ON THE OPERATORS PRESERVING INFORMATION

V. Tsagareishvili

Ivane Javakhishvili Tbilisi State University
University Street 2, Tbilisi, 0186, Georgia

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Abstract

The present work considers the operators which map some space E onto itself. If the operator A, $A(\varphi) = \psi$, then ψ preserves some property of the point $\varphi, \psi \in E$.

In the paper we study the operators preserving some properties of points from the domain of definition of the given operator.

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1 Auxiliary Notation and Theorems

By $C(0,1)$ we denote the class of continuous on $[0,1]$ functions. $\omega(\delta, f)$ is a modulus of continuity of functions $f(x) \in C(0,1)$. If $\omega(\delta, f) = O(\delta^\alpha)$, then $f(x) \in \mathrm{Lip}\,\alpha$, $\alpha \in [0,1]$. $V(0,1)$ is a class of all functions of bounded variation on $[0,1]$.

Assume that (φ_n) is a system, orthonormal on $[0,1]$ (ONS). The numbers

$$\widehat{\varphi}_n(f) = \int_0^1 f(x)\,\varphi_n(x)\,dx \quad (n = 1, 2, \dots) \tag{1}$$

are called the Fourier coefficients of $f(x) \in L(0,1)$.

Definition 1. Let the operator A map the space E onto itself. The operator A preserves information at the point $\varphi \in E$, if $A(\varphi) = \psi$, φ possesses some property B, and the point $\psi \in E$ possesses the same property B.

Definition 2. Assume that P is a space of all complete in $L_2(0,1)$ orthonormal systems (CONS). We say that the point $\varphi \in P$ possesses the property ω if for any $f(x) \in C(0,1)$ the relation

$$\left|\widehat{\varphi}_n(f)\right| < B\,\omega\Big(\frac{1}{n}, f\Big) \tag{2}$$

holds (see (1)), where $B > 0$ does not depend on n.

Definition 3. Let $p > 1$, $\varphi \in P$ be the complete orthonormal system on $[0,1]$. Assume

$$A_p(\varphi) = \Big\{ f : \sum_{n=1}^{\infty} \big|\widehat{\varphi}_n(f)\big|^p < +\infty \Big\}.$$

We say that the point $\varphi \in P$ possesses the property A_p if for any $f(x) \in V(0,1)$ follows $f(x) \in A_p(\varphi)$.

Let $(a_n) \in \ell_q$ be an arbitrary number sequence. Assume

$$Q_m(x,\varphi) = \sum_{k=1}^{m} a_k\, \varphi_k(x)$$

and

$$B_{nm} = \int_0^1 Q_m(x,\psi)\, \varphi_n(x)\, dx.$$

We have the following (see [1], [2])

Theorem 1. *Let* (φ_n) *be the orthonormal on* $[0,1]$ *system and* $\int\limits_0^1 \varphi_n(x)dx = 0$ $(n = 1, 2, \dots)$. *Then for inequality* (2) *to be valid, it is necessary and sufficient that*

$$\sum_{k=1}^{n-1} \left| \int_0^{\frac{k}{n}} \varphi_n(x)\, dx \right| < h,$$

where $h >$ *does not depend on* n.

Theorem 2. *For* $\varphi \in P$ *to possess the property* A_p *for* $p > 1$, *it is necessary and sufficient that for any* $(a_k) \in \ell_q$ *the condition*

$$\max_{x\in[0,1]} \left| \int_0^x Q_m(t,\varphi)\, dt \right| = O(1)$$

be fulfilled.

Theorem A (see [4], p. 433). *If* $(f_n(x))$ *is the sequence of linear on* E (E *is Banach space*) *functionals and for any* $x \in E$

$$\sum_{n=1}^{\infty} |f_n(x)|^p < +\infty \quad (p \geq 1),$$

then there exists $M > 0$ (*absolute constant*) *such that*

$$\sum_{n=1}^{\infty} |f_n(x)|^p \leq M^p \|x\|_E^p.$$

The following equality is valid (see [2]):

$$\int_0^1 f(x)\,\varphi_n(x)\,dx = \sum_{k=1}^{n-1}\Big(f\Big(\frac{k}{n}\Big)-f\Big(\frac{k+1}{n}\Big)\Big)\int_0^{\frac{k}{n}}\varphi_n(x)\,dx+$$

$$+\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Big(f(x)-f\Big(\frac{k}{n}\Big)\Big)\varphi_n(x)\,dx \equiv I_1+I_2, \qquad (3)$$

where the function $f(x)$ takes finite values at every point of the segment $[0,1]$.

