ON CONSTRUCTION OF APPROXIMATE SOLUTIONS OF EQUATIONS OF THE NON-SHALLOW SPHERICAL SHELL FOR THE GEOMETRICALLY NONLINEAR THEORY

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Abstract

In the present paper by means of the I. Vekua method the system of differential equations for the nonlinear theory of non-shallow spherical shells is obtained. Using the method of the small parameter approximate solutions of I. Vekua's equations for approximations $N = 0$ is constructed. The small parameter $\varepsilon = h/R$, where $2h$ is the thickness of the shell, R is the radius of the sphere. Concrete problem is solved.

Key words and phrases: Non-shallow shells, geometrically nonlinear theory, small parameter, spherical shells.

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1 Equations of Equilibrium of an Elastic Medium

Making use of tensor notation, we can write the equilibrium equation of the continuous medium and stress-strain relations (Hooke's law) in the form [1]

$$\hat{\nabla}_i \boldsymbol{\sigma}^i + \boldsymbol{\Phi} = 0, \tag{1}$$

$$\boldsymbol{\sigma}^i = E^{ijpq} e_{pq} \left(\boldsymbol{R}_j + \partial_j \boldsymbol{u}\right), \tag{2}$$

where $\hat{\nabla}_i$ are covariant derivatives with respect to the space coordinates x^i, $\boldsymbol{\Phi}$ is the volume force, $\boldsymbol{\sigma}^i$ are the contravariant constituents of the stress vectors, E^{ijpq} is the contravariant tensor of rank four:

$$E^{ijpq} = \lambda g^{ij} g^{pq} + \mu \left(g^{ip} g^{jq} + g^{iq} g^{ip}\right) \quad \left(g^{ij} = \boldsymbol{R}^i \boldsymbol{R}^j\right),$$

λ and μ are Lamé's constants, e_{pq} are covariant components of the strain tensor:

$$e_{pq} = \frac{1}{2}\Big(\boldsymbol{R}_p \partial_q \boldsymbol{u} + \boldsymbol{R}_q \partial_p \boldsymbol{u} + \partial_p \boldsymbol{u} \partial_q \boldsymbol{u}\Big), \tag{3}$$

$\boldsymbol{u}$ is the displacement vector, $\boldsymbol{R}_i$ and $\boldsymbol{R}^i$ are covariant and contravariant base vectors of the space, g^{ij} are contravariant components of the discriminant g of the metric quadratic form of the space.

2 Non-shallow spherical shells

Let Ω denote a shall and the domain of the space occupied by this shell. Inside the shell, we consider a smooth surface S with respect to which the shell Ω lies symmetrically. The surface S is called the midsurface of the shell Ω. To construct the theory of shells, we use the more convenient coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\boldsymbol{R}$ of any point of the domain Ω can be represented in the form

$$\boldsymbol{R}(x^1, x^2, x^3) = \boldsymbol{r}(x^1, x^2) + x^3\boldsymbol{n}(x^1, x^2),$$

where $\boldsymbol{r}$ and $\boldsymbol{n}$ are the radius-vector and the unit vector of the normal of the surface S $(x^3 = 0)$, respectively, (x^1, x^2) are the Gaussian parameters of the midsurfaces.

For the non-shallow spherical shell of radius R, $\boldsymbol{R}_i$ and $\boldsymbol{R}^i$ are connected with the covariant and contravariant base vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}^i$ of the midsurface S $(x^3 = 0)$ by the following relations:

$$\boldsymbol{R}_\alpha = \left(1 + \frac{x^3}{R}\right)\boldsymbol{r}_\alpha, \quad \boldsymbol{R}^\alpha = \frac{1}{1 + \dfrac{x^3}{R}}\boldsymbol{r}^\alpha, \quad \boldsymbol{R}_3 = \boldsymbol{R}^3 = \boldsymbol{n}. \tag{4}$$

The relation (1) can be written as:

$$\nabla_\alpha(\vartheta\boldsymbol{\sigma}^\alpha) + \partial_3(\vartheta\boldsymbol{\sigma}^3) + \vartheta\boldsymbol{\Phi} = 0, \tag{5}$$

where ∇_α are covariant derivatives on the midsurface S $(x_3 = 0)$ of spherical shell and $\vartheta = \left(1 + \dfrac{x^3}{R}\right)^2$.

