ERROR ESTIMATE OF A SOLUTION OF THE CRANK-NICOLSON SEMIDISCRETE SCHEME FOR A QUASILINEAR EVOLUTIONARY EQUATION IN A BANACH SPACE

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Abstract

The Crank-Nicolson semidiscrete scheme is considered for a quasilinear evolutionary equation with a Lipschitz continuous operator in the Banach space. Let's note that in the scheme value of the nonlinear member we take in the average point. If the main operator A meets conditions: (a) the spectrum of the operator A is contained in a symmetrical open sector with an angle opening less than π, lying in the right-hand half-plane; (b) for any point z $(z \neq 0)$ not belonging to this sector, the resolvent norm is not greater than $c/|z|$. The assessment for the error of the approximate solution is received.

Key words and phrases: Crank-Nicolson semidiscrete scheme, quasilinear evolutionary equation, error estimate.

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1 Introduction

Questions connected with the construction and investigation of approximate solution algorithms of evolutionary problems are considered for example in the well-known books by S. K. Godunov and V. S. Ryabenki [2], G. I. Marchuk [4], R. Richtmayer and K. Morton [6], A. A. Samarski [8], N. N. Yanenko [11]. We also refer to the works by H. A. Alibekov and P. E. Sobolevski [1], A. E. Polichka and P. E. Sobolevski [5], M.Crouzeix [12],

M. Crouzeix and P.-A. Raviart [13] and M.-N. Le Rouxe [16] dedicated to the approximate solution of the Cauchy problem for an abstract parabolic equation. The results obtained in these papers embrace a sufficiently wide class of evolutionary problems.

let's note papers of S. Serdyukova [9] and P. Sobolevski [10]. In paper [9] the stability of linear difference schemes with constant coefficients is investigated in a concrete Banach space, in particular, in the space C. That paper was one of the pioneer works on the study of difference schemes in the Banach space. In paper [10], a logarithmic estimate is given (without proving it) for the resolvent operators of the Crank-Nicolson scheme. It is pointed out that this estimate is proved by the Cauchy-Riesz formula if the initial operator is strongly positive. We further refer to [14], where an explicit estimate of the error of the Crank-Nicolson scheme is obtained in the Hilbert space under the assumption that the initial operator is self-adjoint and positively defined. In the same paper, under the same assumptions for the operator, a lemma is proved, by means of which the obtained results are easily extended to the Banach space.

In the present paper, our investigation relies essentially on the methods developed in the above-mentioned works and in the monograph [7] of one of the authors.

We consider here the Cauchy problem for a quasilinear evolutionary equation with a Lipschitz continuous operator in the Banach space. An Approximate solution of this task is looked for by the Crank-Nicolson scheme. Let's note that in the scheme value of the nonlinear member we take in the average point. If the main operator A meets conditions: (a) the spectrum of the operator A is contained in a symmetrical open sector with an angle opening less than π, lying in the right-hand half-plane; (b) for any point z ($z \neq 0$) not belonging to this sector, the resolvent norm is not greater than $c/|z|$. The assessment for the error of the approximate solution is received

1 Statement of the Problem and Formulation of the Basic Theorem

In the Banach space X, we consider the nonlinear evolutionary problem

$$\frac{du(t)}{dt} + Au\,(t) + M(u) = f(t), \quad t \in \left]0, T\right], \tag{1.1}$$

$$u\,(0) = u_0, \tag{1.2}$$

where $(-A)$ is the generating operator of a strongly continuous semigroup $\exp\,(-tA)$, $t \geq 0$; the nonlinear operator $M(\cdot)$ is Lipschitz continuous; $f(t)$

is a continuously differentiable abstract function with values from X; u_0 is a given vector from X, $u(t)$ is the sought function.

