# ON MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS OF ORNSTEIN-UHLENBECK PROCESSES

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*Abstract*

In a Hilbert space the stochastic differential equation of Ornstein-Uhlenbeck type is considered. Statistical estimation problem of the drift parameter is solved using maximum likelihood method. On the basis of this result estimation of volatility parameter has been obtained using least-squares method.

*Key words and phrases*: Ornstein-Uhlenbeck process, Maximal Likelihood Esitimation, Drift, Volatility.

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The Ornstein-Uhlenbeck type stochastic differential equation is one of the most popular model to describe a variety of phenomena in financial mathematics, biology, chemical reactions, etc. (see [1]). The mathematical theory of such equations is studied by a number of researchers. Many of the questions associated with the Ornstein -Uhlenbeck processes are very well studied (see [2]). But there are problems, particularly related to the statistical evaluation, which require development. Many papers dedicated to parameter estimation of Ornstein-Uhlenbeck process in the finite-dimensional case. In this case, there are a number of methods to obtain such estimations. Maximum likelihood method, method of transition to difference equations, method based on the ergodic theorem and others are especially popular (see [3-7]). Ornsheyna-Uhlenbeck equation generated by Levy process has been studied intensively recently (eg [8-10]). Relatively smaller number of works is dedicated to the infinite-dimensional variant. In [11] Ornstein-Uhlenbeck process in the Hilbert space with bounded linear drift operator is investigated and the parameters estimation problem is solved using maximum likelihood method.

In this paper, we study the Ornstein-Uhlenbeck process in a Hilbert space, in which the drift operator can be unbounded. We obtain estimates of the parameters of this operator, using maximum likelihood method. In this case, we use the method developed in [12]. Based on these estimates the estimation for the volatility is obtained. Here are the asymptotic properties of these estimates.

Let $\{\Omega, \mathcal{F}, P\}$ be a fixed probability space. $H_+ \subset H \subset H_-$ is a rigged Hilbert space with quasi-kernel inclosures. Scalar products and norms will be provided with the indices of the corresponding spaces. The embedding operator $i : H \to H_-$, is the Hilbert-Schmidt operator. The pairing of elements $H_+$ and $H_-$ will be expressed in the scalar product of space $H$. All spaces are assumed to be separable and real. So that $i^* : H_+ \to H$.

Let $A$ be a linear, possibly unbounded, operator with the tight in $H$ domain $\mathcal{D}(A) \subset H$ and let $A$ be the generating operator of strongly continuous semigroup $S(t) = e^{At}$. $B : H_- \to H$ is a linear Hilbert-Schmidt operator, $w_t, t \geq 0$ is a Wiener process in space $H_-$.

Consider Ornstein-Uhlenbeck type stochastic differential equation in the triple of spaces $H_+ \subset H \subset H_-$:

$$dX_t = -AX_t dt + \sigma B dw_t, \quad X_0 = x_0, \ t \geq 0, \tag{1}$$

where $\sigma > 0$ is an unknown parameter (i.e. volatility parameter), which should be estimated by the observations at points $0 \leq t_1 \leq t_2 \leq ... \leq t_n$:

$$X^{(n)} = (X_1, X_2, ..., X_n) \quad where \ X_k = X_{t_k}, k = 1, ..., n.$$

Equation (1) is understood as an equivalent record of

$$X_t = e^{-At} x_0 - \sigma \int_0^t e^{-A(t-s)} B dw_s \tag{2}$$

Certainly, $X_t$ is the Gaussian random process in $H$. Suppose that operator $A$ in (1) is of the form

$$A = \sum_{k=1}^{m} a_k A_k, \tag{3}$$

where $A_k, k = 1, ..., m$ are known linear operators, maybe some of them are unbounded, but each of these operators is defined at least on $\mathcal{D}(A), a_k, k = 1, ..., m$, are unknown real parameters which should be estimated by observations $X^{(n)}$. For the estimation we use maximum likelihood method. As usual (see. [6]), this method doesn't suit the volatility coefficient, but firstly we construct estimations for the parameters $a_k, k = 1, ..., m$, and then using properties of Wiener process we obtain the so called "plug in estimator" for $\sigma$.

Denote by $\theta = (a_1, ..., a_m)^T$ the vector of unknown parameters. Here and below mark $T$ defines transposition operation. We estimate this parameter $\theta$ by maximum likelihood method. Solution of equation (1) generates a measure (probability distribution) in the space of continuous functions $C[0,T]$. In this space as an $\sigma$-algebra we take Borel-$\sigma$ algebra $\mathcal{B}[0,T]\}$. On the measurable space $\{C[0,T], \mathcal{B}[0,T]\}$ distribution of the random process $X = X_t$ generates distribution by equality $P_\theta(A) = P\{X^{-1}(A)\}, A \in \mathcal{B}[0,T]$. According to the general principle of maximum likelihood, it is necessary to calculate the logarithmic derivative with respect to the family of measures $P_\theta$, which is denoted by $\rho(x, \vartheta)$, where $\vartheta$ is the direction vector and the derivative is calculated along this vector. In our case the parameter space is finite dimensional. Therefore calculating logarithmic derivative along some vector isn't essential and denote logarithmic derivative by $r(x;\theta)$. After calculating $r(x;\theta)$ we must find with respect to $\theta$ solution of the following equation

$$\sum_{k=1}^{m} r(x_k;\theta) = 0, \tag{4}$$

where $x_k$ is the observed value of $X_k$. Moreover, we have to check that $\dfrac{dr(x;\theta)}{d\theta}$ is a negatively determined matrix. For calculating $r(x;\theta)$ we use this simple fact

