ANALYTICAL SOLUTION OF CLASSICAL AND NON-CLASSICAL BOUNDARY VALUE CONTACT PROBLEMS OF THERMOELASTICITY FOR SPHERICAL BODIES CONSISTING OF COMPRESSIBLE AND INCOMPRESSIBLE ELASTIC LAYERS

N. Khomasuridze, R. Janjgava, N. Zirakashvili

I.Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2 University Str., Tbilisi 0186, Georgia

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Abstract

Static thermoelastic equilibrium is considered for an N-layer along the radial coordinate body bounded by coordinate surfaces of a spherical system of coordinates. Each layer is isotropic and homogeneous and some of the layers may be composed of an incompressible elastic material. On the spherical surfaces of the involved body changes in the temperature or its normal derivative, stresses, displacements or their combinations are defined while on the remaining part of the boundary special type of homogeneous conditions are given. The stated problems are analytically solved using the method of separation of variables, the general solution being represented by means of harmonic functions. Problem solution is reduced to the solution of systems of algebraic equations with block diagonal matrices.

Key words and phrases: Incompressible elastic material, Spherical body, Method of separation of variables, Analytical solutions.

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1 Introduction

The paper deals with the solution of some application problems of thermoelasticity for a multilayer spherical body. The work has been fulfilled by a financial support of National Science Foundation for Application Research AR/91/5-109/11 (Agreement N 10/17). There is a number of works [1-6] devoted to the solution of three-dimensional problems of elastic equilibrium of spherical bodies. Unlike those works, ours considers a whole class of boundary value and boundary value contact problems of thermoelasticity for multilayer spherical bodies, the great majority of which have been solved for the first time [7-9]. It should be noted that some of the layers may consist of an incompressible elastic material. Analytical solutions are obtained for the above-mentioned problems using the method of separation of variables.

Static thermoelastic equilibrium is considered for a body which is N-layer along the radial coordinate and bounded by coordinate surfaces of a spherical system of coordinates [5],[6]. On spherical boundary surfaces of the body under study changes in the temperature or its normal derivative, stresses, displacements or their combination are considered while on the remaining part of the boundary special type homogeneous conditions are defined [8]. Contact conditions of rigid, sliding or other type of contact may be defined between the layers. The stated problems are analytically solved using the method of separation of variables, the general solution being expressed by means of harmonic functions. Problem solution is reduced to the solution of a system of linear algebraic equations with block-diagonal matrices.

A few words about the solution effectiveness. If it is possible to construct an effective solution of basic boundary value problems of Laplace's equation with zero conditions for $\alpha = \alpha_j$ and $\beta = \beta_j$, where $j = 0, 1$ ($\alpha_0 = 0$) using the method of separation of variables in the domain $\Omega = \{r_0 < r < r_1\ ,\ 0 < \alpha < \alpha_1,\ \ \beta_0 < \beta < \beta_1\}$($r,\ \alpha,\ \beta$ is a spherical system of coordinates), then using the same method in the same domain Ω it will be possible to find with an equal degree of efficiency the thermoelastic equilibrium of the bodies considered in the given paper.

2 Equilibrium Equations. Problem Statement

Consider a multilayer along the radial coordinate spherical body, which in the spherical system of coordinates $r,\ \alpha,\ \beta$ occupies the domain $\Omega = \Omega_1 \bigcup \Omega_2 \bigcup \cdots \bigcup \Omega_k \bigcup \cdots \bigcup \Omega_N$, $\Omega_k = \{r_{k-1} < r < r_k,\ 0 < \alpha < \alpha_1,\ \ \beta_0 < \beta < < \beta_1\}$, $k = \overline{1, N}$, where r_j, α_1, β_1, $j = \overline{0, N}$ are positive constants, while β_0 is a non-negative constant. Hence the domain Ω represents a spherical body which consists of N layers. We assume that $N = \overline{1,\ 12}$, i.e. we can have a one-layer, two-layer, etc. and a twelve-layer body. Note that the maximal number of layers, twelve, is taken for the definiteness of the software in the process of program development for the solution of the problems studied in the paper. If the number of layers exceeds twelve, the program can be easily adjusted, introducing necessary corrections.

Assume that each domain Ω_k is filled with an isotropic homogeneous elastic material. As we know, homogeneous equations of static equilibrium have the following form [7]

$$\partial_r \sigma_{rr}^{(k)} + \frac{1}{r \sin\beta} \partial_\alpha \sigma_{r\alpha}^{(k)} + \frac{1}{r} \partial_\beta \sigma_{r\beta}^{(k)} + \frac{1}{r} \left(2\sigma_{rr}^{(k)} - \sigma_{\alpha\alpha}^{(k)} - \sigma_{\beta\beta}^{(k)} \right) + \frac{\cos\beta}{r \sin\beta} \sigma_{r\beta}^{(k)} = 0,$$

$$\partial_r \sigma_{\alpha r}^{(k)} + \frac{1}{r \sin\beta} \partial_\alpha \sigma_{\alpha\alpha}^{(k)} + \frac{1}{r} \partial_\beta \sigma_{\alpha\beta}^{(k)} + \frac{3}{r} \sigma_{\alpha r}^{(k)} + \frac{2\cos\beta}{r \sin\beta} \sigma_{\alpha\beta}^{(k)} = 0, \qquad (1)$$

$$\partial_r \sigma_{\beta r}^{(k)} + \frac{1}{r \sin\beta} \partial_\alpha \sigma_{\beta\alpha}^{(k)} + \frac{1}{r} \partial_\beta \sigma_{\beta\beta}^{(k)} + \frac{\cos\beta}{r \sin\beta} \left(\sigma_{\beta\beta}^{(k)} - \sigma_{\alpha\alpha}^{(k)} \right) + \frac{3}{r} \sigma_{\beta r}^{(k)} = 0,$$

where $\partial_r \equiv \frac{\partial}{\partial r}$, $\partial_\alpha \equiv \frac{\partial}{\partial \alpha}$, $\partial_\beta \equiv \frac{\partial}{\partial \beta}$; $\sigma_{rr}^{(k)}$, $\sigma_{\alpha\alpha}^{(k)}$, $\sigma_{\beta\beta}^{(k)}$ are normal stresses, $\sigma_{\alpha r}^{(k)} = \sigma_{r\alpha}^{(k)}$, $\sigma_{\beta r}^{(k)} = \sigma_{r\beta}^{(k)}$, $\sigma_{\alpha\beta}^{(k)} = \sigma_{\beta\alpha}^{(k)}$ are tangential stresses. Duhamel-Neumann's relations can be written as [7]

$$\sigma_{rr}^{(k)} = \frac{E^{(k)}}{\left(1+\nu^{(k)}\right)\left(1-2\nu^{(k)}\right)} \left\{ \left(1-\nu^{(k)}\right) \partial_r u^{(k)} + \frac{\nu^{(k)}}{r \sin\beta} \left[\partial_\alpha v^{(k)} + 2\sin\beta u^{(k)} + \partial_\beta (\sin\beta w^{(k)}) \right] \right\} - \beta^{(k)} T^{(k)},$$

$$\sigma_{\alpha\alpha}^{(k)} = \frac{E^{(k)}}{1+\nu^{(k)}} \left\{ \frac{1-\nu^{(k)}}{1-2\nu^{(k)}} \left[\partial_\alpha v^{(k)} + 2\sin\beta u^{(k)} + \partial_\beta (\sin\beta w^{(k)}) \right] - \frac{1}{r} \left(\partial_\beta w^{(k)} + u^{(k)} \right) + \frac{\nu^{(k)}}{1-2\nu^{(k)}} \partial_r u^{(k)} \right\} - \beta^{(k)} T^{(k)}, \qquad (2)$$