The Hooke's law (2) can be written as:

$$\begin{aligned}\boldsymbol{\sigma}^\alpha = \lambda &\left[\frac{\boldsymbol{r}^\beta\partial_\beta\boldsymbol{u}}{1 + \dfrac{x^3}{R}} + \boldsymbol{n}\partial_3\boldsymbol{u} + \frac{\partial^\beta\boldsymbol{u}\partial_\beta\boldsymbol{u}}{2\left(1 + \dfrac{x^3}{R}\right)^2} + \frac{1}{2}\partial^3\boldsymbol{u}\partial_3\boldsymbol{u}\right] \\ &\times\left(\frac{\boldsymbol{r}^\alpha}{1 + \dfrac{x^3}{R}} + \frac{\partial^\alpha\boldsymbol{u}}{\left(1 + \dfrac{x^3}{R}\right)^2}\right)\end{aligned} \tag{6}$$

$$+\mu\left[\frac{\boldsymbol{r}^{\alpha}\partial^{\beta}\boldsymbol{u}+\boldsymbol{r}^{\beta}\partial^{\alpha}\boldsymbol{u}}{\left(1+\dfrac{x^3}{R}\right)^3}+\frac{\partial^{\alpha}\boldsymbol{u}\partial^{\beta}\boldsymbol{u}}{\left(1+\dfrac{x^3}{R}\right)^4}\right]\left(\left(1+\frac{x^3}{R}\right)\boldsymbol{r}_{\beta}+\partial_{\beta}\boldsymbol{u}\right)$$
$$+\mu\left[\frac{\boldsymbol{r}^{\alpha}\partial^{3}\boldsymbol{u}}{1+\dfrac{x^3}{R}}+\frac{(\boldsymbol{n}+\partial^{3}\boldsymbol{u})\partial^{\alpha}\boldsymbol{u}}{\left(1+\dfrac{x^3}{R}\right)^2}\right](\boldsymbol{n}+\partial_{3}\boldsymbol{u}),$$

$$\begin{aligned}\boldsymbol{\sigma}^3=\lambda&\left[\frac{\boldsymbol{r}^{\beta}\partial_{\beta}\boldsymbol{u}}{1+\dfrac{x^3}{R}}+\boldsymbol{n}\partial_{3}\boldsymbol{u}+\frac{\partial^{\beta}\boldsymbol{u}\partial_{\beta}\boldsymbol{u}}{2\left(1+\dfrac{x^3}{R}\right)^2}+\frac{1}{2}\partial^{3}\boldsymbol{u}\partial_{3}\boldsymbol{u}\right](\boldsymbol{n}+\partial_{3}\boldsymbol{u})\\ +\mu&\left[\frac{(\boldsymbol{n}+\partial^{3}\boldsymbol{u})\partial^{\beta}\boldsymbol{u}}{\left(1+\dfrac{x^3}{R}\right)^2}+\frac{\boldsymbol{r}^{\beta}\partial^{3}\boldsymbol{u}}{1+\dfrac{x^3}{R}}\right]\left(\left(1+\frac{x^3}{R}\right)\boldsymbol{r}_{\beta}+\partial_{\beta}\boldsymbol{u}\right)\\ +\mu&\left[2\boldsymbol{n}\partial^{3}\boldsymbol{u}+\partial^{3}\boldsymbol{u}\partial^{3}\boldsymbol{u}\right](\boldsymbol{n}+\partial_{3}\boldsymbol{u}).\end{aligned}\tag{7}$$