Let us introduce $[0, T]$ the grid $t_k = k\tau$, $k = 0, 1, \ldots, n$, with step $\tau = T/n$. For problem (1.1)–(1.2) we consider the Crank-Nikolson scheme

$$\frac{u_{k+1} - u_{k-1}}{2\tau} + A\,\frac{u_{k+1} + u_{k-1}}{2} + M(u_k) = f_k, \quad k = 1, \ldots, n. \tag{1.3}$$

where $f_k \approx f\,(t_k)$.

The following theorem is true.

Theorem 1.1. *Assume that the following conditions are fulfilled:*

(a) The solution $u(t)$ of problem (1.1)–(1.2) is twice continuously differentiable and $u''(t)$ satisfies the Lifshitz condition ;

(b) $u'(t) \in D(A)$ for every t from $[0, T]$ and function $Au'(t)$ satisfies Lipshitz condition;

(c) A is a linear, densely defined closed operator in the Banach space X, whose spectrum is wholly contained in the sector $|arg(z)| < \varphi_0$, $0 < \varphi_0 < \pi/2$ and the condition

$$\left\|(zI - A)^{-1}\right\| \le \frac{c_0}{|z|}, \quad c_0 = const > 0.$$

is fulfilled for any z, $(z \neq 0)$ not belonging to this sector;

(d) The nonlinear operator $M(\cdot)$ satisfies Lipshitz condition.

Then the estimate

$$\|z_{k+1}\| \le c\left(\ln\frac{et_k}{\tau}\left(\|z_0\| + \|z_1\|\right) + t_k \max_{1\le i\le k}\|f\,(t_i) - f_i\| + t_k\tau^2\right),$$

$$k = 1, \ldots, n-1,$$

is valid, where $z_k = u(t_k) - u_k$, $c = const > 0$.

To prove this theorem, we need some auxiliary statements, which, in our opinion, are of independent interest.

2 Auxiliary statements

Lemma 2.1. *Let us assume that the operator A satisfies the conditions of Theorem 1.1. Then the following estimate is valid:*

$$\left\|\tau\,A\,(I - \tau\,A)^k\,(I + \tau\,A)^{-(k+1)}\,(I + 2\tau A)^{-j}\right\| \le \frac{c_1(\lambda)}{k+j}, \tag{2.1}$$

where k and j are natural numbers, $\tau > 0$,

$$c_1(\lambda) = \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \frac{1}{\lambda\lambda_1}\right)\frac{c_0}{2\pi},$$

$$\lambda_1 = \min\left(16\lambda^2, \left(\ln\frac{9}{4\lambda^3}\right)^{-1}\right),$$

$$\lambda = \cos(\varphi), \quad \varphi_0 \le \varphi < \frac{\pi}{2}.$$

Proof. Applying the Danford–Taylor integral (see e.g. [3]), we have

$$\tau A\,(I - \tau A)^k\,(I + \tau A)^{-(k+1)}\,(I + 2\tau A)^{-j}$$
$$= \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{z(1-z)^k}{(1+z)^{k+1}(1+2z)^j}\,(zI - \tau A)^{-1}\,dz, \tag{2.2}$$

where Γ is the boundary of a sector $|arg(z)| < \varphi$ (the integral is taken along the positive direction).

By virtue of Theorem 1.1 we have

$$\left\|(zI - \tau A)^{-1}\right\| = \frac{1}{\tau}\left\|\left(\frac{z}{\tau}I - A\right)^{-1}\right\| \le \frac{c_0}{|z|}. \tag{2.3}$$

Passing in (2.2) to the norm and taking (2.3) into account, we obtain

$$\left\|\tau A\,(I - \tau A)^k\,(I + \tau A)^{-(k+1)}\,(I + 2\tau A)^{-j}\right\|$$
$$\le \frac{c_0}{2\pi}\int\limits_0^{+\infty} \frac{|1-z|^k}{|1+z|^{k+1}|1+2z|^j}\,d\rho, \tag{2.4}$$

where $z = \rho(\cos\varphi + i\sin\varphi)$.