$$\sigma_{\beta\beta}^{(k)} = \frac{E^{(k)}}{1+\nu^{(k)}} \left\{ \frac{\nu^{(k)}}{1-2\nu^{(k)}} \left[\partial_\alpha v^{(k)} + 2\sin\beta u^{(k)} + \partial_\beta (\sin\beta w^{(k)}) \right] + \frac{1}{r} \left(\partial_\beta w^{(k)} + u^{(k)} \right) + \frac{\nu^{(k)}}{1-2\nu^{(k)}} \partial_r u^{(k)} \right\} - \beta^{(k)} T^{(k)},$$

$$\sigma_{r\alpha}^{(k)} = \frac{E^{(k)}}{2(1+\nu^{(k)})}\left[r\partial_r\left(\frac{v^{(k)}}{r}\right)+\frac{1}{r\,\sin\beta}\partial_\alpha u^{(k)}\right],$$

$$\sigma_{r\beta}^{(k)} = \frac{E^{(k)}}{2(1+\nu^{(k)})}\left[\frac{1}{r}\partial_\beta u^{(k)}+r\partial_r\left(\frac{w^{(k)}}{r}\right)\right], \qquad (2)$$

$$\sigma_{\alpha\beta}^{(k)} = \frac{E^{(k)}}{2(1+\nu^{(k)})}\left[\partial_\alpha w^{(k)}+\sin^2\beta\partial_\beta\left(\frac{v^{(k)}}{\sin\beta}\right)\right],$$

where $\nu^{(k)}$ is Poisson's ratio of the k-th layer, $E^{(k)}$ is Young's modulus of the k- th layer; $\vec{U}^{(k)} = \left(u^{(k)},\, v^{(k)},\, w^{(k)}\right)$ is the displacement vector of the k–th layer; $\beta^{(k)}$ is a coefficient depending on thermal characteristics of the k–th layer; $T^{(k)}$ is the change in the temperature in the k–th layer satisfying Laplace's equation

$$\left\{\frac{1}{r^2\,\sin^2\beta}\left[\sin\beta\,\partial_\beta\left(\sin\beta\,\partial_\beta\right)+\partial_{\alpha\alpha}\right]+\partial_{rr}+\frac{2}{r}\partial_r\right\}T^{(k)}=0. \qquad (3)$$

We also consider cases when some of the layers of the body under study are of incompressible thermoelastic materials, assuming that the incompressible layers are also isotropic and homogeneous. The incompressible layers are also assumed to be isotropic and homogeneous. Material incompressibility implies a quality enabling the material to preserve its volume unchanged at any point and any strain. Besides, when we investigate static thermoelastic equilibrium we assume that the change in the temperature in incompressible layers, similar to the classical case, satisfies Laplace's equation (3).

Let the j–th layers of the parallelepiped under study represent an incompressible thermoelastic material. Then the equilibrium equation, written in an invariant form, will look as follows [11]

$$\begin{cases} grad\,\left(s^{(j)}+4\nu^{(j)}T^{(j)}\right)-rot\ rot\overrightarrow{U}^{(j)}=0, \\ \quad div\,\overrightarrow{U}^{(j)}=3\nu^{(j)}T^{(j)}, \end{cases} \quad \text{in} \quad \Omega_j \qquad (4)$$

where $s^{(j)}$ is the so called hydrostatic pressure of the$j$–th layer.

On flat boundaries of each layer the following homogeneous boundary conditions are defined:

$$\text{For } \alpha=\alpha_j: \qquad a)\ \sigma^{(k)}_{\alpha\alpha}=0,\ u^{(k)}=0,\ w^{(k)}=0,\ T^{(k)}=0$$

or

$$b)\ v^{(k)}=0,\ \sigma^{(k)}_{\alpha r}=0,\quad \sigma^{(k)}_{\alpha\beta}=0,\ \partial_\alpha T^{(k)}=0, \tag{5}$$

$$r_{k-1}<r<r_k,\quad k=\overline{1,\,N},\quad 0<\beta<\tfrac{\pi}{2};$$

On conical boundary surfaces of each layer the following homogeneous boundary conditions are defined:

$$\text{For } \beta=\beta_j: \qquad a)\ D^{(k)}=0,\ u^{(k)}=0,\ v^{(k)}=0,\ T^{(k)}=0$$

or

$$b)\ w^{(k)}=0,\ \sigma^{(k)}_{\beta r}=0,\quad K^{(k)}_r=0,\ \partial_\beta T^{(k)}=0, \tag{6}$$

$$r_{k-1}<r<r_k,\quad k=\overline{1,\,N},\quad 0<\alpha<\alpha_1\,,$$

where $j=0,\,1$, and $\alpha_0=0$;

$$D^{(k)}=\frac{E^{(k)}(1-\nu^{(k)})}{(1+\nu^{(k)})(1-2\nu^{(k)})}\left\{\frac{1}{r\sin\beta}\right.$$

$$\left.\times\Big(\partial_\alpha v^{(k)}+2u^{(k)}\,\sin\beta+\partial_\beta(w^{(k)}\,\sin\beta)\Big)+\partial_r u^{(k)}\right\}-\tfrac{\gamma^{(k)}E^{(k)}}{1-2\nu^{(k)}}T^{(k)};$$

$$K^{(k)}_r=\frac{E^{(k)}}{2(1+\nu^{(k)})}\frac{1}{r^2\sin\beta}\left\{\partial_\alpha(rw^{(k)})-\partial_\beta(rv^{(k)}\sin\beta\right\}\,.$$

A technical interpretation of boundary conditions (5) and (6) is given in [10]. Conditions (5a) and (6a) are called anti-symmetry conditions and anti-symmetry type conditions, respectively, while conditions (5a) and (5b) are referred to as symmetry conditions and symmetry-type conditions, respectively.

On spherical boundary surfaces the following boundary conditions are defined.

$$\text{For } r = r_{\delta_j}: \quad a) \ \sigma_{rr}^{(j)} = F_1^{(j)}(\alpha,\beta), \quad \sigma_{r\alpha}^{(j)} = \tilde{F}_2^{(j)}(\alpha,\beta),$$
$$\sigma_{r\beta}^{(j)} = \tilde{F}_3^{(j)}(\alpha,\beta), \ T^{(j)} = \tau^{(j)}(\alpha,\beta) \ \text{ or } \ \partial_r T^{(j)} = t^{(j)}(\alpha,\beta)$$
$$\text{or } b) \ u^{(j)} = f_1^{(j)}(\alpha,\beta), \quad v^{(j)} = \tilde{f}_2^{(j)}(\alpha,\beta),$$
$$w^{(j)} = \tilde{f}_3^{(j)}(\alpha,\beta), \ T^{(j)} = \tau^{(j)}(\alpha,\beta) \ \text{ or } \ \partial_r T^{(j)} = t^{(j)}(\alpha,\beta) \tag{7}$$
$$\text{or } c) \ u^{(j)} = f_1^{(j)}(\alpha,\beta), \quad \sigma_{r\alpha}^{(j)} = \tilde{F}_2^{(j)}(\alpha,\beta),$$
$$\sigma_{r\beta}^{(j)} = \tilde{F}_3^{(j)}(\alpha,\beta), \ T^{(j)} = \tau^{(j)}(\alpha,\beta) \ \text{ or } \ \partial_r T^{(j)} = t^{(j)}(\alpha,\beta)$$
$$\text{or } d) \ \sigma_{rr}^{(j)} = F_1^{(j)}(\alpha,\beta), \quad v^{(j)} = \tilde{f}_2^{(j)}(\alpha,\beta),$$
$$w^{(j)} = \tilde{f}_3^{(j)}(\alpha,\beta), \ T^{(j)} = \tau^{(j)}(\alpha,\beta) \ \text{ or } \ \partial_r T^{(j)} = t^{(j)}(\alpha,\beta),,$$

where $j = 1,\ N$, $\delta_j = \begin{cases} 0, & j = 1, \\ N, & j = N, \end{cases}$ and $r_0 = 0$; $F_1^{(j)}(\alpha,\ \beta)$, $\tilde{F}_i^{(j)}(\alpha,\ \beta)$, $f_1^{(j)}(\alpha,\ \beta)$ $\tilde{f}_i^{(j)}(\alpha,\ \beta)$, $t^{(j)}(\alpha,\ \beta)$ $\tau^{(j)}(\alpha,\ \beta)$, $i = 2, 3$ are defined functions satisfying matching conditions on the facets of the curvilinear coordinate parallelepiped under study. Extra conditions imposed on these functions will be considered in what follows.

