AN ITERATION METHOD OF SOLUTION OF A NONLINEAR EQUATION FOR A STATIC BEAM

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Abstract

The paper deals with the boundary value problem for the nonlinear integro-differential equation $u'''' - m(\int_0^l u'^2\,dx)u'' = f(x)$, $m(z) \geq \alpha > 0$, $0 \leq z < \infty$, modeling the static state of the Kirchhoff beam. The problem is reduced to an integral equation which is solved using the iteration method. The convergence of the iteration process is established and the error estimate is obtained.

Key words and phrases: Kirchhoff type equation, Green's function, iteration method, error estimate.

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1 Statement of the problem

Let us consider the nonlinear beam equation

$$u''''(x) - m\left(\int_0^\ell u'^2(x)\,dx\right)u''(x) = f(x), \quad 0 < x < \ell, \tag{1}$$

with the conditions

$$u(0) = u(\ell) = 0, \quad u''(0) = u''(\ell) = 0. \tag{2}$$

Here $u = u(x)$ is the displacement function of the length ℓ of the beam subjected to a force given by the function $f(x)$, the function $m(z)$,

$$m(z) \geq \alpha > 0, \qquad 0 \leq z < \infty, \tag{3}$$

describes the type of relationship between stress and strain. Namely, if the function $m(z)$ is linear, this means that the latter relation is consistent with Hooke's linear law, while otherwise case we deal with material nonlinearities.

Equation (1) is the stationary problem associated with the equation

$$u_{tt} + u_{xxxx} - m\Big(\int_0^\ell u_x^2\,dx\Big)u_{xx} = f(x,t),$$
$$m(z) \geq const > 0,$$

which for the case where $m(z) = m_0 + m_1 z$, $m_0, m_1 > 0$, and $f(x,t) = 0$, was proposed by Woinowsky–Krieger [11] as a model of deflection of an extensible dynamic beam with hinged ends. The nonlinear term $\int_0^\ell u_x^2\,dx$ was for the first time used by Kirchhoff [2] who generalized D'Alembert's classical linear model. Therefore (1) is frequently called a Kirchhoff type equation for a static beam.

The topic of solvability of equations of (1) type is studied in [3]–[5] and [10], while the problem of construction of numerical algorithms and estimation of their accuracy is investigated in [1], [4], [6], [8] and [9].

In the present paper, in order to obtain an approximate solution of problem (1), (2), an approach different from the ones applied in the above references is used. It consists in reducing problem (1), (2) by means of Green's function to a nonlinear integral equation, to solve which we use an iteration method. A condition for convergence of the method is established and its accuracy is estimated.

The Green's function method with a further iteration procedure has been applied by us previously also to a nonlinear problem for the axially symmetric Timoshenko plate [7].

2 Assumptions

Suppose that $f(x) \in L_2(0,\ell)$ and the function $m(z)$ satisfies, in addition to requirement (3), the Lipschitz condition on an interval

$$\big|m(z_2) - m(z_1)\big| \leq L|z_2 - z_1|, \qquad (4)$$
$$0 \leq z_1, z_2 \leq c,$$

where $L = const > 0$ and

$$c = \frac{\ell^2}{2}\Big(\alpha + \frac{2}{\ell^2}\Big)\int_0^\ell f^2(x)\,dx. \qquad (5)$$

3 The Method

Applying the Green's function of the problem $u''''(x) - au''(x) = f(x)$, $0 < x < \ell$, $u(0) = u(\ell) = 0$, $u''(0) = u''(\ell) = 0$, $a = const > 0$, and performing some transformations, from problem (1), (2) we come as a result to the equivalent nonlinear integral equation

$$u(x) = \int_0^\ell G(x,\xi) f(\xi)\, d\xi + \frac{1}{\tau}\,\varphi(x), \tag{6}$$