3 I. Vekua's reduction method

In the present paper we use I. Vekua's reduction method for the nonlinear theory of non-shallow shells (I. Vekua used the method for linear theory of shallow shells) the essence of which consists, without going into detals, in the following: since the system of Legendre polynomials $P_m(\frac{x_3}{h})$ is complete in the interval $[-h,h]$, for equation (5) the equivalent infinite system of 2-D equations is obtained [2-3]

$$\nabla_{\alpha}\overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha}-\frac{2m+1}{h}\left(\overset{(m-1)}{\boldsymbol{\sigma}}{}^{3}+\overset{(m-3)}{\boldsymbol{\sigma}}{}^{3}+...\right)+\overset{(m)}{\boldsymbol{F}}=0,\tag{8}$$

where

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^{i},\overset{(m)}{\boldsymbol{\Phi}}\right)=\frac{2m+1}{2h}\int_{-h}^{h}\left(\vartheta\boldsymbol{\sigma}^{i},\vartheta\boldsymbol{\Phi}\right)P_m\left(\tfrac{x_3}{h}\right)dx_3,$$

$$\overset{(m)}{\boldsymbol{F}}=\overset{(m)}{\boldsymbol{\Phi}}+\frac{2m+1}{2h}\left(\overset{(+)}{\vartheta}\overset{(+)}{\boldsymbol{\sigma}}{}_{3}-(-1)^m\overset{(-)}{\vartheta}\overset{(-)}{\boldsymbol{\sigma}}{}_{3}\right).$$

Thus we have obtained the infinite system of 2-D equations (8), for which the boundary conditions of the face surfaces ($x_3 = \pm h$) are satisfied, i.e. $\overset{(\pm)}{\boldsymbol{\sigma}}{}^3 = \boldsymbol{\sigma}^3(x^1, x^2, \pm h)$ is the preassigned vector field and is entered in the equilibrium equations.

The equations of the state (6) may be write as:

$$
\begin{aligned}
\overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha} = {} & \lambda\Big(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m)}{\boldsymbol{u}}\Big)\boldsymbol{r}^{\alpha} + \mu\Big(\boldsymbol{r}^{\alpha}\partial_{\beta}\overset{(m)}{\boldsymbol{u}}\Big)\boldsymbol{r}^{\beta} + \mu\partial^{\alpha}\overset{(m)}{\boldsymbol{u}} \\
& + \sum_{m_1=0}^{\infty}\Bigg\{B^{m}_{m_1}\left[\lambda \overset{(m_1)}{u}{}'^{3}\boldsymbol{r}^{\alpha} + \mu \overset{(m_1)}{u}{}'^{\alpha}\boldsymbol{n}\right] \\
& + \sum_{m_2=0}^{\infty}\Bigg\{A^{m}_{m_1 m_2}\left[\frac{\lambda}{2}\Big(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\,\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}}\Big)\boldsymbol{r}^{\alpha} + \lambda\Big(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m_1)}{\boldsymbol{u}}\Big)\partial^{\alpha}\overset{(m_2)}{\boldsymbol{u}}\right. \\
& \left. + \mu\Big(\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\,\partial^{\beta}\overset{(m_2)}{\boldsymbol{u}}\Big)\boldsymbol{r}_{\beta} + \mu\Big(\boldsymbol{r}^{\alpha}\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}} + \boldsymbol{r}^{\beta}\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\Big)\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}}\right] \\
& + B^{m}_{m_1 m_2}\left[\frac{\lambda}{2}\Big(\overset{(m_1)}{\boldsymbol{u}}{}'\,\overset{(m_2)}{\boldsymbol{u}}{}'\Big)\boldsymbol{r}^{\alpha} + \mu \overset{(m_1)}{u}{}'^{\alpha}\,\overset{(m_2)}{\boldsymbol{u}}{}'\right] \qquad (9) \\
& + C^{m}_{m_1 m_2}\left[\lambda \overset{(m_1)}{u}{}'^{3}\partial^{\alpha}\overset{(m_2)}{\boldsymbol{u}} + \mu\Big(\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\,\overset{(m_2)}{\boldsymbol{u}}{}'\Big)\boldsymbol{n} + \mu\Big(\boldsymbol{n}\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\Big)\overset{(m_2)}{\boldsymbol{u}}{}'\right] \\
& + \sum_{m_3=0}^{\infty}\Bigg\{\bar{A}^{m m_3}_{m_1 m_2}\left[\frac{\lambda}{2}\Big(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\,\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}} + \overset{(m_1)}{\boldsymbol{u}}{}'\,\overset{(m_2)}{\boldsymbol{u}}{}'\Big)\partial^{\alpha}\overset{(m_3)}{\boldsymbol{u}}\right. \\
& \left. + \mu\Big(\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\,\partial^{\alpha}\overset{(m_2)}{\boldsymbol{u}}\Big)\partial_{\beta}\overset{(m_3)}{\boldsymbol{u}}\right] + \mu C^{m m_3}_{m_1 m_2}(\partial^{\alpha}\overset{(m_1)}{\boldsymbol{u}}\,\overset{(m_2)}{\boldsymbol{u}}{}')\,\overset{(m_3)}{\boldsymbol{u}}{}'\Bigg\}\Bigg\}\Bigg\}
\end{aligned}
$$