As for contact conditions, the authors consider cases when contact conditions of rigid or sliding contact between the layers can be defined.

Rigid contact conditions have the following form:

$$r = r_j: \ u^{(j)} = u^{(j+1)}, \quad v^{(j)} = v^{(j+1)}, \quad w^{(j)} = w^{(j+1)};$$
$$\sigma_{r\alpha}^{(j)} = \sigma_{r\alpha}^{(j+1)}, \quad \sigma_{r\beta}^{(j)} = \sigma_{r\beta}^{(j+1)}, \quad \sigma_{rr}^{(j)} = \sigma_{rr}^{(j+1)}; \tag{8}$$
$$T^{(j)} = T^{(j+1)}, \ \mathrm{k}^{(\mathrm{j})}\partial_{\mathrm{r}} T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)}, \ j = \overline{1, N-1}.$$

The form of sliding contacts is the following:

$$r = r_j: \ \sigma_{rr}^{(j)} = \sigma_{rr}^{(j+1)}, \quad u^{(j)} = u^{(j+1)}, \quad \sigma_{r\alpha}^{(j)} = 0,$$

$$\sigma_{r\alpha}^{(j+1)} = 0, \quad \sigma_{r\beta}^{(j)} = 0, \quad \sigma_{r\beta}^{(j+1)} = 0;$$

$$T^{(j)} = T^{(j+1)}, \ \mathrm{k}^{(\mathrm{j})}\partial_r T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)}, \ j = \overline{1, N-1}. \tag{9}$$

In formulas (8) and (9) $k^{(j)}$ is the heat conductivity coefficient of the j-th layer.

Hence, contact conditions of type (8) (rigid contact) or contact conditions of type (9) (sliding contact) may be defined between the layers $\Omega_j$and Ω_{j+1}. Although the paper deals with contact conditions (8) or (9), we can also solve analytically boundary value contact problems when other types of contact conditions between neighboring layers may be defined, e.g. one can study the following type of contact conditions:

$$r = r_j: \ v^{(j)} = v^{(j+1)}, \ w^{(j)} = w^{(j+1)}, \ \sigma_{r\beta}^{(j)} = \sigma_{r\beta}^{(j+1)}, \ \sigma_{r\alpha}^{(j)} = \sigma_{r\alpha}^{(j+1)},$$

$$\sigma_{rr}^{(j)} = 0, \quad \sigma_{rr}^{(j+1)} = 0; \tag{10}$$

$$T^{(j)} = T^{(j+1)}, \ \mathrm{k}^{(\mathrm{j})}\partial_{\mathrm{r}} T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)}, \ j = \overline{1, N-1}.$$

Furthermore, using the method of separation of variables, the function $T^{(k)}$ in the domain $\Omega_k$is represented as

$$T^{(k)} = t_0^{(k)} + \frac{t_1^{(k)}}{r} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{Tmn}^{(k)}\left(\frac{r_{k-1}}{r}\right)^{n_T+1} + B_{Tmn}^{(k)}\left(\frac{r}{r_k}\right)^{n_T}\right]\psi_{mn}(\alpha, \beta). \tag{11}$$

where $t_0^{(k)}$, $t_1^{(k)}$, $A_{Tmn}^{(k)}$, $B_{Tmn}^{(k)}$, n_T are constants. ψ_{mn} is a non-trivial solution of the following regular Sturm-Liouville problem [12]

$$r^2\Delta_2\psi_{mn} + n_T(n_T+1)\psi_{mn} = 0; \tag{12}$$

$$\text{for } \alpha = \alpha_j: \quad a)\ \psi_{mn} = 0 \ \text{ or } \ b)\ \partial_\alpha\psi_{mn} = 0; \tag{13}$$

$$\text{for } \beta = \beta_j: \quad a)\ \psi_{mn} = 0\text{or} \ \ b)\ \partial_\beta\psi_{mn} = 0; \tag{14}$$

$$\Delta_z = \frac{1}{r^2\sin^2\beta}\left[\sin\beta\ \partial_\beta\left(\sin\beta\frac{\partial}{\partial\beta}\right) + \partial_{\alpha\alpha}\right]$$

The function ψ_{mn} will be represented as

$$\psi_{mn} = A(\alpha,\ m) \cdot B(\beta,\ m,\ n),$$

In this case possible expressions for the functions $A(\alpha,\ m)$ and $B(\beta,\ m,\ n)$ have the following form.

1) If conditions (5a) are satisfied, we have

$$A(m,\ n) = \sin\left(\frac{m\pi\alpha}{\alpha_1}\right);$$

If conditions (5b) are satisfied, we have

$$A(m,\ n) = \cos\left(\frac{m\pi\alpha}{\alpha_1}\right);$$

If for $\alpha = 0$ conditions (5a) are satisfied, while for $\alpha = \alpha_1$ conditions (5b) hold, we have

$$A(m,\ n) = \sin\left(\frac{(2m+1)\pi\alpha}{2\alpha_1}\right);$$

And if $\alpha_1 = 2\pi$, then periodicity conditions imply

$$A(m,\ n) = a_{1m}\cos(m\alpha) + a_{2m}\sin(m\alpha),$$

where a_{1m} and a_{2m} are constants.

2) If $0 < \beta_0 < \beta_1 < \pi$ and conditions (6b) are satisfied, then

$$B(\beta,\ m,\ n) = P_{\tilde{n}}^{-\tilde{m}}(\cos\beta) \cdot \left[\frac{d}{d\beta} Q_{\tilde{n}}^{-\tilde{m}}(\cos\beta)\right]_{\beta=\beta_0} -$$

$$-Q_{\tilde{n}}^{-\tilde{m}}(\cos\beta) \cdot \left[\frac{d}{d\beta} P_{\tilde{n}}^{-\tilde{m}}(\cos\beta)\right]_{\beta=\beta_0},$$

where $P_{\tilde{n}}^{-\tilde{m}}(\cos\beta)$ and $Q_{\tilde{n}}^{-\tilde{m}}(\cos\beta)$are first and second order Legendre functions [13], $\tilde{m} = \frac{\pi m}{\alpha_1}$ or $\tilde{m} = \frac{\pi(2m+1)}{2\alpha_1}$, or $\tilde{m} = m$, and $\tilde{n}$ is the root of number n of the equation $\left[\frac{d}{d\beta} B(\beta,\ m,\ n)\right]_{\beta=\beta_1} = 0$. And if for $\beta = \beta_0$ conditions (6b) are satisfied, while for $\beta = \beta_1$ conditions (6a) hold, then $\tilde{n}$ is the root of the number n of the equation $B(\beta_1,\ m,\ n) = 0$. We have

$$B(\beta,\ m,\ n) = P_{\tilde{n}}^{-\tilde{m}}(\cos\beta) \cdot Q_{\tilde{n}}^{-\tilde{m}}(\cos\beta_0) - Q_{\tilde{n}}^{-\tilde{m}}(\cos\beta) \cdot P_{\tilde{n}}^{-\tilde{m}}(\cos\beta_0),$$

If conditions (6a) are satisfied, and $\tilde{n}$ is the root of number n of the equation $B(\beta_1,\ m,\ n) = 0$.And if for $\beta = \beta_0$ conditions (6a) are satisfied, while for

$\beta = \beta_1$ conditions (6b) hold, then $\tilde{n}$ is the root of number n of the equation $\left[\frac{d}{d\beta} B(\beta,\, m,\, n)\right]_{\beta=\beta_1} = 0.$

If $\beta_0 = 0,\ \beta_1 \neq \frac{\pi}{2}$ and $\beta_1 \neq \pi$, then the boundedness in any cases (only two of which remain) implies

$$B(\beta,\, m,\, n) = P_{\tilde{n}}^{-\tilde{m}}(\cos\beta).$$

For an integer $\tilde{m}$ instead of the index $-\tilde{m}$index $\tilde{m}$ is taken.