2 The Basic Results

Theorem 3. *Let the operator A map the space P onto P and $A(\varphi)=\psi$. The operator A at the point φ preserves information A_p if for any $(a_n)\in\ell_q$ $\big(\frac{1}{p}+\frac{1}{q}=1\big)$ the condition (see B_{mn})*

$$\lim_{s\to\infty}\sum_{m=1}^{N_s}\big|B_{ms}\big|^q < C \qquad (4)$$

is fulfilled, and $N_s\uparrow\infty$ is some sequence of natural numbers.

Proof. Assume the for any $(a_n)\in\ell_q$ the condition (4) is fulfilled.

The equality (see B_{mn})

$$Q_s(x,\psi)\overset{L_2}{=}\sum_{m=1}^{\infty}B_{ms}\,\varphi_m(x),\ \ \big(\overset{L_2}{=}\ \text{ is equality in the sense of } L_2\big).$$

is valid, whence we have

$$\int_0^x Q_s(t,\psi)\,dt=\int_0^x\sum_{m=1}^{\infty}B_{ms}\,\varphi_m(t)\,dt. \qquad (5)$$

Using Parseval's identity we have for all $x\in[0,1]$

$$\sum_{m=1}^{\infty}\Big(\int_0^x\varphi_m(t)\,dt\Big)^2=x^2.$$

Consequently, according to Dini's theorem about the uniform convergence this series above is uniformly convergent on $[0,1]$.

Then we can choose the number N_s such that

$$\sum_{m=N_s+1}^{\infty}\Big(\int_0^x\varphi_m(t)\,dt\Big)^2\le\frac{1}{s}\,, \qquad (6)$$

uniformly on $[0,1]$.

By continuity there exists $x_s \in [0,1]$ such that

$$\max_{x\in[0,1]}\left|\int_0^x \sum_{m=1}^{N_s} B_{ms}\,\varphi_m(t)\,dt\right| = \left|\int_0^{x_s} \sum_{m=1}^{N_s} B_{ms}\,\varphi_m(t)\,dt\right|.$$

From here of $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we obtain (see (4))

$$\left|\sum_{m=1}^{N_s} B_{ms}\int_0^{x_s}\varphi_m(t)\,dt\right| \le \left(\sum_{m=1}^{N_s}|B_{ms}|^q\right)^{\frac{1}{q}}\left(\sum_{m=1}^{N_s}\left|\int_0^{x_s}\varphi_m(t)\,dt\right|^p\right)^{\frac{1}{p}} \le$$
$$\le c^{\frac{1}{q}}\left(\sum_{m=1}^{\infty}\left|\int_0^1 \chi^{(s)}(t)\varphi_m(t)\,dt\right|^p\right)^{\frac{1}{p}},$$

where $\chi^{(s)}(t)=\chi_{(0,x_s)}(t)$.

It should be noted that $\chi^{(s)}(t)\in V(0,1)$ and $\|\chi^{(s)}(t)\|_V \le 1$ $(s=1,2,\dots)$. Consequently by statement of Theorems A and 3

$$\sum_{m=1}^{\infty}\left|\int_0^1 \chi^{(s)}\varphi_m(t)\,dt\right|^p \le M^p\cdot\|\chi^{(s)}\|_V \le M^p.$$

From here it follows

$$\max_{x\in[0,1]}\left|\int_0^x \sum_{m=1}^{N_s} B_{ms}\varphi_m(t)\,dt\right|^{\frac{1}{p}} \le c^{\frac{1}{q}}M.$$