where

$$
\overset{(m)}{\boldsymbol{u}} = \frac{2m+1}{2h}\int_{-h}^{h}\boldsymbol{u}P_m\Big(\frac{x_3}{h}\Big)dx_3, \quad \overset{(m)}{\boldsymbol{u}}{}' = \frac{2m+1}{h}\Big(\overset{(m+1)}{\boldsymbol{u}} + \overset{(m+3)}{\boldsymbol{u}} + \ldots\Big),
$$

$$
\begin{aligned}
B^{m}_{m_1} &= \frac{2m+1}{2h}\int_{-h}^{h}\Big(1+\frac{x_3}{R}\Big)P_{m_1}\Big(\frac{x_3}{h}\Big)P_m\Big(\frac{x_3}{h}\Big)dx_3 \\
&= \delta^{m}_{m_1} + \frac{h}{R}\left[\frac{m+1}{2m+3}\delta^{m+1}_{m_1} + \frac{m}{2m-1}\delta^{m-1}_{m_1}\right], \\
B^{m}_{m_1 m_2} &= \frac{2m+1}{2h}\int_{-h}^{h}\Big(1+\frac{x_3}{R}\Big)P_{m_1}\Big(\frac{x_3}{h}\Big)P_{m_2}\Big(\frac{x_3}{h}\Big)P_m\Big(\frac{x_3}{h}\Big)dx_3 \\
&= \sum_{s=0}^{\min(m_1,m_2)}\alpha_{m_1 m_2 s}B^{m}_{m_1+m_2-2s},
\end{aligned}
$$

$$A^m_{m_1m_2} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\frac{P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)}{1+\frac{x_3}{R}}dx_3$$

$$= \begin{cases} -\dfrac{R(2m+1)}{h}\dfrac{\partial}{\partial y}\sum\limits_{r=0}^{\min(m_1,m_2)}\alpha_{m_1m_2r}P_{m_1+m_2-2r}\left(-\dfrac{R}{h}\right) \\ \times Q_m\left(-\dfrac{R}{h}\right), \qquad m_1+m_2-2r \le m \\ -\dfrac{R(2m+1)}{h}\dfrac{\partial}{\partial y}\sum\limits_{r=0}^{\min(m_1,m_2)}\alpha_{m_1m_2r}P_m\left(-\dfrac{R}{h}\right) \\ \times Q_{m_1+m_2-2r}\left(-\dfrac{R}{h}\right), \qquad m_1+m_2-2r > m \end{cases}$$

$$\bar{A}^{mm_3}_{m_1m_2} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\frac{P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_{m_3}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)}{\left(1+\frac{x_3}{R}\right)^2}dx_3$$