Let us estimate the improper integral contained in the right-hand part of (2.4). It is obvious that

$$\int\limits_0^{+\infty} \frac{|1-z|^k}{|1+z|^{k+1}|1+2z|^j}\,d\rho$$
$$= \int\limits_0^{+\infty} \frac{\left(1 - 2\lambda\rho + \rho^2\right)^{\frac{k}{2}}}{\left(1 + 2\lambda\rho + \rho^2\right)^{\frac{k+1}{2}}\left(1 + 4\lambda\rho + 4\rho^2\right)^{\frac{j}{2}}}\,d\rho. \tag{2.5}$$

We represent integral (2.5) as a sum of three integrals

$$\int\limits_0^{+\infty} \frac{(1 - 2\lambda\rho + \rho^2)^{\frac{k}{2}}}{(1 + 2\lambda\rho + \rho^2)^{\frac{k+1}{2}}(1 + 4\lambda\rho + 4\rho^2)^{\frac{j}{2}}}\,d\rho = \int\limits_0^{2\lambda} + \int\limits_{2\lambda}^{k+1} + \int\limits_{k+1}^{+\infty}. \tag{2.6}$$

The validity of the following inequalities is obvious:

$$1 + 2\lambda\rho + \rho^2 \geq (1 + \lambda\rho)^2,$$
$$1 + 4\lambda\rho + 4\rho^2 \geq (1 + \lambda\rho)^2,$$
$$1 - 2\lambda\rho + \rho^2 \leq 1, \quad 0 \leq \rho \leq 2\lambda.$$

Taking these inequalities into account, for the first integral from the right-hand part of equality (2.6) we obtain the estimate

$$\int_0^{2\lambda} \frac{(1 - 2\lambda\rho + \rho^2)^{\frac{k}{2}}}{(1 + 2\lambda\rho + \rho^2)^{\frac{k+1}{2}}(1 + 4\lambda\rho + 4\rho^2)^{\frac{j}{2}}}\, d\rho$$
$$\leq \int_0^{2\lambda} \frac{1}{(1 + \lambda\rho)^{k+j+1}}\, d\rho \leq \int_0^{+\infty} \frac{d\rho}{(1 + \lambda\rho)^{k+j+1}}$$
$$= -\frac{1}{\lambda(k+j)(1 + \lambda\rho)^{k+j}}\bigg|_0^{+\infty} = \frac{1}{\lambda(k+j)}\,. \quad (2.7)$$

Let us estimate the second integral in the right-hand part of equality (2.6).

For any $j \geq 1$ the following inequality is valid

$$\int_{2\lambda}^{k+1} \frac{(1 - 2\lambda\rho + \rho^2)^{\frac{k}{2}}}{(1 + 2\lambda\rho + \rho^2)^{\frac{k+1}{2}}(1 + 4\lambda\rho + 4\rho^2)^{\frac{j}{2}}}\, d\rho$$
$$\leq \int_{2\lambda}^{k+1} \frac{(1 - 2\lambda\rho + \rho^2)^{\frac{k}{2}}}{(1 + 2\lambda\rho + \rho^2)^{\frac{k+2}{2}}}\, d\rho = \int_{2\lambda}^{k+1} \frac{\chi(\rho)(1 - \chi(\rho))^m}{2\lambda\rho(1 + \chi(\rho))^{m+1}}\, d\rho, \quad (2.8)$$

where

$$m = \frac{k}{2}\,, \quad \chi(\rho) = \frac{2\lambda\rho}{(1 + \rho^2)}\,.$$

Since $0 \leq \chi(\rho) < 1$ for $2\lambda \leq \rho < +\infty$, we have the estimate

$$\chi(\rho)\,(1 - \chi(\rho))^m \leq \frac{1}{m+1}\left(1 - \frac{1}{m+1}\right)^m \leq \frac{1}{2(m+1)}\,. \quad (2.9)$$

The inequality

$$\rho(1 + \chi(\rho))^{m+1} \geq \rho(1 + (m+1)\chi(\rho)) = \rho + 2\lambda(m+1)\frac{\rho^2}{1 + \rho^2}$$
$$\geq \rho + \lambda(k+2)\frac{4\lambda^2}{1 + 4\lambda^2} = \rho + \lambda\lambda_0(k+2), \quad \lambda_0 = \frac{4\lambda^2}{1 + 4\lambda^2} \quad (2.10)$$

is also valid by virtue of the Bernoulli inequality.