From equality (5), by virtue of (6) and the statement of Theorem 2, we obtain

$$\left|\int_0^x Q_s(t,\psi)dt\right| \le \left|\int_0^x\sum_{m=1}^{N_s} B_{ms}\,\varphi_m(t)\,dt\right| + \sum_{m=N_s+1}^{\infty}|B_{ms}|\left|\int_0^x\varphi_m(t)\,dt\right| \le$$
$$\le O(1)+\left(\sum_{m=N_s+1}^{\infty}(B_{ms})^2\right)^{1/2}\left(\sum_{m=N_s+1}^{\infty}\left(\int_0^x\varphi_m(t)\,dt\right)^2\right)^{1/2}\le$$
$$\le O(1)+\left(\int_0^1 Q_s^2(x)\,dx\right)^{1/2}\frac{1}{\sqrt{s}} \le O(1)+\left(\sum_{n=1}^{s}a_n^2\right)^{1/2}\frac{1}{\sqrt{s}}\le$$
$$\le O(1)+\sqrt{s}\max_{1\le n\le s}|a_n|\frac{1}{\sqrt{s}} = O(1). \tag{7}$$

By equality (7) and the statement of Theorem 2 we can see that Theorem 3 is valid.

Lemma 1. *If for some $(a_k) \in \ell_q$,*

$$\lim_{s\to\infty} \left(\sum_{m=1}^{N_s} \left|B_{ms}\right|^q \right)^{1/q} = +\infty, \tag{8}$$

then there exists the function $f(x) \in A_p(\varphi)$ such that $f(x) \notin A_p(\psi)$ ((N_s) depends on (φ_n), see (6)*).*

Proof. It follows form equality (8) that there exists the sequence $(b_m) \in \ell_p$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ such that

$$\lim_{s\to\infty} \sum_{m=1}^{N_s} \left|b_m\, B_{ms}\right| = +\infty. \tag{9}$$

Consider the sequence of functions

$$f_s(x) = \sum_{m=1}^{N_s} b_m\, \varphi_m(x). \tag{10}$$

This implies that

$$\widehat{\varphi}_n(f_s) = \int_0^1 \sum_{m=1}^{N_s} b_m\, \varphi_m(x)\, \varphi_n(x)\, dx = \begin{cases} b_n & \text{for} \quad n \le N_s\,, \\ 0 & \text{for} \quad n > N_s\,. \end{cases}$$

Consequently,

$$\left\|f_s\right\|_{A_p(\varphi)} = \sum_{n=1}^{\infty} \left|\widehat{\varphi}_n(f_s)\right|^p = \sum_{m=1}^{N_s} |b_m|^p < M < +\infty,$$

where $M > 0$ does not depend on s. Thus $f_s(x) \in A_p(\varphi)$ $(p > 1)$.

It should be noted that $A_p(\varphi)$ is the Banach space with the norm $\left\|f\right\|_{A_p(\varphi)} = \sum\limits_{n=1}^{\infty} \left|\widehat{\varphi}_n(f)\right|^p$, when (φ_n) is the complete orthonormal system (see [3], p. 51).

From (10) follows

$$\int_0^1 f_s(x)\, Q_s(x,\psi)\, dx = \sum_{m=1}^{N_s} b_m \int_0^1 Q_s(x,\psi)\, \varphi_m(x)\, dx = \sum_{m=1}^{N_s} b_m\, B_{ms}.$$

whence (see (9))

$$\lim_{s\to\infty} \left| \int_0^1 f_s(x)\, Q_s(x,\psi)\, dx \right| = \lim_{s\to\infty} \left| \sum_{m=1}^{N_s} b_m\, B_{ms} \right| = +\infty. \tag{11}$$

