varphi(z) - z\,\overline{\varphi'(z)} - \dfrac{1}{2}\,\overline{\psi(z)} + h\overline{f'(z)}\right], \\ \overset{(1)}{\sigma}_{33} = \lambda\overset{(1)}{\theta} = \dfrac{2\mu\lambda}{\lambda+\mu}\left[\varphi'(z) + \overline{\varphi'(z)}\right]; \end{cases} \tag{6.4}$$

4) the boundary conditions on the border of the circle $|z| = R$ are written as

$$\begin{cases} \overset{(1)}{\sigma}_{(rr)} + \overset{(1)}{\sigma}_{(r\vartheta)} = 2\mu\left[\varphi'(z) + \overline{\varphi'(z)} - \left(z\,\overline{\varphi''(z)} + \overline{\psi'(z)}\right)e^{-2i\vartheta}\right], \\ \overset{(0)}{\sigma}_{(r3)} = \dfrac{2\mu}{h}\,\mathrm{Re}\left[\left(\dfrac{\mu}{\lambda+\mu}\,\varphi(z) - z\,\overline{\varphi'(z)} - \dfrac{1}{2}\,\overline{\psi(z)} + h\overline{f'(z)}\right)e^{-i\vartheta}\right]. \end{cases} \tag{6.5}$$

7 Problem I′. *Uni-Directional Bending of a Plate, Weakened by a Circular Hole.*

We have conditions at infinity

$$(\overset{(1)}{\sigma}_{11})^\infty = \boldsymbol{p} = const, \quad \big(\overset{(1)}{\sigma}_{12} = \overset{(1)}{\sigma}_{22} = \overset{(0)}{\sigma}_{+} = \overset{(1)}{\sigma}_{33}\big)^\infty = 0,$$

$$\begin{cases} \big(\overset{(1)}{\sigma}_{11} - \overset{(1)}{\sigma}_{22} + 2i\overset{(1)}{\sigma}_{12}\big)^\infty = \boldsymbol{p} = 4\mu\bar{b}_0 \\ \big(\overset{(1)}{\sigma}_{11} + \overset{(1)}{\sigma}_{22}\big)^\infty = \boldsymbol{p} = 8\mu a_0 \\ (\overset{(0)}{\sigma}_{+})^\infty = \dfrac{2\mu}{h}\Big[hf'(z) - \dfrac{\lambda}{\lambda+\mu}\,a_0 z - \dfrac{1}{2}\,b_0\bar{z}\Big]^\infty \neq 0 \\ (\overset{(1)}{\sigma}_{33})^\infty = 4\,\dfrac{\mu\lambda}{\lambda+\mu}\,a_0 \neq 0 \end{cases} \tag{7.1}$$

because the conditions at infinity are not satisfied therefore we consider the same problem for the approximation $N = 2$.

We have

1) the system of the equilibrium equations in the components of the displacement vector

$$\begin{cases} \mu\Delta\overset{(1)}{u}_{+} + 2(\lambda+\mu)\partial_{\bar{z}}\Big(\overset{(1)}{\theta} + \dfrac{2}{h}\overset{(2)}{u}_3\Big) = 0, \\ \mu\Delta\overset{(0)}{u}_2 + \dfrac{\lambda+\mu}{3}\overset{(1)}{\theta} + \dfrac{2(\lambda+2\mu)}{h^2}\overset{(2)}{u}_3 = 0, \\ \mu\Delta\overset{(2)}{u}_3 = 0, \end{cases} \tag{7.2}$$

where

$$\overset{(1)}{\theta} = \frac{2\mu}{\lambda+\mu}\left[\varphi_1'(z) + \overline{\varphi_1'(z)}\right] - 2\big[f_2'(z) + \overline{f_2'(z)}\big]; \tag{7.3}$$