In all cases that have been studied for $\tilde{m} = p - \frac{1}{2}$, where $p = 1,\, 2,\, 3,\, ...\,,$ $B(\beta,\, m,\, n)$ is an elementary function.

In order to make it possible to use the method of separation of variables for the solution of the boundary value contact problems considered in the given paper boundary conditions on the boundary spherical surfaces $r = r_0$ and $r = r_N$ are represented in the following manner.

$$\begin{aligned}
&\text{For } r = r_{\delta_j}: \quad a)\ \sigma_{rr}^{(j)} = F_1^{(j)}(\alpha,\beta)\,, \quad \Gamma_1\left(\sigma_{r\alpha}^{(j)}, \sigma_{r\beta}^{(j)}\right) = F_2^{(j)}(\alpha,\beta)\,,\\
&\qquad \Gamma_2\left(\sigma_{r\beta}^{(j)}, \sigma_{r\alpha}^{(1)}\right) = F_3^{(j)}(\alpha,\beta)\,,\ \ T^{(j)} = \tau^{(j)}(\alpha,\beta)\\
&\qquad\qquad \text{or } \partial_r T^{(j)} = t^{(j)}(\alpha,\beta)\\
&\text{or } b)\ u^{(j)} = f_1^{(j)}(\alpha,\beta)\,, \quad \Gamma_1\left(v^{(j)}, w^{(j)}\right) = f_2^{(j)}(\alpha,\beta)\,,\\
&\qquad \Gamma_2\left(w^{(j)}, v^{(j)}\right) = f_3^{(j)}(\alpha,\beta)\,,\ \ T^{(j)} = \tau^{(j)}(\alpha,\beta)\\
&\qquad\qquad \text{or } \partial_r T^{(j)} = t^{(j)}(\alpha,\beta)\\
&\text{or } c)\ u^{(j)} = f_1^{(j)}(\alpha,\beta)\,, \quad \Gamma_1\left(v^{(j)}, w^{(j)}\right) = f_2^{(j)}(\alpha,\beta)\,,\\
&\qquad \Gamma_2\left(\sigma_{r\beta}^{(j)}, \sigma_{r\alpha}^{(j)}\right) = F_3^{(j)}(\alpha,\beta)\,,\ \ T^{(j)} = \tau^{(j)}(\alpha,\beta)\\
&\qquad\qquad \text{or } \partial_r T^{(j)} = t^{(j)}(\alpha,\beta)\\
&\text{or } d)\ \sigma_{rr}^{(j)} = F_1^{(j)}(\alpha,\beta)\,, \quad \Gamma_1\left(\sigma_{r\alpha}^{(j)}, \sigma_{r\beta}^{(j)}\right) = F_2^{(j)}(\alpha,\beta)\,,\\
&\qquad \Gamma_2\left(w^{(j)}, v^{(j)}\right) = f_3^{(j)}(\alpha,\beta)\,,\ \ T^{(j)} = \tau^{(j)}(\alpha,\beta)\\
&\qquad\qquad \text{or } \partial_r T^{(j)} = t^{(j)}(\alpha,\beta)\,,
\end{aligned} \tag{15}$$

where$j = 0$ or $j = N$; $\Gamma_1(g_1, g_2)$ and $\Gamma_2(g_2, g_1)$ are the following differential operators: $\Gamma_1(g_1, g_2) = \frac{1}{\sin\beta}\left[\partial_\alpha g_1 + \partial_\beta(\sin\beta\ g_2)\right]$, $\Gamma_2(g_2, g_1) = \frac{1}{\sin\beta}\left[\partial_\alpha g_2 - \partial_\beta(\sin\beta\ g_1)\right]$, and $g_1 = \sigma_{r\alpha}^{(j)}$ or $g_1 = v^{(j)}$, $g_2 = \sigma_{r\beta}^{(j)}$ or $g_2 = w^{(j)}$. It is assumed that the functions $t^{(j)}(\alpha,\ \beta)$, $F_2^{(j)}(\alpha,\ \beta)$, $F_3^{(j)}(\alpha,\ \beta)$themselves, the functions $\tau^{(j)}(\alpha,\ \beta)$, $F_1^{(j)}(\alpha,\ \beta)$, $f_2^{(j)}(\alpha,\ \beta)$ and $f_3^{(j)}(\alpha,\ \beta)$ together with their first derivatives, the functions $f_1^{(j)}(\alpha,\ \beta)$ together with their first and second derivatives can be expanded into uniformly converging Fourier series with respect to their eigenfunctions of the corresponding regular Sturm-Liouville problem (12), (13), (14).

Contact conditions (8), (9) and (10) are also replaced with the following equivalent conditions.

Conditions of rigid contact

$$
\begin{aligned}
r = r_j :\ & u^{(j)} = u^{(j+1)},\ \Gamma_1\left(v^{(j)}, w^{(j)}\right) = \Gamma_1\left(v^{(j+1)}, w^{(j+1)}\right);\\
& \Gamma_2\left(w^{(j)}, v^{(j)}\right) = \Gamma_2\left(w^{(j+1)}, v^{(j+1)}\right)\\
& \sigma_{rr}^{(j)} = \sigma_{rr}^{(j+1)}, \Gamma_1\left(\sigma_{r\alpha}^{(j)}, \sigma_{r\beta}^{(j)}\right) = \Gamma_1\left(\sigma_{r\alpha}^{(j+1)}, \sigma_{r\beta}^{(j+1)}\right),\\
& \Gamma_2\left(\sigma_{r\beta}^{(j)}, \sigma_{r\alpha}^{(j)}\right) = \Gamma_2\left(\sigma_{r\beta}^{(j+1)}, \sigma_{r\alpha}^{(j+1)}\right),\\
& T^{(j)} = T^{(j+1)},\ \ k^{(\mathrm{j})}\partial_r T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)},\ \ j = \overline{1, N-1}.
\end{aligned}
\tag{16}
$$

Conditions of sliding contact

$$
\begin{aligned}
r = r_j :\quad & u^{(j)} = u^{(j+1)};\ \ \sigma_{rr}^{(j)} = \sigma_{rr}^{(j+1)},\\
& \Gamma_1\left(\sigma_{r\alpha}^{(j)}, \sigma_{r\beta}^{(j)}\right) = 0,\ \ \Gamma_2\left(\sigma_{r\beta}^{(j)}, \sigma_{r\alpha}^{(j)}\right) = 0,\\
& \Gamma_1\left(\sigma_{r\alpha}^{(j+1)}, \sigma_{r\beta}^{(j+1)}\right) = 0,\quad \Gamma_2\left(\sigma_{r\beta}^{(j+1)}, \sigma_{r\alpha}^{(j+1)}\right) = 0.\\
& T^{(j)} = T^{(j+1)},\ k^{(j)}\partial_r T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)},\ j = \overline{1, N-1}.
\end{aligned}
\tag{17}
$$

Contact conditions (10) are replaced with the following equivalent conditions

$$
\begin{aligned}
r = r_j:\ & \Gamma_1\left(v^{(j)}, w^{(j)}\right) = \Gamma_1\left(v^{(j+1)}, w^{(j+1)}\right);\\
& \Gamma_2\left(w^{(j)}, v^{(j)}\right) = \Gamma_2\left(w^{(j+1)}, v^{(j+1)}\right),\\
& \Gamma_1\left(\sigma_{r\alpha}^{(j)}, \sigma_{r\beta}^{(j)}\right) = \Gamma_1\left(\sigma_{r\alpha}^{(j+1)}, \sigma_{r\beta}^{(j+1)}\right),\\
& \Gamma_2\left(\sigma_{r\beta}^{(j)}, \sigma_{r\alpha}^{(j)}\right) = \Gamma_2\left(\sigma_{r\beta}^{(j+1)}, \sigma_{r\alpha}^{(j+1)}\right),\\
& \sigma_{rr}^{(j)} = 0, \quad \sigma_{rr}^{(j+1)} = 0;\\
& T^{(j)} = T^{(j+1)},\ \mathrm{k}^{(\mathrm{j})}\partial_{\mathrm{r}}T^{(j)} = k^{(j+1)}\partial_r T^{(j+1)}, j = \overline{1, N-1}.
\end{aligned}
\tag{18}
$$

Paper [8] proves the equivalence of boundary conditions (7) and (15) under the condition that matching conditions are satisfied on the facets of the curvilinear coordinate parallelepiped under study. Contact conditions (8) and (16), (9) and (17), (10) and (18) will be equivalent, respectively.