Since

$$F_s(f) = \int_0^1 f(x)\, Q_s(x,\psi)\, dx, \quad s = 1, 2, \dots$$

is the sequence of linear on $A_p(\varphi)$ functionals, and $\|f_s\|_{A_p(\varphi)} \le B$, from the condition (11), by virtue of the Banach-Steinhaus theorem, there exists the function $f_0(x) \in A_p(\varphi)$ for which the condition

$$\overline{\lim_{s\to\infty}} \left| \int_0^1 f_0(x)\, Q_s(x,\psi)\, dx \right| = +\infty \tag{12}$$

is fulfilled. Using Hölder's inequality, we obtain

$$\left| \int_0^1 f_0(x)\, Q_s(x,\psi)\, dx \right| = \left| \sum_{m=1}^{s} a_m \int_0^1 f_0(x)\, \psi_m(x)\, dx \right| =$$

$$= \left| \sum_{m=1}^{s} a_m\, \widehat{\psi}_m(f_0) \right| \le \left(\sum_{m=1}^{s} |a_m|^q \right)^{1/q} \left(\sum_{m=1}^{s} |\widehat{\psi}_m(f_0)|^p \right)^{1/p},$$

whence

$$\sum_{m=1}^{s} |\widehat{\psi}_m(f_0)|^p \ge \frac{1}{M^p} \left| \int_0^1 f_0(x)\, Q_s(x,\psi)\, dx \right|^p, \tag{13}$$

where

$$M = \left(\sum_{m=1}^{\infty} |a_m|^p \right)^{1/q}.$$

(12) and (13) result in

$$\lim_{s\to\infty} \sum_{m=1}^{s} |\widehat{\psi}_m(f_0)|^p = +\infty,$$

i.e., $f_0(x) \notin A_p(\psi)$.

Lemma 2. *Let* $f_n(x) \in \mathrm{Lip}\ 1$ $(n=1,2,\dots)$ *and* $\lim_{n\to\infty} \|f_n(x) - f_0(x)\|_{\mathrm{Lip}\ 1} = 0$. *Then*

$$\lim_{n\to\infty} \|f_n(x) - f_0(x)\|_{A_p(\varphi)} = 0,$$

if the orthonormal system (φ_n) *possesses the property* ω.

Proof. If (φ_n) possesses the property ω, then for any $f(x) \in C(0,1)$ (see [1]) we have

$$|\widehat{\varphi}_n(f)| \le C\, \omega\Big(\frac{1}{n}, f\Big), \tag{14}$$

Thus if $f(x) \in \mathrm{Lip}\ 1$, then it follows from (14) that

$$|\widehat{\varphi}_n(f)| < C \cdot n^{-1},$$

where $C > 0$ does not depend on n.

This implies that for $p > 1$,

$$\|f\|_{A_p(\varphi)} = \sum_{n=1}^{\infty} |\widehat{\varphi}_n(f)|^p \le C^p \sum_{m=1}^{\infty} \frac{1}{n^p} < +\infty.$$

Consequently, $f_n(x) \in A_p(\varphi)$ $(n = 1, 2, \dots)$ and $f_0(x) \in A_p(\varphi)$.

Analogously,

$$I_2 \le \frac{1}{n} \sum_{k=1}^{n} \max_{x\in[\frac{k-1}{n},\frac{k}{n}]} \frac{\left|f(x) - f\left(\frac{k}{n}\right)\right|}{\frac{1}{n}} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |\varphi_n(x)|\, dx \le$$
$$\le \frac{1}{n} \|f\|_{\mathrm{Lip}\ 1} \left(\int_0^1 \varphi_n^2(x)\, dx \right)^{1/2} = \frac{1}{n} \|f\|_{\mathrm{Lip}\ 1}. \tag{15}$$

Now in equality (3) we put $f(x) = F_m(x) = f_m(x) - f_0(x)$ and from (16) and (17) we find that

$$|\widehat{\varphi}_n(F_m)| = \left| \int_0^1 F_m(x)\, \varphi_n(x)\, dx \right| \le \frac{h+1}{n} \|F_m\|_{\mathrm{Lip}\ 1},$$

whence for $p > 1$ we obtain

$$\|F_m\|_{A_p(\varphi)} = \sum_{n=1}^{\infty} |\widehat{\varphi}_n(F_m)|^p \le (h+1)^p \|F_m\|_{\mathrm{Lip}\ 1}^p \sum_{n=1}^{\infty} \frac{1}{n^p} <$$
$$< 2(h+1)^p \|F_m\|_{\mathrm{Lip}\ 1}^p.$$

Consequently, if $\lim\limits_{m\to\infty} \|F_m\|_{\mathrm{Lip}\ 1} = 0$, then $\lim_{m\to\infty} \|F_m\|_{A_p(\varphi)} = 0$.