Taking (2.9) and (2.10) into account, from (2.8) we obtain

$$
\begin{aligned}
&\int_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}}\,d\rho \\
\leq \frac{1}{\lambda(k+2)} &\int_{2\lambda}^{k+1} \frac{d\rho}{\rho+\lambda\lambda_0(k+2)} = \frac{1}{\lambda(k+2)} \ln\frac{k+1+\lambda\lambda_0(k+2)}{2\lambda+\lambda\lambda_0(k+2)} \\
&\leq \frac{1}{\lambda(k+2)} \ln\frac{1+\lambda\lambda_0}{\lambda\lambda_0} = \frac{1}{\lambda(k+2)} \ln\left(\frac{1}{4\lambda^3}+\frac{1}{\lambda}+1\right) \\
&\qquad \leq \frac{1}{\lambda(k+2)} \ln\frac{9}{4\lambda^3}\,. \qquad (2.11)
\end{aligned}
$$

Let us show that the inequality

$$
\begin{aligned}
&\int_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}}\,d\rho \\
&\qquad \leq \frac{1}{16\lambda^3(k+j+2)} \qquad (2.12)
\end{aligned}
$$

is fulfilled for any $j > 2$.

By virtue of the Bernoulli inequality, the inequality

$$
(1+4\lambda\rho+4\rho^2)^{\frac{j-1}{2}} \geq (1+4\rho^2)^{\frac{j-1}{2}} \geq 1+2(j-1)\rho^2
$$

holds for $j > 2$.

Taking this inequality into account, we obtain

$$\int\limits_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}}\, d\rho$$

$$\leq \int\limits_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+2}{2}}(1+2(j-1)\rho^2)}\, d\rho$$

$$= \frac{1}{2\lambda}\int\limits_{2\lambda}^{k+1} \frac{\chi(\rho)(1-\chi(\rho))^m}{\rho(1+\chi(\rho))^{m+1}(1+2(j-1)\rho^2)}\, d\rho$$

$$\leq \frac{1}{\lambda(k+2)}\int\limits_{2\lambda}^{k+1} \frac{d\rho}{\rho(1+2(j-1)\rho^2)} \leq \frac{1}{\lambda(k+2)}\int\limits_{2\lambda}^{+\infty} \frac{d\rho}{\rho(1+2(j-1)\rho^2)}$$

$$= \frac{1}{\lambda(k+2)}\int\limits_{2\lambda}^{+\infty} \left(\frac{1}{\rho} - \frac{2(j-1)\rho}{1+2(j-1)\rho^2}\right) d\rho$$

$$= \frac{1}{\lambda(k+2)} \ln \frac{\rho}{\sqrt{1+2(j-1)\rho^2}}\Bigg|_{2\lambda}^{+\infty} = \frac{1}{\lambda(k+2)} \ln \frac{\sqrt{1+8\lambda^2(j-1)}}{2\lambda\sqrt{2(j-1)}}$$

$$= \frac{1}{2\lambda(k+2)} \ln\left(1+\frac{1}{8\lambda^2(j-1)}\right) \leq \frac{1}{16\lambda^3(k+2)(j-1)}$$

$$= \frac{1}{16\lambda^3(k(j-1)+2j-2)} \leq \frac{1}{16\lambda^3(2k+j+(j-2))}$$

$$\leq \frac{1}{16\lambda^3(2k+j+1)} \leq \frac{1}{16\lambda^3(k+k+j+1)} \leq \frac{1}{16\lambda^3(k+j+2)}\,.$$

We have thus proved inequality (2.12).