2) the general solution of this system has the form

$$\begin{cases} \overset{(1)}{u}_{+} = \dfrac{\lambda+3\mu}{\lambda+\mu}\,\varphi_1(z) - z\,\overline{\varphi_1'(z)} - \overline{\psi_1(z)} - \big(f_2(z) + z\,\overline{f_2'(z)}\big), \\ \overset{(0)}{u}_3 = f_0(z) + \overline{f_0(z)} - \\ \qquad -\dfrac{1}{2h}\Big[\bar{z}\varphi_1(z) + z\,\overline{\varphi_1(z)} + \dfrac{2\lambda+3\mu}{\lambda+\mu}\big(\bar{z}f_2(z) + z\,\overline{f_2(z)}\big)\Big], \\ \overset{(2)}{u}_3 = \dfrac{\lambda+2\mu}{\lambda+\mu}\,h\big[f_2'(z) + \overline{f_2'(z)}\big], \end{cases} \tag{7.4}$$

where $\varphi_1(z)$, $\psi_1(z)$, $f_0(z)$, $f_2(z)$ are the analytic functions of the complex variable $z = x_1 + ix_2$;

3) Hooke's law has the form

$$\begin{cases} \overset{(1)}{\sigma}_{11} - \overset{(1)}{\sigma}_{22} + 2i\overset{(1)}{\sigma}_{12} = 4\mu\Big[z\,\overline{\varphi_1''(z)} + \overline{\psi_1'(z)} + z\,\overline{f_2''(z)}\Big], \\ \overset{(1)}{\sigma}_{11} + \overset{(1)}{\sigma}_{22} = 2(\lambda+\mu)\overset{(1)}{\theta} + \dfrac{4\lambda}{h}\overset{(2)}{u}_3 = \\ \qquad = 4\mu\Big[\varphi_1'(z) + \overline{\varphi_1'(z)} - \dfrac{\mu}{\lambda+\mu}\big(f_2'(x) + \overline{f_2'(z)}\,\big)\Big], \\ \overset{(0)}{\sigma}_{+} = \mu\Big[2\partial_{\overline{z}}\overset{(0)}{u}_3 + \dfrac{1}{h}\overset{(1)}{u}_{+}\Big] = \\ \quad = \dfrac{\mu}{h}\Big[2h\overline{f_0'(z)} + \dfrac{2\mu}{\lambda+\mu}\varphi_1(z) - 2z\,\overline{\varphi_1'(z)} - \overline{\psi_1(z)} - \\ \qquad - \dfrac{3\lambda+4\mu}{\lambda+\mu}\big(f_2(z) + z\,\overline{f_2'(z)}\,\big)\Big], \\ \overset{(2)}{\sigma}_{+} = 2\mu\partial_{\overline{z}}\overset{(2)}{u}_3 = 2\mu\,\dfrac{\lambda+2\mu}{\lambda+\mu}\,h\overline{f}_2'', \\ \overset{(1)}{\sigma}_{33} = \lambda\overset{(1)}{\theta} + (\lambda+2\mu)\,\dfrac{2}{h}\overset{(2)}{u}_3 = \\ \quad = 2\mu\Big[\dfrac{\lambda}{\lambda+\mu}\big(\varphi'(z)+\overline{\varphi'(z)}\,\big) + \dfrac{3\lambda+4\mu}{\lambda+\mu}\big(f_2'(z)+\overline{f_2'(z)}\,\big)\Big]; \end{cases} \tag{7.5}$$

4) the boundary conditions on the border of the circle $|z| = R$

$$\begin{cases} \overset{(1)}{\sigma}_{(rr)} + i\overset{(1)}{\sigma}_{(r\vartheta)} = 2\mu\Big[\varphi_1'(z) + \overline{\varphi_1'(z)} - \dfrac{\mu}{\lambda+\mu}\big(f_2'(z) + f_2'(z)\big) - \\ \qquad - \Big(z\,\overline{\varphi_1''(z)} + \overline{\psi'(z)} + z\,\overline{f_2''(z)}\Big)e^{-2i\vartheta}\Big\}, \\ \overset{(0)}{\sigma}_{(r3)} = \dfrac{\mu}{h}\,\mathrm{Re}\Big\{\Big[2hf_0'(z) + \dfrac{2\mu}{\lambda+\mu}\varphi_1(z) - \\ \quad -2z\,\overline{\varphi_1'(z)} - \overline{\psi_1(z)} - \dfrac{3\lambda+4\mu}{\lambda+\mu}\big(f_2(z) + z\,\overline{f_2'(z)}\,\big)\Big]e^{-i\vartheta}\Big\}, \\ \overset{(2)}{\sigma}_{(r3)} = 2\mu\,\dfrac{\lambda+2\mu}{\lambda+\mu}\,h\,\mathrm{Re}\big[\,\overline{f_2''(z)}e^{-i\vartheta}\big]. \end{cases} \tag{7.6}$$

8 Lurie's Method (N=2)

Let's solve the above-mentioned problem.