$$= \begin{cases} -\dfrac{R^2(2m+1)}{h^2}\dfrac{\partial}{\partial y}\sum\limits_{r=0}^{\min(m_1,m_2)}\alpha_{m_1m_2r}\sum\limits_{s=0}^{\min(m_3,m)}\alpha_{m_3ms}P_{m_1+m_2-2r}\left(-\dfrac{R}{h}\right) \\ \times Q_{m_3+m-2s}\left(-\dfrac{R}{h}\right), \qquad m_1+m_2-2r \le m_3+m-2s \\ -\dfrac{R^2(2m+1)}{h^2}\dfrac{\partial}{\partial y}\sum\limits_{r=0}^{\min(m_1,m_2)}\alpha_{m_1m_2r}\sum\limits_{s=0}^{\min(m_3,m)}\alpha_{m_3ms}P_{m_3+m-2s}\left(-\dfrac{R}{h}\right) \\ \times Q_{m_1+m_2-2r}\left(-\dfrac{R}{h}\right), \qquad m_1+m_2-2r > m_3+m-2s \end{cases}$$

$$C^m_{m_1m_2} = \frac{2m+1}{2h}\int\limits_{-h}^{h}P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3$$

$$= \sum_{r=0}^{\min(m_1,m_2)}\alpha_{m_1m_2r}\delta^r_{m_1+m_2-2r},$$

$$C^{mm_3}_{m_1m_2} = \frac{2m+1}{2h}\int\limits_{-h}^{h}P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_{m_3}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3$$

$$= \sum_{r=0}^{\min(m_1,m_2)}\alpha_{m_1m_2r}\sum_{s=0}^{\min(m_3,m)}\alpha_{m_3ms}\delta^{m_3+m-2s}_{m_1+m_2-2r},$$

The equations of the state (7) may be write as:

$$
\begin{aligned}
&\overset{(m)}{\boldsymbol{\sigma}}{}^{3} = \sum_{m_1=0}^{\infty} \Bigg\{ B_{m_1}^{m} \left[\lambda(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m_1)}{\boldsymbol{u}})\boldsymbol{n} + \mu(\boldsymbol{n}\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}})\boldsymbol{r}_{\beta} \right] \\
&+\bar{B}_{m_1}^{m} \left[(\lambda+\mu)\overset{(m_1)}{u}{}'_{3}\boldsymbol{n} + \mu\overset{(m_1)}{\boldsymbol{u}}{}' \right] + \sum_{m_2=0}^{\infty} \Bigg\{ B_{m_1 m_2}^{m} \left[\lambda(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m_1)}{\boldsymbol{u}})\overset{(m_2)}{\boldsymbol{u}}{}' \right. \\
&\left. +\mu(\boldsymbol{r}^{\beta}\overset{(m_1)}{\boldsymbol{u}}{}')\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}} + \mu(\overset{(m_1)}{\boldsymbol{u}}{}'\partial^{\beta}\overset{(m_2)}{\boldsymbol{u}})\boldsymbol{r}_{\beta} \right] \\
&+\bar{B}_{m_1 m_2}^{m} \left[(\lambda+\mu)(\overset{(m_1)}{\boldsymbol{u}}{}'\overset{(m_2)}{\boldsymbol{u}}{}')\boldsymbol{n} + (\lambda+2\mu)(\boldsymbol{n}\overset{(m_1)}{\boldsymbol{u}}{}')\overset{(m_2)}{\boldsymbol{u}}{}' \right] \qquad (10) \\
&+C_{m_1 m_2}^{m} \left[\frac{\lambda}{2}(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}})\boldsymbol{n} + \mu(\boldsymbol{n}\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}})\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}} \right] \\
&+\sum_{m_3=0}^{\infty} \Bigg\{ \left(\frac{\lambda}{2}+\mu\right) \bar{B}_{m_1 m_2}^{m m_3}(\overset{(m_1)}{\boldsymbol{u}}{}'\overset{(m_2)}{\boldsymbol{u}}{}')\overset{(m_3)}{\boldsymbol{u}}{}' \\
&+C_{m_1 m_2}^{m m_3} \left[\frac{\lambda}{2}(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\partial_{\beta}\overset{(m_2)}{\boldsymbol{u}})\overset{(m_3)}{\boldsymbol{u}}{}' + \mu(\partial^{\beta}\overset{(m_1)}{\boldsymbol{u}}\overset{(m_2)}{\boldsymbol{u}}{}')\partial_{\beta}\overset{(m_3)}{\boldsymbol{u}} \right] \Bigg\} \Bigg\} \Bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{B}_{m_1}^{m} &= \frac{2m+1}{2h}\int_{-h}^{h}\left(1+\frac{x_3}{R}\right)^2 P_{m_1}\Big(\frac{x_3}{h}\Big)P_m\Big(\frac{x_3}{h}\Big)dx_3 = \delta_{m_1}^{m} \\
&+ \frac{2h}{R}\left[\frac{m+1}{2m+3}\delta_{m_1}^{m+1} + \frac{m}{2m-1}\delta_{m_1}^{m-1}\right] + \frac{h^2}{R^2}\left[\frac{m(m+1)\delta_{m_1}^{m-2}}{(2m-1)(2m-3)}\right. \\
&+ \left.\frac{2m^2+2m-1}{(2m-1)(2m+3)}\delta_{m_1}^{m} + \frac{(m+1)(m+2)}{(2m+3)(2m+5)}\delta_{m_1}^{m+2}\right],
\end{aligned}
$$