When we solve boundary value problems admitting a rigid displacement of an elastic body the boundary conditions also have the requirement of the main vector and main moment being equal to zero.

Thus the aim of the given paper is to construct a regular solution of the stated boundary value contact problems. The solution of system (1), (2) defined by the functions $u^{(k)}$, $v^{(k)}$, $w^{(k)}$is called regular if the functions $u^{(k)}$, $v^{(k)}$, $w^{(k)}$ are three times continuously differentiable in the domain $\tilde{\Omega}^{(k)}$, where $\tilde{\Omega}^{(k)}$ is the domain $\Omega^{(k)}$ together with the boundaries $\alpha = 0$, $\alpha = \alpha_1$, $\beta = 0$ and $\beta = \beta_1$, and on the surfaces $r = r_{k-1}$ and $r = r_k$ can be represented together with their first and second derivatives by absolutely and uniformly converging Fourier series with respect to the eigenfunctions of the corresponding Sturm-Liouville regular problem (12), (13), (14).

3 Solution of the Stated Boundary Value Contact Problems

Paper [8] proves that the general solution of equation system (1), (2) can be represented by means of four harmonic functions. Components of the displacement vector are expressed by the following formulas:

$$
\begin{aligned}
u^{(k)} &= \partial_r\varphi_3^{(k)} - \frac{r^3}{4\left(1-\nu^{(k)}\right)}\left[\partial_{rr}\varphi_2^{(k)} - \frac{4}{r}\partial_r\varphi_2^{(k)}\right.\\
&\left.-2\left(1-2\nu^{(k)}\right)\frac{1}{r^2}\partial_r\varphi_2^{(k)}\right] + \tfrac{\gamma^{(k)}}{2}\partial_r\left(r^2\tilde{T}^{(k)}\right),\\
v^{(k)} &= \frac{1}{\sin\beta}\partial_\alpha\left[\frac{\varphi_3^{(k)}}{r} - \frac{1}{4\left(1-\nu^{(k)}\right)}\partial_r\left(r^2\varphi_2^{(k)}\right) - r\varphi_2^{(k)}\right.\\
&\left.+\frac{\gamma^{(k)}}{2}r\tilde{T}^{(k)}\right] + \partial_\beta\varphi_1^{(\mathrm{k})}\\
w^{(k)} &= \partial_\beta\left[\tfrac{\varphi_3^{(k)}}{r} - \tfrac{1}{4\left(1-\nu^{(k)}\right)}\partial_r\left(r^2\varphi_2^{(k)}\right) - r\varphi_2^{(k)} + \tfrac{\gamma^{(k)}}{2}r\tilde{T}^{(k)}\right]\\
&-\frac{1}{\sin\beta}\partial_\alpha\varphi_1^{(\mathrm{k})},\ k=\overline{1,N}.
\end{aligned}
\tag{19}
$$

where $\varphi_1^{(k)}$, $\varphi_2^{(k)}$, $\varphi_3^{(k)}$ are arbitrary harmonic functions in the domain Ω_k; $\gamma^{(k)}$is a linear thermal expansion coefficient of the k-th layer, which is expressed through $\beta^{(k)}$ by means of the formula

$$\gamma^{(k)} = \frac{1-2\nu^{(k)}}{2\mu_k\left(1+\nu^{(k)}\right)}\beta^{(k)};$$

$\tilde{T}^{(k)}$ is also a harmonic function, which is connected to the function $T^{(k)}$ by means of the relation

$$T^{(k)} = \frac{1-\nu^{(k)}}{1+\nu^{(k)}}\left(2r\partial_r\tilde{T}^{(k)} + 3\tilde{T}^{(k)}\right).$$

If we substitute formulas (10) in Duhamel-Neumann relation (2), we obtain stress representations by means of the introduced harmonic functions. In particular, for stresses $\sigma_{rr}^{(k)}$, $\sigma_{r\alpha}^{(k)}$ and $\sigma_{r\beta}^{(k)}$ we will have formulas

$$\frac{1}{2\mu^{(k)}}\sigma_{rr}^{(k)}=\partial_{rr}\left[\varphi_3^{(k)}-\frac{r}{4\left(1-\nu^{(k)}\right)}\partial_r\left(r^2\varphi_2^{(k)}\right)-r^2\varphi_2^{(k)}\right.$$

$$\left.+\frac{\gamma^{(k)}}{2}r^2\tilde{T}^{(k)}\right]+\frac{2-\nu^{(k)}}{2\left(1-\nu^{(k)}\right)}\partial_r\left(2r^2\partial_r\varphi_2^{(k)}+3r\varphi_2^{(k)}\right)$$

$$-\gamma^{(k)}\left(2r\partial_r\tilde{T}^{(k)}+3\tilde{T}^{(k)}\right),$$

$$\frac{1}{\mu^{(k)}}\sigma_{r\alpha}^{(k)}=\frac{1}{\sin\beta}\partial_\alpha\left[2\partial_{\mathrm{r}}\left(\frac{\varphi_3^{(k)}}{r}-\frac{1}{4\left(1-\nu^{(k)}\right)}\partial_r\left(r^2\varphi_2^{(k)}\right)\right.\right. \tag{20}$$

$$\left.\left.+\frac{\gamma^{(k)}}{2}r\tilde{T}^{(k)}\right)+\varphi_2^{(k)}\right]+r\partial_{r\beta}\left(\frac{\varphi_1^{(k)}}{r}\right),$$

$$\frac{1}{\mu^{(k)}}\sigma_{r\beta}^{(k)}=\partial_\beta\left[2\partial_{\mathrm{r}}\left(\frac{\varphi_3^{(k)}}{r}-\frac{1}{4\left(1-\nu^{(k)}\right)}\partial_r\left(r^2\varphi_2^{(k)}\right)\right.\right.$$

$$\left.\left.+\frac{\gamma^{(k)}}{2}r\tilde{T}^{(k)}\right)+\varphi_2^{(k)}\right]-\frac{r}{\sin\beta}\partial_{r\alpha}\left(\frac{\varphi_1^{(k)}}{r}\right).$$

It is proved that the general solution of the system of equilibrium equations for an incompressible material (3) results from formulas (9) and (10), if $\nu^{(k)}$ is substituted by the value $\frac{1}{2}$. Note that in the case of an incompressible material as well boundary conditions (4)-(6) are imposed. The same is also true for contact conditions, i.e. in the case when one of the two neighboring layers is incompressible, contact conditions (7) or (8) still hold.