We have

(I_1') conditions at infinity

$$(\overset{(1)}{\sigma}_{11})^{\infty} = \boldsymbol{p} = const, \quad (\overset{(1)}{\sigma}_{12} = \overset{(1)}{\sigma}_{22} = \overset{(0)}{\sigma}_{+} = \overset{(2)}{\sigma}_{+} = \overset{(1)}{\sigma}_{33})^{\infty} = 0,$$

$$\begin{cases} \big(\overset{(1)}{\sigma}_{11} - \overset{(1)}{\sigma}_{22} + 2i\overset{(1)}{\sigma}_{12}\big)^{\infty} = \boldsymbol{p} = -4\mu\bar{b}_0 \\ \big(\overset{(1)}{\sigma}_{11} + \overset{(1)}{\sigma}_{22}\big)^{\infty} = \boldsymbol{p} = 8\mu\Big(a_0 - \dfrac{\mu}{\lambda+\mu}\,d_0\Big) \\ (\overset{(0)}{\sigma}_{+})^{\infty} = \dfrac{\mu}{h}\Big[2h\overline{f}_0' - \dfrac{2\lambda}{\lambda+\mu}\,a_0 z - b_0\overline{z} - 2\,\dfrac{3\lambda+4\mu}{\lambda+\mu}\,d_0 z\Big] = 0 \\ (\overset{(2)}{\sigma}_{+})^{\infty} = 2\mu\,\dfrac{\lambda+2\mu}{\lambda+\mu}\,h[\,\overline{f}_2''\,]^{\infty} = 0 \\ (\overset{(1)}{\sigma}_{11})^{\infty} = \dfrac{4\mu}{\lambda+\mu}\big[\lambda a_0 + (3\lambda+4\mu)d_0\big] = 0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} b_0 = \bar{b}_0 = -\dfrac{\boldsymbol{p}}{4\mu} \\ a_0 = \overline{a}_0 = \dfrac{\boldsymbol{p}}{8\mu}\cdot\dfrac{(\lambda+\mu)(3\lambda+4\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\ d_0 = \overline{d}_0 = -\dfrac{\boldsymbol{p}}{8\mu}\cdot\dfrac{\lambda(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\ f_0' = -\dfrac{\boldsymbol{p}}{8\mu h}\,z + \dfrac{c_1}{z} + \dfrac{c_3}{z^3} \end{cases} ; \quad (8.1)$$

(I_2') the boundary conditions on the border of the circle $|z| = R$

$$\begin{cases} \overset{(1)}{\sigma}_{(rr)} + i\overset{(1)}{\sigma}_{(r\vartheta)}\big|_{r=R} = 0 \\ \overset{(0)}{\sigma}_{(r3)}\big|_{r=R} = 0 \\ \overset{(2)}{\sigma}_{(r3)}\big|_{r=R} = 0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} a_2 = -\dfrac{\boldsymbol{p}R^2}{4\mu} \\ b_2 = -\dfrac{\boldsymbol{p}R^2}{4\mu}\,, \quad b_4 = -\dfrac{3\boldsymbol{p}R^4}{4\mu} \\ d_n = 0, \quad n \geq 1 \\ c_1 = -\dfrac{\boldsymbol{p}R^2}{8\mu h}\,, \quad c_3 = -\dfrac{\boldsymbol{p}R^4}{8\mu h}\cdot\dfrac{\lambda+3\mu}{\lambda+\mu} \end{cases} . \quad (8.2)$$

(8.1) and (8.2) $\Longrightarrow$

$$\begin{cases} \varphi'(z) = z_0 + \dfrac{a_2}{z^2} \\ \psi'(z) = b_0 + \dfrac{b_2}{z^2} + \dfrac{b_4}{z^4} \\ f_0(z) = -\dfrac{\boldsymbol{p}}{8\mu h}\, z + \dfrac{c_1}{z} + \dfrac{c_3}{z^3} \\ f_2'(z) = d_0 \end{cases} . \tag{8.3}$$