Let us now estimate the third integral in the right-hand part of equality (2.6). Like in the case of the second integral of (2.6), we have

$$\int\limits_{k+1}^{+\infty} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}}\, d\rho$$

$$\leq \int\limits_{k+1}^{+\infty} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+2}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j-1}{2}}}\, d\rho$$

$$\leq \int\limits_{k+1}^{+\infty} \left(\frac{\chi(\varrho)(1-\chi(\varrho))^k}{2\lambda\rho(1+\rho^2)(1+\chi(\varrho))^{k+2}}\right)^{\frac{1}{2}} \frac{1}{(1+\rho^2)^{\frac{j-1}{2}}}\, d\rho$$

$$\leq \frac{1}{2\sqrt{\lambda(k+1)}} \int\limits_{k+1}^{+\infty} \frac{1}{\rho^{\frac{2j+1}{2}}}\, d\rho = \frac{1}{\sqrt{\lambda}(2j-1)(k+1)} \leq \frac{1}{\sqrt{\lambda}(k+j)}\,. \tag{2.13}$$

With estimates (2.7), (2.11), (2.12) and (2.13) taken into account, equality (2.6) implies the estimate

$$\int\limits_{0}^{+\infty} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}}\, d\rho \leq \frac{c_2(\lambda)}{k+j}\,, \tag{2.14}$$

where

$$c_2(\lambda) = \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \frac{1}{\lambda\lambda_1}\right).$$

Taking (2.5) and (2.14) into account, from (2.4) we obtain estimate (2.1). Lemma 2.1 is proved. □

Remark 2.2. It is not difficult to observe that estimate (2.1) remains valid if the operator $(I+2\tau A)^{-1}$ is replaced by the operator $(I+\tau\, A)^{-1}$, i.e. the following estimate is fulfilled

$$\left\|\tau\, A\,(I-\tau\, A)^k\,(I+\tau\, A)^{-(k+j+1)}\right\| \leq \frac{c_1(\lambda)}{k+j}\,. \tag{2.15}$$

Remark 2.3. Using the Danford-Taylor integral, it is easy to prove the estimate

$$\left\|(\tau A)(I+\tau A)^{-k}\right\| \leq \frac{c}{k}\,,\quad c = const > 0.$$

Theorem 2.4. *Assume that the operator A satisfies the conditions of Theorem 1.1. Then for the transition operator $L = (I-\tau A)\,(I+\tau A)^{-1}$ of the Crank-Nicolson scheme the estimate*

$$\left\|L^k\right\| \leq c\ln\frac{et_k}{\tau},\quad k = 2,\ldots,n, \tag{2.16}$$

is valid, where $\tau = T/n$ and the constant $c > 0$ does not depend on τ.

Proof. It is obvious that

$$L+I = 2S_0,\quad L-I = -2\tau A S_0, \tag{2.17}$$

where $S_0 = (I+\tau A)^{-1}$.

From equalities (2.17) it follows that

$$L^2 = -4\tau A S_0^2 + I. \tag{2.18}$$

Multiplying both parts of equality (2.18) by L^2, we obtain

$$L^4 = -4\tau AS_0^2L^2+L^2 = -4\tau AS_0^2L^2+\left(-4\tau AS_0^2 + I\right) = -4\tau AS_0^2\left(L^2+I\right)+I.$$

Further, by induction, we have

$$L^{2m} = -4\tau AS_0^2\left(L^{2m-2}+L^{2m-4}+\cdots+I\right)+I. \tag{2.19}$$

Hence it follows that

$$L^{2m+1} = -4\tau AS_0^2\left(L^{2m-1}+L^{2m-3}+\cdots+L\right)+L. \tag{2.20}$$