where

$$G(x,\xi) =$$
$$= \frac{1}{\tau\sqrt{\tau}\,\sinh(\sqrt{\tau}\,\ell)} \begin{cases} \sinh(\sqrt{\tau}\,(x-\ell))\sinh(\sqrt{\tau}\,\xi), & 0 < \xi \le x < \ell, \\ \sinh(\sqrt{\tau}\,(\xi-\ell))\sinh(\sqrt{\tau}\,x), & 0 < x \le \xi < \ell, \end{cases}$$

$$\tau = m\Bigg(\int_0^\ell u'^2(x)\, dx\Bigg),$$

$$\varphi(x) = \frac{1}{\ell}\Bigg((\ell - x)\int_0^x \xi f(\xi)\, d\xi + x\int_x^l (\ell-\xi) f(\xi)\, d\xi\Bigg).$$

Equation (6) is solved by the method of ordinary iterations. After choosing a function $u_0(x)$, $0 \le x \le \ell$, which together with its second derivative vanishes for $x = 0$ and $x = \ell$, we find subsequent approximations by the formula

$$u_{k+1}(x) = \int_0^\ell G_k(x,\xi) f(\xi)\, d\xi + \frac{1}{\tau_k}\,\varphi(x), \quad 0 < x < \ell, \tag{7}$$
$$k = 0, 1, \ldots,$$

where

$$G_k(x,\xi) = \frac{1}{\tau_k\sqrt{\tau_k}\,\sinh(\sqrt{\tau_k}\,\ell)} \times$$
$$\times \begin{cases} \sinh(\sqrt{\tau_k}\,(x-\ell))\sinh(\sqrt{\tau_k}\,\xi), & 0 < \xi \le x < \ell, \\ \sinh(\sqrt{\tau_k}\,(\xi-\ell))\sinh(\sqrt{\tau_k}\,x)), & 0 < x \le \xi < \ell, \end{cases} \tag{8}$$

$$\tau_k = m\Bigg(\int_0^\ell {u'_k}^2(x)\, dx\Bigg), \tag{9}$$
$$k = 0, 1, \ldots,$$

and $u_k(x)$ is the k-th approximation of the solution of equation (6).

At the $(k+1)$-th iteration step, having the k-th approximation $u_k(x)$, to find $u_{k+1}(x)$ we first calculate the parameter τ_k by (9) and then, taking (8) into account, find the function $G_k(x,\xi)$, which is used together with τ_k in formula (7). The numerical integration technique can be used for calculating definite integrals in the general case.

4 Estimation of the Method Error

Our aim is to estimate the error of the method, by which we understand the difference between the approximate and the exact solution

$$\delta u_k(x) = u_k(x) - u(x), \qquad k = 0,1,\dots . \tag{10}$$

For this, it is advisable to use not formula (7), but the system of equalities

$$u''''_{k+1}(x) - m\Big(\int_0^\ell {u'_k}^2(x)\,dx\Big)u''_{k+1}(x) = f(x), \;\; 0<x<\ell, \tag{11}$$

$$u_k(0) = u_k(\ell) = 0, \quad u''_k(0) = u''_k(\ell) = 0, \tag{12}$$
$$k = 0,1,\dots,$$

from which formula (7) follows.

If we subtract the respective equalities in (1) and (2) from (11) and (12), we get

$$\begin{gathered}\delta u''''_k(x) - \frac{1}{2}\Bigg(\bigg(m\Big(\int_0^\ell u'^2_{k-1}(x)\,dx\Big) + m\Big(\int_0^\ell u'^2(x)\,dx\Big)\bigg)\delta u''_k(x) + \\ + \bigg(m\Big(\int_0^\ell u'^2_{k-1}(x)\,dx\Big) - m\Big(\int_0^\ell u'^2(x)\,dx\Big)\bigg)\big(u''_k(x) + u''(x)\big)\Bigg) = 0,\end{gathered} \tag{13}$$

$$\delta u_k(0) = \delta u_k(\ell) = 0, \quad \delta u''_k(0) = \delta u''_k(\ell) = 0, \tag{14}$$
$$k = 1,2,\dots .$$