General solutions of (10) and (11) lead to the following formulas, which prove the appropriateness of the substitution of conditions (7)-(10) by conditions (15)-(18)

$$\frac{1}{2\mu^{(k)}}\sigma_{rr}^{(k)} = \partial_{rr}\left[\varphi_3^{(k)} - \frac{r}{4\left(1-\nu^{(k)}\right)}\partial_r\left(r^2\varphi_2^{(k)}\right) - r^2\varphi_2^{(k)}\right.$$

$$\left.+\frac{k^{(k)}}{2}r^2\tilde{T}^{(k)}\right] + \frac{2-\nu^{(k)}}{2\left(1-\nu^{(k)}\right)}\partial_r\left(2r^2\partial_r\varphi_2^{(k)} + 3r\varphi_2^{(k)}\right)$$

$$-\gamma^{(k)}\left(2r\partial_r\tilde{T}^{(k)} + 3\tilde{T}^{(k)}\right),$$

$$\Gamma_2\left(w^{(k)}, v^{(k)}\right) = \partial_r\left(r^2\partial_r\varphi_1^{(k)}\right),$$

$$\Gamma_1\left(v^{(k)}, w^{(k)}\right) = \frac{1}{4\left(1-\nu^{(k)}\right)}\partial_r\left[r^2\partial_r\left(r^2\partial_r\varphi_2^{(k)}\right)\right] \tag{21}$$

$$+r^2\partial_{rr}\left(r\varphi_2^{(k)}\right) - \partial_{rr}\left(r\varphi_3^{(k)}\right) - \frac{\gamma^{(k)}}{2}r^2\partial_{rr}\left(r\tilde{T}^{(k)}\right),$$

$$\frac{1}{\mu^{(k)}}\Gamma_1\left(\sigma_{r\alpha}^{(k)}, \sigma_{r\beta}^{(k)}\right) = \frac{1}{2\left(1-\nu^{(k)}\right)}\partial_{rr}\left[r^2\partial_r\left(r^2\partial_r\varphi_2^{(k)}\right)\right]$$

$$-\partial_r\left(r^2\partial_r\varphi_2^{(k)}\right) - 2\partial_{rrr}\left(r\varphi_3^{(k)}\right) - \gamma^{(k)}\frac{\partial}{\partial r}\left[r^2\partial_{rr}\left(r\tilde{T}^{(k)}\right)\right],$$

$$\frac{1}{\mu^{(k)}}\Gamma_2\left(\sigma_{r\beta}^{(k)}, \sigma_{r\alpha}^{(k)}\right) = r\partial_{rrr}\left(r\varphi_1^{(k)}\right).$$

In formulas (19), (20) and (21) the functions $\varphi_1^{(k)}$, $\varphi_2^{(k)}$, $\varphi_3^{(k)}$ and $\tilde{T}^{(k)}$have the following form

$$\varphi_j^{(k)} = b_{j0}^{(k)} + \frac{b_{j1}^{(k)}}{r} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{jmn}^{(k)}\left(\frac{r_{k-1}}{r}\right)^{\tilde{n}+1} + B_{jmn}^{(k)}\left(\frac{r}{r_k}\right)^{\tilde{n}}\right]\psi_{mn}(\alpha,\,\beta),$$

$$\varphi_1^{(k)} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{1mn}^{(k)}\left(\frac{r_{k-1}}{r}\right)^{\tilde{n}_1+1} + B_{Tmn}^{(k)}\left(\frac{r}{r_k}\right)^{\tilde{n}_1}\right]\bar{\psi}_{mn}(\alpha,\,\beta), \tag{22}$$

where $j = 0,\, 2,\, 3$, and $\varphi_0 = \tilde{T}$; $\bar{\psi}_{mn}(\alpha,\,\beta)$ is complementary to the function $\psi_{mn}(\alpha,\,\beta)$ in the sense that if $\psi_{mn}|_{\alpha=\alpha_i} = 0$, then $\partial_\alpha\,\bar{\psi}_{mn}\big|_{\alpha=\alpha_i} = 0$, and if $\partial_\alpha\,\psi_{mn}|_{\alpha=\alpha_i} = 0$, then $\bar{\psi}_{mn}\big|_{\alpha=\alpha_i} = 0$ and vice versa; everything is similar for $\beta = \beta_i$ ($i = 0,\, 1$,and $\alpha_0 = 0$).

Similar to the reasoning in paper [8], it can be easily proved that all of the stated boundary value contact problems do not admit more than one regular solution.

Hence, using the above-stated algorithm, an analytical solution of the stated boundary value contact problems can be easily constructed by the method of separation of variables.

As an example consider a multilayer spherical body occupying the domain $\Omega = \Omega_1 \bigcup \Omega_2 \bigcup \cdots \bigcup \Omega_k \bigcup \cdots \bigcup \Omega_N$, $\Omega_k = \{r_{k-1} < r < r_k,\ 0 < \alpha < < \alpha_1,\ 0 < \beta < \frac{\pi}{2}\}$, $k = \overline{1, N}$ (Fig. 1).

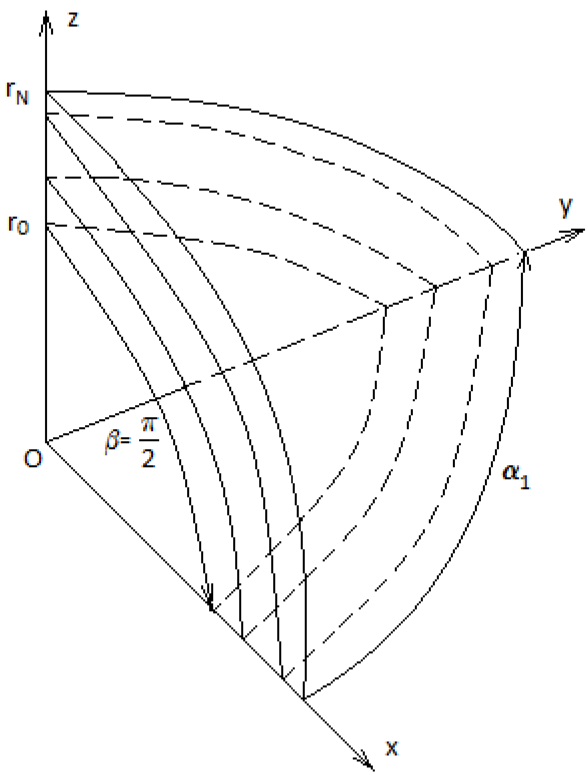

Fig.1. Multilayer spherical body

In the given case $\beta_0 = 0$ and the conical surface $\beta = \beta_0$ degenerates into an interval. We assume that matching conditions are satisfied on this interval. As for the boundary surface $\beta = \beta_1$, it changes into a flat one since in this case $\beta_1 = \frac{\pi}{2}$. In the given case boundary conditions (6a) and (6b) on the flat boundary surface $\beta = \frac{\pi}{2}$ represent, respectively anti-symmetry and symmetry conditions

$$\text{For } \beta = \frac{\pi}{2}: \quad \begin{array}{l} a)\ \sigma_{\beta\beta}^{(k)} = 0, \ \ u^{(k)} = 0, \ \ v^{(k)} = 0, \ \ T^{(k)} = 0 \\ \text{or} \\ b)\ w^{(k)} = 0, \ \ \sigma_{\beta r}^{(k)} = 0, \ \ \sigma_{\beta\alpha}^{(k)} = 0, \ \ \partial_\beta T^{(k)} = 0. \end{array} \tag{23}$$

Taking into account the above-mentioned, we can have six different combinations of symmetry and anti-symmetry boundary conditions on flat boundary surfaces of the involved elastic body (Fig. 2).