(1.6) and (8.3) $\Longrightarrow$

$$\begin{cases} \overset{(1)}{\sigma}_{(rr)} = \dfrac{\boldsymbol{p}}{2}\left[1 - \dfrac{R^2}{r^2}\right] + \dfrac{\boldsymbol{p}}{2}\left[1 - \dfrac{4R^2}{r^2} + \dfrac{3R^4}{r^4}\right]\cos 2\vartheta \\ \overset{(1)}{\sigma}_{(\vartheta\vartheta)} = \dfrac{\boldsymbol{p}}{2}\left[1 + \dfrac{R^2}{r^2}\right] - \dfrac{\boldsymbol{p}}{2}\left[1 + \dfrac{3R^4}{r^4}\right]\cos 2\vartheta \\ \overset{(1)}{\sigma}_{(r\vartheta)} = -\dfrac{\boldsymbol{p}}{2}\left[1 + \dfrac{2R^2}{r^2} - \dfrac{3R^4}{r^4}\right]\sin 2\vartheta \end{cases} . \tag{8.4}$$

At the internal boundary

$$\overset{(1)}{\sigma}_{(rr)} = \overset{(1)}{\sigma}_{(r\vartheta)} = 0,$$

while the value of $\overset{(1)}{\sigma}_{(\vartheta\vartheta)}$ is given by

$$\overset{(1)}{\sigma}_{(\vartheta\vartheta)} = \boldsymbol{p}[1 - 2\cos 2\vartheta] \ \text{ on } L. \tag{8.5}$$

The maximum value of $\overset{(1)}{\sigma}_{(\vartheta\vartheta)}$ thus occurs for $\cos 2\vartheta = -1$, i.e. for $\vartheta = \pm\frac{\pi}{2}$, where

$$\max \overset{(1)}{\sigma}_{(\vartheta\vartheta)} = 3\boldsymbol{p} \tag{8.6}$$

so that the value of the tensile is increased.

9 Problem II′. *Bi-Axial Bending.*

We have

(II_1') conditions at infinity

$$(\overset{(1)}{\sigma}_{11})^{\infty} = (\overset{(1)}{\sigma}_{22})^{\infty} = \boldsymbol{p} = const, \quad (\overset{(1)}{\sigma}_{12} = \overset{(0)}{\sigma}_{+} = \overset{(2)}{\sigma}_{+} = \overset{(1)}{\sigma}_{33})^{\infty} = 0,$$

$$\begin{cases} \big(\overset{(1)}{\sigma}_{11} - \overset{(1)}{\sigma}_{22} + 2i\overset{(1)}{\sigma}_{12}\big)^{\infty} = 0 \\ \big(\overset{(1)}{\sigma}_{11} + \overset{(1)}{\sigma}_{22}\big)^{\infty} = 2\boldsymbol{p} \\ (\overset{(0)}{\sigma}_{+})^{\infty} = \dfrac{\mu}{h}\left[2h\overline{f}'_0 - \dfrac{2\lambda}{\lambda+\mu}\, a_0 z - 2\,\dfrac{3\lambda+4\mu}{\lambda+\mu}\, d_0 z\right] = 0 \\ (\overset{(2)}{\sigma}_{+})^{\infty} = 0 \\ (\overset{(1)}{\sigma}_{33})^{\infty} = \lambda a_0 + (3\lambda+4\mu) d_0 = 0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} a_0 = \dfrac{\boldsymbol{p}}{4\mu}\cdot\dfrac{(\lambda+\mu)(3\lambda+4\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\ b_0 = 0 \\ d_0 = -\dfrac{\boldsymbol{p}}{4\mu}\cdot\dfrac{\lambda(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\ f'(0) \equiv 0 \end{cases} ; \quad (9.1)$$

(II$'_2$) the boundary conditions on the border of the circle $|z| = R$

$$\begin{cases} \overset{(1)}{\sigma}_{(rr)} + i\overset{(1)}{\sigma}_{(r\vartheta)}\big|_{r=R} = 0 \\ \overset{(0)}{\sigma}_{(r3)}\big|_{r=R} = 0 \\ \overset{(2)}{\sigma}_{(r3)}\big|_{r=R} = 0 \end{cases} \Longrightarrow \begin{cases} a_2 = 0 \\ b_2 = \dfrac{\boldsymbol{p}R^2}{4\mu}\,, \quad b_4 = 0 \\ d_0, \quad n \geq 1 \\ c_1 = -\dfrac{\boldsymbol{p}R^2}{2\mu h} \end{cases} . \quad (9.2)$$