System (13) and conditions (14) are the starting point of estimation of the method error. We shall have to derive preliminarily several a priori estimates. Let us denote the norms in $\overset{\circ}{W}{}^1_2(0,\ell) \cap W^2_2(0,\ell)$ as

$$\|u(x)\|_p = \Big(\int_0^\ell \Big(\frac{d^p u}{dx^p}(x)\Big)^2 dx\Big)^{\frac{1}{2}}, \;\; p = 0,1,2, \;\; \|u(x)\| = \|u(x)\|_0.$$

The symbol $(\,\cdot\,,\,\cdot\,)$ is understood as a scalar product in $L_2(0,\ell)$.

Lemma 1. *The inequalities*

$$\frac{\sqrt{2}}{\ell}\|u(x)\| \le \|u(x)\|_1 \le \frac{\ell}{\sqrt{2}}\,\|u(x)\|_2 \tag{15}$$

are valid for $u(x) \in \overset{\circ}{W}{}_2^2(0,\ell)$.

Proof. Using the equality $u(x) = \int\limits_0^x u'(\xi)\,d\xi$ we obtain

$$|u(x)| \le \Big(\int\limits_0^x d\xi\Big)^{\frac{1}{2}} \Big(\int\limits_0^x u'^2(\xi)\,d\xi\Big)^{\frac{1}{2}} \le x^{\frac{1}{2}}\|u(x)\|_1,$$

which implies the left inequality of (15). Applying the latter and taking into account that

$$\|u(x)\|_1^2 = u(x)u'(x)\big|_0^\ell - \big(u(x),u''(x)\big) = \\ = -\big(u(x),u''(x)\big) \le \|u(x)\|\,\|u(x)\|_2,$$

we complete the proof. □

Lemma 2. *For the solution of problem* (1), (2) *the inequality*

$$\|u(x)\|_1 \le c_1, \tag{16}$$

where

$$c_1 = \max z, \quad 0 < \frac{\sqrt{2}}{\ell}\,z\Big(\frac{2}{\ell^2} + m(z^2)\Big) \le \|f(x)\|, \tag{17}$$

is true.

Proof. We multiply equation (1) by $u(x)$ and integrate the resulting equality with respect to x from 0 to ℓ. Using (2), we get the relation

$$\|u(x)\|_2^2 + m\big(\|u(x)\|_1^2\big)\|u(x)\|_1^2 = \big(f(x),u(x)\big).$$

By (15) we obtain

$$\Big(\frac{2}{\ell^2} + m\big(\|u(x)\|_1^2\big)\Big)\|u(x)\|_1^2 \le \frac{\ell}{\sqrt{2}}\,\|f(x)\|\,\|u(x)\|_1,$$

which means the fulfillment of (16). □

Note that when we deal with the problem of finding c_1 by (17), by virtue of (3) c_1 can be defined by $c_1 = \max z$, $0 < \frac{\sqrt{2}}{\ell}\,z(\frac{2}{\ell^2}+\alpha) \le \|f(x)\|$. As a result we get $c_1 = \frac{\ell}{\sqrt{2}}\,(\frac{2}{\ell^2}+\alpha)^{-1}\|f(x)\|$, but, speaking in general, this solution is not the best way to cope with the situation.