If in equality (2.19) we take the norms, use estimate (2.15) and Remark 2, then we obtain

$$\begin{aligned}\left\|L^{2m}\right\| &\le 4\left(\left\|\tau AL^{2m-2}S_0^2\right\|+\left\|\tau AL^{2m-4}S_0^2\right\|+\cdots+\left\|\tau AS_0^2\right\|\right)+1\\ &\le c\left(\frac{1}{2m-1}+\frac{1}{2m-3}+\cdots+\frac{1}{2}\right)+1\le c\ln(2m)+1. \end{aligned}\tag{2.21}$$

Analogously, from (2.20) we obtain

$$\left\|L^{2m+1}\right\| \le c\left(\ln\left((2m+1)\right)+1\right). \tag{2.22}$$

(2.21) and (2.22) yield estimate (2.16).

Lemma 2.5 (see e.g. [15, ch. I]). *Let the operator A satisfy the conditions of Theorem 1.1. Then for any $\tau > 0$ and natural k we have the inequality*

$$\left\|\left(I+\tau A\right)^{-k}\right\| \le c, \quad c = const > 0.$$

Remark 2.6. For any natural k the estimate

$$\left\|(L^k - S^k)u_0\right\| \le c_1(\lambda)\tau\left\|Au_0\right\|, \quad u_0\in D(A),$$

is valid, where

$$L = (I-\tau A)(I+\tau A)^{-1}, \quad S = (I+2\tau A)^{-1}.$$

Indeed, by Lemma 2.1 the representation

$$\begin{aligned}L^k - S^k &= (L-S)\left(L^{k-1}+L^{k-2}S+\cdots+LS^{k-2}+S^{k-1}\right)\\ &= -2\tau^2A^2S_0S\left(L^{k-1}+L^{k-2}S+\cdots+LS^{k-2}+S^{k-1}\right),\end{aligned}$$

implies the estimate

$$\begin{aligned}\left\|(L^k - S^k)u_0\right\| &\le 2\tau \left\|Au_0\right\| \sum_{i=1}^{k} \left\|\tau\, AL^{k-i} S_0 S^i\right\| \\ &\le \tau \left\|Au_0\right\| \sum_{i=1}^{k} \frac{c_1(\lambda)}{k} \\ &= c_1(\lambda)\tau \left\|Au_0\right\|.\end{aligned}$$

Here we have used the representation

$$\begin{aligned}L - S &= (I - \tau\, A)\,(I + \tau\, A)^{-1} - (I + 2\tau A)^{-1} \\ &= [(I - \tau\, A)\,(I + 2\tau A) - (I + \tau\, A)]\,(I + \tau\, A)^{-1}\,(I + 2\tau A)^{-1} \\ &= -2\,(\tau A)^2\,(I + \tau\, A)^{-1}\,S.\end{aligned}$$

Lemma 2.7. *For any $\tau > 0$ and natural k ($k \le n$, $\tau = T/n$), the following estimate is valid*

$$\left\|S_0 L^k\right\| \le c, \qquad c = const > 0. \tag{2.23}$$

Proof. By virtue of Lemma 2 and Remark 2 we have

$$\begin{aligned}\left\|L^k u\right\| = \left\|(L^k - S^k)u + S^k u\right\| &\le \left\|(L^k - S^k)u\right\| + \left\|S^k u\right\| \\ &\le c\tau\|Au\| + \|S^k\|\|u\| \le c\,(\tau\|Au\| + \|u\|)\,.\end{aligned}$$

Taking this inequality into account, we obtain

$$\begin{aligned}\left\|S_0 L^k u\right\| = \left\|L^k S_0 u\right\| = \left\|L^k (I + \tau A)^{-1} u\right\| &\le c\left(\tau\left\|A(I + \tau A)^{-1}u\right\| + \left\|(I + \tau A)^{-1}u\right\|\right) \\ &\le c\left(\left\|(\tau A)(I + \tau A)^{-1}\right\|\|u\| + \left\|(I + \tau A)^{-1}\right\|\|u\|\right) \le c\,\|u\|\,.\end{aligned}$$

Clearly, this implies (2.23).