(9.1) and (9.2) $\Longrightarrow$

$$\begin{cases} \varphi'(z) = a_0 \\ \psi'(z) = \dfrac{b_2}{z^2} \\ f'_0(z) = \dfrac{c_1}{z} \\ f'_2(z) = d_0 \end{cases} . \quad (9.3)$$

(1.6) and (9.3) $\Longrightarrow$

$$\begin{cases} \overset{(1)}{\sigma}_{(rr)} = \boldsymbol{p}\Big[1 - \dfrac{R^2}{r^2}\Big] \\ \overset{(1)}{\sigma}_{(\vartheta\vartheta)} = \boldsymbol{p}\Big[1 + \dfrac{R^2}{r^2}\Big] \\ \overset{(1)}{\sigma}_{(r\vartheta)} = 0 \end{cases} . \quad (9.4)$$

At the internal boundary

$$\overset{(1)}{\sigma}_{(rr)} = \overset{(1)}{\sigma}_{(r\vartheta)} = 0, \quad \overset{(1)}{\sigma}_{(\vartheta\vartheta)} = 2\boldsymbol{p}.$$

Thus we have considered and solved Kirsch's type problems: case of uni-directional and bi-axial tensions, as well as the uniform normal pressure, applied to the edge of a circular hole.

For A. Lurie first approximation the coefficient of concentration $k = 3$, whereas I. Vekua's coefficient of concentration depends on Lame's constants λ and μ ($\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$, $\mu = \frac{E}{2(1+\sigma)}$, E is the modulus of elasticity, σ is the Poisson's coefficient) and $\frac{R}{h}$ value (R is the radius of a circular hole, h is the thickness of a plate).

As to the bending, besides A. Lurie and I. Vekua's refined theories, we consider E. Reissner's well-known theory, which refers only to the bending.

For the first approximation according to Lurie $\sigma_{+}^{\infty} \neq 0$ and $\sigma_{33}^{\infty} \neq 0$ and according to I. Vekua $\sigma_{+}^{\infty} = 0$ and $\sigma_{33}^{\infty} \neq 0$, therefore we consider the second approximation $N = 2$.

In Lurie's case the coefficient of concentration equals to three $k = 3$, but it should be noted, that γ parameter was introduced in to the theory of plates by Prof. T. Vashakmadze, which connects different refined theories.

10 The Stress Coefficient

In this case the stress coefficient of concentration has the following form

$$K = \frac{\max \sigma_{(\vartheta\vartheta)}}{\boldsymbol{p}} = 1 + 2\,\frac{(1+\sigma)k_2(\varkappa R)}{2k_0(\varkappa R) + (1+\sigma)k_2(\varkappa R)}\,, \tag{10.1}$$

where

$$\varkappa R = \sqrt{\frac{3}{1+2\gamma}\,\frac{R}{h}}\,. \tag{10.2}$$

From this formula it is clear that the coefficient of concentration K depends not only on the material σ, but on $\frac{R}{h}$ relation and on γ as well.

Assume that

$$X = \varkappa R = \sqrt{\frac{3}{1+2\gamma}\,\frac{R}{h}}\,.$$

For X (with a large radius or small thickness) taking into consideration the asymptotic formulas

$$K_n(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}\Big(1 + o\Big(\frac{1}{x}\Big)\Big), \tag{10.3}$$

we obtain the coefficient of concentration for the classical case $k = \frac{5+3\sigma}{3+\sigma}$, σ is Poisson's coefficient.

When $\gamma = -0.5$ we obtain the same result.

When $\gamma = 0$ we obtain the same result as in I. Vekua's theory.

When $\gamma = 0.1$ we obtain the same result as in E. Reissner's theory.

The formula (10.2) for small X (with a small radius or large thickness) gives the coefficient of concentration K, which is equal to 3, that is the same result as Lurie's theory, when $N = 2$ (the second approximation).

Note: for small X the classical result isn't obtained.

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