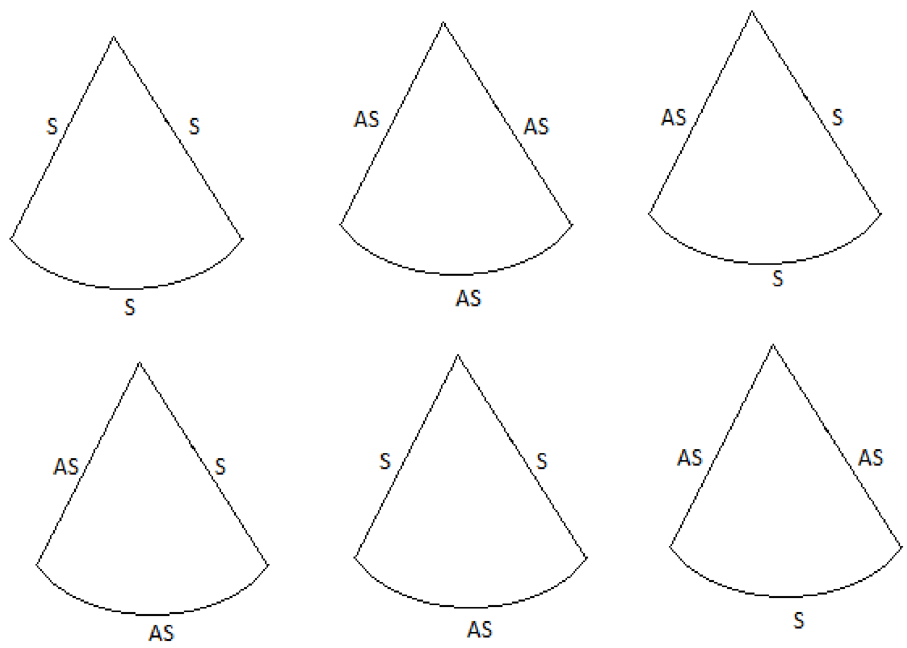

Fig. 2. Boundary conditions on flat boundaries

Then using the method of separation of variables and bearing in mind boundary conditions defined on the flat boundaries of the cylindrical body we can establish the form of the harmonic functions $\varphi_1^{(k)}$, $\varphi_2^{(k)}$, $\varphi_3^{(k)}$ and $\tilde{T}^{(k)}$.

1. When on all the three flat boundary surfaces anti-symmetry conditions are defined the functions $\varphi_1^{(k)}$, $\varphi_2^{(k)}$, $\varphi_3^{(k)}$ and $\tilde{T}^{(k)}$ will have the following form:

$$
\begin{aligned}
\varphi_1^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1}+B_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}\right)\\
&\qquad\times P_{\tilde{m}+2n}^{-\tilde{m}}\left(\cos\beta\right)\cos\left(\tilde{m}\alpha\right),\\
\varphi_2^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(C_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2}+D_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n+1}\right)\\
&\qquad\times P_{\tilde{m}+2n+1}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right),\\
\varphi_3^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(E_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2}+F_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n+1}\right)\\
&\qquad\times P_{\tilde{m}+2n+1}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right),\\
\tilde{T}^{(k)} &= \sum_{m=1}^{\infty}\sum_{m=0}^{\infty}\left(T_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2}+t_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n+1}\right)\\
&\qquad\times P_{\tilde{m}+2n+1}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right),
\end{aligned}
\tag{24}
$$

where

$$
\tilde{m}=\frac{\pi m}{\alpha_1}.
$$

2. When on the boundaries $\alpha=0$ and $\alpha=\alpha_1$ anti-symmetry conditions are given, while on the boundary $\beta=\frac{\pi}{2}$ we have symmetry conditions the functions $\varphi_1^{(k)},\varphi_2^{(k)},\varphi_3^{(k)}$ and $\tilde{T}^{(k)}$ are represented by the series

$$
\begin{aligned}
\varphi_1^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2}+B_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n+1}\right)\\
&\qquad\times P_{\tilde{m}+2n+1}^{-\tilde{m}}\left(\cos\beta\right)\cos\left(\tilde{m}\alpha\right),\\
\varphi_2^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(C_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1}+D_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}\right)\\
&\qquad\times P_{\tilde{m}+2n}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right),\\
\varphi_3^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(E_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1}+F_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}\right)\\
&\qquad\times P_{\tilde{m}+2n}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right),\\
\tilde{T}^{(k)} &= \sum_{m=1}^{\infty}\sum_{m=0}^{\infty}\left(T_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1}+t_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}\right)\\
&\qquad\times P_{\tilde{m}+2n}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right).
\end{aligned}
\tag{25}
$$

3. And when on all of the three flat boundary surfaces symmetry conditions are defined the functions $\varphi_1^{(k)}$, $\varphi_2^{(k)}$, $\varphi_3^{(k)}$ and $\tilde{T}^{(k)}$ take the form

$$\begin{aligned}
\varphi_1^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2} + B_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n+1}\right) \\
&\quad \times P_{\tilde{m}+2n+1}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right), \\
\varphi_2^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(C_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1} + D_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}\right) \\
&\quad \times P_{\tilde{m}+2n}^{-\tilde{m}}\left(\cos\beta\right)\cos\left(\tilde{m}\alpha\right), \\
\varphi_3^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(E_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1} + F_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}\right) \\
&\quad \times P_{\tilde{m}+2n}^{-\tilde{m}}\left(\cos\beta\right)\cos\left(\tilde{m}\alpha\right), \\
\tilde{T}^{(k)} &= \sum_{m=1}^{\infty}\sum_{m=0}^{\infty}\left(T_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1} + t_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}\right) \\
&\quad \times P_{\tilde{m}+2n}^{-\tilde{m}}\left(\cos\beta\right)\cos\left(\tilde{m}\alpha\right).
\end{aligned} \tag{26}$$

4. If on the boundaries $\alpha = 0$ and $\alpha = \alpha_1$ symmetry conditions are defined while on the boundary $\beta = \frac{\pi}{2}$ we have anti-symmetry conditions the functions $\varphi_1^{(k)}, \varphi_2^{(k)}, \varphi_3^{(k)}$ and $\tilde{T}^{(k)}$ will have the form

$$\begin{aligned}
\varphi_1^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1} + B_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}\right) \\
&\quad \times P_{\tilde{m}+2n}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right), \\
\varphi_2^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(C_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2} + D_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n+1}\right) \\
&\quad \times P_{\tilde{m}+2n+1}^{-\tilde{m}}\left(\cos\beta\right)\cos\left(\tilde{m}\alpha\right), \\
\varphi_3^{(k)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(E_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2} + F_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n+1}\right) \\
&\quad \times P_{\tilde{m}+2n+1}^{-\tilde{m}}\left(\cos\beta\right)\cos\left(\tilde{m}\alpha\right), \\
\tilde{T}^{(k)} &= \sum_{m=1}^{\infty}\sum_{m=0}^{\infty}\left(T_{mn}^{(k)}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2} + t_{mn}^{(k)}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n+1}\right) \\
&\quad \times P_{\tilde{m}+2n+1}^{-\tilde{m}}\left(\cos\beta\right)\sin\left(\tilde{m}\alpha\right).
\end{aligned} \tag{27}$$

5. When on the boundaries $\alpha = 0$ and $\beta = \frac{\pi}{2}$ anti-symmetry conditions are defined, while on the boundary $\alpha = \alpha_1$ symmetry conditions are given the functions $\varphi_1^{(k)}, \varphi_2^{(k)}, \varphi_3^{(k)}$ and $\tilde{T}^{(k)}$ will have the form (24), where $\tilde{m}$ is expressed by the formula

$$\tilde{m} = \frac{(2m+1)\,\pi}{2\alpha_1}. \tag{28}$$

6. When on the boundaries $\alpha = \alpha_1$ and $\beta = \frac{\pi}{2}$ symmetry conditions are given, while on the boundary $\alpha = 0$ anti-symmetry conditions are defined the functions $\varphi_1^{(k)}, \varphi_2^{(k)}, \varphi_3^{(k)}$ and $\tilde{T}^{(k)}$ will have the form (25), where $\tilde{m}$ is expressed by formula (28).