**Lemma.** Let the family $\{P_\theta, \theta \in \Theta, \Theta$ is *open in* $R^l, l \in N\}$ be absolutely continuous with respect to some $\sigma$-finite measure $\mu$ and the Radon-Nycodym derivative $\dfrac{dP_\theta(x)}{d\mu(x)} = \pi(\theta, x)$ is continuously differentiable by $\theta$. Then this family has the logarithmic derivative $r(x;\theta)$ by parameter and

$$r(x;\theta) = \frac{grad\pi(\theta, x)}{\pi(\theta, x)} \text{ (here assumed that } \frac{0}{0} = 0) \tag{5}$$

The proof follows from relationship

$$\frac{dP_\theta(x)}{d\mu(x)} = \pi(\theta, x) \Rightarrow \frac{dd_\theta P_\theta(x)}{d\mu(x)} = grad\pi(\theta, x).$$

For equation (1) ([13]) conditions are well known under which the measure $P_\theta$ (it is distribution of process $X_t$) is equivalent to Gaussian measure $\mu_{\sigma Bw}$. We apply this theorem and obtain the Radon-Nikodym density. Transformation in the space $H$ is

$$\xi_t - x_0 = X_t + \int_0^t AX_s ds,$$

where $\xi_t = \sigma B \xi_t^0$, $\xi_t^0$-"white noise" in $H_-$. The correlation operator of the process $\xi_t$ is $\sigma^2 BB^*$ and is a kernel operator. For this transformation, if the linear operator $AB$ is bounded then measures $P_\theta$ and $\mu$ generated by $X_t$ and $\xi_t - x_0$ respectively, are equivalent and

$$\pi(X;\theta) = \frac{dP_\theta}{d\mu}(X) = \exp\left\{-\frac{1}{\sigma^2}\int_0^T (B^*AX_s, dX_s)_H - \frac{1}{2\sigma^2}\int_0^T \|B^*AX_s\|_H^2 ds\right\} \tag{6}$$

We now apply Lemma and compute vector logarithmic derivative. Denote $b = (b_1, ..., b_m)^T, C = (C_{ij})_{i,j=1}^m, a = (a_1, ..., a_m)^T$, where

$$b_i = \int_0^T \langle B^*A_iX_s, dX_s\rangle_H, \quad i = 1, ..., m.$$

$$C_{ij} = \int_0^T \langle B^*A_iX_s, B^*A_jX_s\rangle_H ds, \quad i, j = 1, ..., m.$$

Then

$$r(X;\theta) = -\frac{1}{\sigma^2}b - \frac{1}{\sigma^2}Ca.$$

It is easy to check that the matrix

$$\frac{d}{d\theta}r(X;\theta) = -\frac{1}{\sigma^2}\begin{pmatrix} C_{11} & 0 & \cdots & 0 \\ 0 & C_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & C_{mm} \end{pmatrix}$$

is negatively defined. Therefore, it remains only to solve the equation

$$E\left\{-\frac{1}{\sigma^2}b - \frac{1}{\sigma^2}C\hat{a}_n | X^{(n)}\right\} = 0. \tag{7}$$

Solution of the equation (7) is

$$\hat{a}_n = -\{E\{C|X^{(n)}\}\}^{-1}E\{b|X^{(n)}\}. \tag{8}$$

It is clear that we have bult a consistent estimation for the operator $A$:

$$\hat{A}_n = \sum_{k=1}^m \hat{a}_k A_k.$$

For instance if $H = R, m = 1, \sigma B = 1, A_1 = 1, x_0 = 0, a_1 = \theta$ then we obtain classic formula ([14])

$$\hat{\theta}_n = \frac{E(\int_0^T X_s dX_s | X^{(n)})}{E(\int_0^T X_s^2 ds | X^{(n)})}.$$

It remains to construct estimate of the volatility $\sigma$. To do this, we substitute in (1) an estimation $\hat{A}_n$ instead of $A$ and rewrite the equation in an integral form at $t = T$:

$$X_T + x_0 + \int_0^T \hat{A}_n X_s ds = \sigma B w_T.$$

Hence we write

$$\hat{\sigma}_n^2 = \frac{E\|\{X_T + x_0 + \int_0^T \hat{A}_n X_s ds | X^{(n)}\}\|_H^2}{E\|\{B w_T | X^{(n)}\}\|_H^2}.$$

Because of the consistency $\hat{A}_n$, when $n \to \infty$ we shall have $\hat{\sigma}_n^2 \to \sigma_0$, where $\sigma_0$ is the true value of volatility.

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