Lemma 3. *Approximations of the iteration method* (7) *satisfy the inequality*

$$\|u_k(x)\|_1 \le c_2, \quad k = 1, 2, \ldots, \tag{18}$$

where

$$c_2 = \frac{\ell}{\sqrt{2}} \Big(\frac{2}{\ell^2} + \alpha\Big)^{-1} \|f(x)\|. \tag{19}$$

Proof. We multiply equation (11) by $u_k(x)$ and integrate the resulting relation with respect to x from 0 to ℓ. Taking (12) into account, we get

$$\|u_k(x)\|_2^2 + m\big(\|u_{k-1}(x)\|_1^2\big)\|u_k(x)\|_1^2 = \big(f(x), u_k(x)\big),$$
$$k = 1, 2, \ldots .$$

Applying (3) and (15) we have

$$\Big(\frac{2}{\ell^2} + \alpha\Big)\|u_k(x)\|_1^2 \le \frac{\ell}{\sqrt{2}} \|f(x)\| \, \|u_k(x)\|_1,$$

which implies (18). □

Let us compare the parameters c, c_1 and c_2. By (5), (17), (3) and (19)

$$c_1 \le c_2 = \sqrt{c}\,. \tag{20}$$

5 Convergence of the Method

Multiplying (13) by $\delta u_k(x)$, integrating the resulting equality with respect to x from 0 to ℓ and using (14), we come to the relation

$$\|\delta u_k(x)\|_2^2 + \frac{1}{2}\bigg(\Big(m\big(\|u_{k-1}(x)\|_1^2\big) + m\big(\|u(x)\|_1^2\big)\Big)\|\delta u_k(x)\|_1^2 +$$
$$+ \Big(m\big(\|u_{k-1}(x)\|_1^2\big) - m\big(\|u(x)\|_1^2\big)\Big)\Big(u_k'(x) + u'(x), \delta u_k'(x)\Big)\bigg) = 0.$$

Using (3), (4), (5) together with (16), (18) and (20) we obtain

$$\|\delta u_k(x)\|_2^2 + \alpha\|\delta u_k(x)\|_1^2 \le$$
$$\le \frac{1}{2} L \prod_{p=0}^{1} \Big|\Big(u_{k-p}'(x) + u'(x), \delta u_{k-p}'(x)\Big)\Big| \le$$
$$\le \frac{1}{2} L \prod_{p=0}^{1} \Big(\|u_{k-p}(x)\|_1 + \|u(x)\|_1\Big)\|\delta u_{k-p}(x)\|_1.$$

By (15), (16) and (18) we get

$$\|\delta u_k(x)\|_1 \leq \frac{1}{2} L\Big(\frac{2}{\ell^2}+\alpha\Big)^{-1} \|\delta u_{k-1}(x)\|_1 \prod_{p=0}^{1} \big(\|u_{k-p}(x)\|_1 + \|u(x)\|_1\big) \leq$$
$$\leq q\|\delta u_{k-1}(x)\|_1,$$

where

$$q = \frac{1}{2} L\Big(\frac{2}{\ell^2}+\alpha\Big)^{-1} (c_1 + c_2)^2.$$

Taking (10), (15), (17) and (19) into consideration, we come to the following result.

Theorem. *Let the assumptions* (3) *and formulated in Section* 2 *be fulfilled and besides*

$$q = \frac{1}{2} L\Big(\frac{2}{\ell^2}+\alpha\Big)^{-1} \bigg(\max z + \frac{\ell}{\sqrt{2}} \Big(\frac{2}{\ell^2}+\alpha\Big)^{-1} \|f(x)\|\bigg)^2 < 1,$$

where

$$0 < \frac{\sqrt{2}}{\ell} z\Big(\frac{2}{\ell^2} + m(z^2)\Big) \leq \|f(x)\|.$$

Then approximations of the iteration method (7) *converge to the exact solution of problem* (1), (2) *and for the error the following estimate*

$$\|u_k(x) - u(x)\|_p \leq \Big(\frac{\ell}{\sqrt{2}}\Big)^{1-p} q^k \|u_0(x) - u(x)\|_1,$$
$$k = 1, 2, \ldots, \quad p = 0, 1,$$

is true.

In conclusion, note that since relations (11), (12) do not necessarily lead to (7), the theorem will also be valid in the case of another extension of (11), (12), for instance, when the current iteration approximation in (11), (12) is defined by finite difference, finite element or other methods.

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