In order to find the desired displacements and stresses representations (24)-(27) of the functions $\varphi_1^{(k)}, \varphi_2^{(k)}, \varphi_3^{(k)}$ and $\tilde{T}^{(k)}$, for example, are taken in the corresponding formulas (19) or (20). In particular, when on all of the three flat boundary surfaces anti-symmetry conditions are defined we have the following expressions for the displacements

$$u^{(k)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{-\frac{r}{4(1-\nu^{(k)})}\left[(\tilde{m}+2n+1)(\tilde{m}+2n+2)\right.\right.$$
$$\left.+4(\tilde{m}+2n+1)-2(1-2\nu^{(k)})\right]\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1} C_{mn}^{(k)}$$
$$-\tfrac{r}{4(1-\nu^{(k)})}\left[(\tilde{m}+2n)(\tilde{m}+2n-1)-4(\tilde{m}+2n)\right.$$
$$\left.-2(1-2\nu^{(k)})\right]\left(\frac{r}{r_1}\right)^{\tilde{m}+2n} D_{mn}^{(k)} - (\tilde{m}+2n+1)\frac{1}{r_0}$$
$$\times\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2} E_{mn}^{(k)} + (\tilde{m}+2n)\frac{1}{r_1}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n-1} F_{mn}^{(k)}$$
$$+\frac{\gamma^{(k)}}{2}\left[(-\tilde{m}-2n+1)r\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1} T_{mn}^{(k)} + (\tilde{m}+2n+2)r\right.$$
$$\left.\left.\times\left(\frac{r}{r_1}\right)^{\tilde{m}+2n} t_{mn}^{(k)}\right]\right\} P_{\tilde{m}+2n}^{-\tilde{m}}(\cos\beta)\sin(\tilde{m}\alpha),$$

$$\Gamma_2\left(w^{(k)}, v^{(k)}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{[-2(\tilde{m}+2n+2)+(\tilde{m}+2n+2)$$
$$\times(\tilde{m}+2n+3)]\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+2} A_{mn}^{(k)} + [2(\tilde{m}+2n+1)+$$
$$\left.+(\tilde{m}+2n+1)(\tilde{m}+2n)]\left(\tfrac{r}{r_1}\right)^{\tilde{m}+2n+1} B_{mn}^{(k)}\right\}$$
$$\times P_{\tilde{m}+2n+1}^{-\tilde{m}}(\cos\beta)\cos(\tilde{m}\alpha),$$

$$
\begin{aligned}
\Gamma_1\left(v^{(k)}, w^{(k)}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Bigg\{ &\frac{r}{2(1-\nu^{(k)})}\left[-\frac{1}{2}(\tilde{m}+2n+3)\right.\\
&\times(\tilde{m}+2n+2)(\tilde{m}+2n+1)+(5-2\nu^{(k)})(\tilde{m}+2n+1)\\
&\left.\times(\tilde{m}+2n+2)-(7-4\nu^{(k)})(\tilde{m}+2n+1)\right]\left(\tfrac{r_0}{r}\right)^{\tilde{m}+2n+1}C_{mn}^{(k)}\\
&+\frac{r}{2(1-\nu^{(k)})}\left[\frac{1}{2}(\tilde{m}+2n)\,(\tilde{m}+2n-1)(\tilde{m}+2n-2)\right.\\
&\left.+(5-2\nu^{(k)})(\tilde{m}+2n)(\tilde{m}+2n-1)+(7-4\nu^{(k)})(\tilde{m}+2n)\right]\\
&\times\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}D_{mn}^{(k)}-(\tilde{m}+2n)(\tilde{m}+2n+1)\frac{1}{r}\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1}E_{mn}^{(k)}\\
&-(\tilde{m}+2n)(\tilde{m}+2n+1)\frac{1}{r}\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}F_{mn}^{(k)}\\
&-\frac{\gamma^{(k)}}{2}\left[(\tilde{m}+2n)(\tilde{m}+2n+1)r\left(\frac{r_0}{r}\right)^{\tilde{m}+2n+1}T_{mn}^{(k)}+(\tilde{m}+2n)\right.\\
&\left.\left.\times(\tilde{m}+2n+1)r\left(\frac{r}{r_1}\right)^{\tilde{m}+2n}t_{mn}^{(k)}\right]\right\}P_{\tilde{m}+2n}^{-\tilde{m}}(\cos\beta)\sin(\tilde{m}\alpha).
\end{aligned}
$$

The defined functions on the spherical boundary surfaces of a multilayer spherical body are expanded into corresponding series. The expressions for the functions defined on the boundary and contact surfaces and the appropriate expansions into series of functions defined on the given spherical boundary surfaces of the body are used for the corresponding conditions (7), (8) or (9) and expressions with identical combinations of trigonometric functions and Legendre's functions are equated. As a result, with respect to the desired coefficients of harmonic functions, we obtain systems of linear algebraic equations the main matrix of which has a block diagonal form for a fixed $m = \bar{m}$, which is shown in Fig.3.

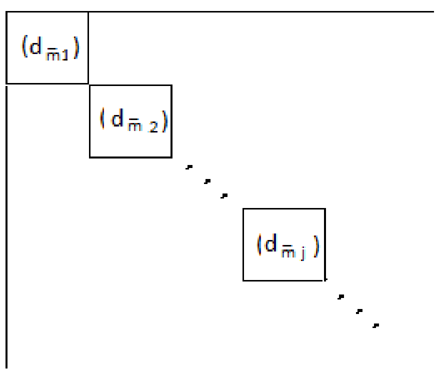

Fig.3. Form of the main matrix

In all of the six cases of boundary conditions on the flat boundaries of the spherical body that we have considered the main matrix corresponding to the linear algebraic equations of any of the above-stated problems will have the form shown in Fig.3. Each of its (d_{ij}) blocks represents a matrix of the following form ($\times$ denotes non-zero elements of the matrix):

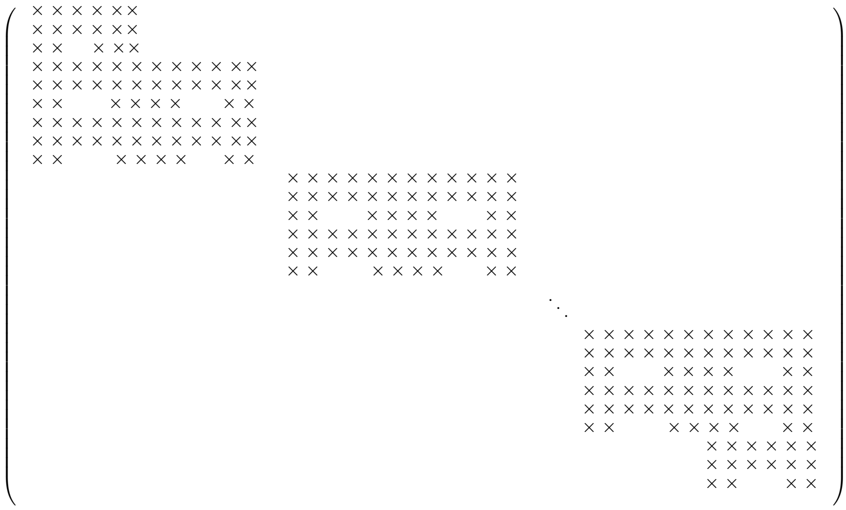

Fig. 4. Form of the matrix $(d_{ij})(i = 1, 2, \ldots; \quad j = 1, 2, \ldots)$

The N-layer spherical body under study has $N-1$ spherical contact surfaces the contact conditions on which generate $6\,(N-1)$ equations while boundary conditions on the spherical boundary surfaces of the body give another three equations each. Consequently, for every fixed m and n system of $6N$ equations is obtained with the same number of unknowns, i.e. the main matrix represents a square matrix of the dimension $6N \times 6N$.

As for the changes in the temperature, this problem is reduced to integration of Laplace's equation for a multilayer spherical domain and it has been thoroughly studied in mathematical physics.

For the numerical implementation and visual representation of the solutions of the above-mentioned problems a computer program is developed using MATLAB language[14], [15]. This program has a very simple form, it is user-friendly and, if necessary, can be supplemented with a required computation program modulus.

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