Theorem 4. *Let $\varphi \in P$ possess the properties ω and A_p $(p > 1)$. $A(\varphi) = \psi$, where A is the operator mapping P onto P. If for some $(a_k) \in \ell_q$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$,*

$$\lim_{s\to\infty} \sum_{m=1}^{N_s} |B_{ms}|^q = +\infty \tag{16}$$

then the operator A does not preserve information A_p at the point φ (N_s see (6)).

Proof. Equality (18) and Lemma 1 imply that there exists the function $f_0(x) \in A_p(\varphi)$ such that $f_0 \notin A_p(\psi)$. From the proof of Lemma 1 and from the condition of the Banach-Steinhaus theorem it follows that there exists the set $B \subset A_p(\varphi)$ such that B is the set of the second category and $A_p(\varphi) \backslash B$ is that of the first category. Consequently, for any $f \in B$ we have $f \notin A_p(\psi)$.

Since (φ_n) possesses the property ω, therefore $\text{Lip } 1 \subset A_p(\varphi)$, and by Lemma 2, if

$$\lim_{m\to\infty} \left\| f_m(x) - f(x) \right\|_{\text{Lip } 1} = 0, \tag{17}$$

then

$$\lim_{m\to\infty} \left\| f_m(x) - f(x) \right\|_{A_p(\varphi)} = 0. \tag{18}$$

From here $f(x) \in \text{Lip } 1$ and $f(x) \in A_p(\varphi)$. Next, there exists a sequence $B_m(x) \in B$ such that

$$\lim_{m\to\infty} \left\| B_m(x) - f(x) \right\|_{A_p(\varphi)} = 0$$

and

$$\sum_{n=1}^{\infty} \left| \widehat{\psi}_n(B_m) \right|^p = +\infty. \tag{21}$$

Now suppose the contrary that $f(x) \in A_p(\psi)$. We have ($1 < p < 2$, (φ_n) is ONCS)

$$\begin{aligned} \left\| B_m(x) - f(x) \right\|_{L_2}^2 &= \sum_{n=1}^{\infty} \widehat{\varphi}_n^2(B_m - f) \le \sum_{n=1}^{\infty} \left| \widehat{\varphi}(B_m - f) \right|^p = \\ &= \left\| B_m(x) - f(x) \right\|_{A_p(\varphi)}. \end{aligned}$$

From here if

$$\lim_{m\to\infty} \left\| B_m(x) - f(x) \right\|_{A_p(\varphi)} = 0,$$

then

$$\lim_{m\to\infty} \left\| B_m(x) - f(x) \right\|_{L_2} = 0.$$

Consequently

$$\lim_{m\to\infty} \int_0^1 B_m(x)\psi_n(x)\,dx = \int_0^1 f(x)\psi_n(x)\,dx. \tag{22}$$

Using (22) for any N we obtain ($f(x) \in A_p(\psi)$)

$$\lim_{m\to\infty} \sum_{n=1}^{N} \left| \widehat{\psi}_n(B_m) \right|^p \le \sum_{n=1}^{N} \left| \int_0^1 f(x)\psi_n(x)\,dx \right|^p \le \left\| f \right\|_{A_p(\psi)} < +\infty.$$

This implies that

$$\sum_{n=1}^{\infty} \left| \widehat{\psi}_n(B_m) \right|^p \le M_0 \tag{23}$$

(where M_0 is an absolute constant), but (23) contradicts (21).

Thus there exists the function $f(x) \in \text{Lip } 1$ (i.e., $f(x) \in V(0,1)$) such that $f(x) \in A_p(\varphi)$ and $f(x) \notin A_p(\psi)$. Consequently, the operator A does not preserve information A_p at the point φ.

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