3 Proof of the Basic Theorem

Let us proceed to proving the basic Theorem 1.1 (in what follows, c always denotes a positive constant).

Equation (1.1) at the point $t = t_k = k\tau$ is written in the form

$$\frac{u(t_{k+1}) - u(t_{k-1})}{2\tau} + A\,\frac{u(t_{k+1}) + u(t_{k-1})}{2} + M(u(t_k))$$

$$= f(t_k) + \frac{1}{2\tau}\,\varphi_k^{(1)} + \frac{1}{2}\,A\varphi_k^{(2)}, \tag{3.1}$$

where

$$\varphi_k^{(1)} = \int\limits_{t_k}^{t_{k+1}}\int\limits_{t_k}^{t}\Big(u^{''}(s) - u^{''}(t_k)\Big)\,ds\,dt + \int\limits_{t_{k-1}}^{t_k}\int\limits_{t}^{t_k}\Big(u^{''}(t_k) - u^{''}(s)\Big)\,ds\,dt,$$

$$\varphi_k^{(2)} = \int\limits_{t_k}^{t_{k+1}}\Big(u^{'}(t) - u^{'}(t_k)\Big)\,dt + \int\limits_{t_{k-1}}^{t_k}\Big(u^{'}(t_k) - u^{'}(t)\Big)\,dt.$$

From (1.3) and (3.1) we have

$$\frac{z_{k+1} - z_{k-1}}{2\tau} + A\,\frac{z_{k+1} + z_{k-1}}{2} + (M(u(t_k)) - M(u_k)) = \varphi_k, \tag{3.2}$$

where

$$z_k = u(t_k) - u_k, \qquad \varphi_k = \frac{1}{2\tau}\,\varphi_k^{(1)} + \frac{1}{2}\,A\varphi_k^{(2)} + \left(f\left(t_k\right) - f_k\right).$$

From (3.2) it follows that

$$z_{k+1} = Lz_{k-1} + 2\tau S_0\widetilde{\varphi}_k, \tag{3.3}$$

where

$$L = \left(I - \tau\,A\right)\left(I + \tau\,A\right)^{-1}, \quad S_0 = \left(I + \tau\,A\right)^{-1},$$
$$\widetilde{\varphi}_k = \varphi_k - (M(u(t_k)) - M(u_k)).$$

From the recurrent relation (3.3) we obtain

$$z_{2k+m} = L^k z_m + 2\tau S_0\sum_{i=1}^{k} L^{k-i}\widetilde{\varphi}_{2i+m-1}, \quad m = 0, 1. \tag{3.4}$$

It is obvious that for $\widetilde{\varphi}_i$ the following estimate is true

$$\left\|\widetilde{\varphi}_i\right\| \le \alpha_i, \quad \alpha_i = c\tau^2 + \left\|f\left(t_i\right) - f_i\right\| + c\left\|z_i\right\|. \tag{3.5}$$

If in equality (3.4) we take the norms and substituting (3.5), then we obtain

$$\left\|z_{2k+m}\right\| \le \left\|L^k\right\|\left\|z_m\right\| + 2\tau\sum_{i=1}^{k}\left\|S_0L^{k-i}\right\|\alpha_{2i+m-1}, \quad m = 0, 1.$$

Hence, by Lemma 2.7 and Theorem 2.4, it follows that

$$\|z_{k+1}\| \leq c\delta_k + c\tau \sum_{i=1}^{k} \|z_i\|, \tag{3.6}$$

where

$$\delta_k = \ln \frac{et_k}{\tau} (\|z_0\| + \|z_1\|) + t_k\tau^2 + t_k \max_{1\leq i\leq k} \|f(t_i) - f_i\|.$$

From (3.6), according to the discrete analog of Gronwall's lemma, we obtain

$$\|z_{k+1}\| \leq ce^{ct_{k-1}} (\tau \|z_1\| + \delta_k).$$

This completes the proof of Theorem 1.1.

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