$$
\begin{aligned}
\bar{B}_{m_1 m_2}^{m} &= \frac{2m+1}{2h}\int_{-h}^{h}\left(1+\frac{x_3}{R}\right)^2 P_{m_1}\Big(\frac{x_3}{h}\Big)P_{m_2}\Big(\frac{x_3}{h}\Big)P_m\Big(\frac{x_3}{h}\Big)dx_3 \\
&= \sum_{s=0}^{\min(m_1,m_2)} \alpha_{m_1 m_2 s}\bar{B}_{m_1+m_2-2s}^{m},
\end{aligned}
$$

$$
\begin{aligned}
\bar{B}_{m_1 m_2}^{m_3 m} &= \frac{2m+1}{2h}\int_{-h}^{h}\left(1+\frac{x_3}{R}\right)^2 P_{m_1}\Big(\frac{x_3}{h}\Big)P_{m_2}\Big(\frac{x_3}{h}\Big)P_{m_3}\Big(\frac{x_3}{h}\Big)P_m\Big(\frac{x_3}{h}\Big)dx_3 \\
&= \sum_{s=0}^{\min(m_1,m_2)} \alpha_{m_1 m_2 s}\sum_{r=0}^{\min(m_3,m)} \alpha_{m_3 m_r}\bar{B}_{m_1+m_2-2s}^{m_3+m-2r}.
\end{aligned}
$$

Here we have used the formulas of F. Neumann and J. Adams:

$$\int_{-1}^{1} \frac{P_m(t)dt}{x-t} = 2Q_m(x), \quad (|x| > 1),$$

$$P_m(x)P_n(x) = \sum_{r=0}^{\min(m,n)} \alpha_{mnr} P_{m+n-2r}(x).$$

respectively, where $Q_m(x)$ is the Legender function of second order, and

$$\alpha_{mnr} = \frac{A_{m-r}A_r A_{n-r}}{A_{m+n-r}} \frac{2m+2n-4r+1}{2m+2n-2r+1}, \quad A_m = \frac{1\cdot 3\cdots(2m-1)}{m!}.$$

4 Approximation of Order $N = 0$

Introduce the notations

$$\overset{(0)}{\boldsymbol{\sigma}}_i = \boldsymbol{T}_i, \quad \overset{(0)}{\boldsymbol{F}} = \boldsymbol{X}.$$

To find components of the displacements vector and stress tensor, we take the following series of expansions with respect to the small parameter ε [4-5]:

$$(u_i,\ \boldsymbol{T}_i,\ X_i) = \sum_{k=1}^{\infty} (\overset{(k)}{u}_i, \overset{(k)}{\boldsymbol{T}}_i, \overset{(k)}{X}_i)\varepsilon^k. \tag{11}$$

Substituting the above expansions into relations (8), (9), (10) and than equalizing the coefficients of expansions for ε^n, we obtain the following system of equations:

$$\begin{aligned} &4\mu\partial_{\bar z}\left(\frac{1}{\Lambda}\partial_z \overset{(k)}{u}_+\right) + 2(\lambda+\mu)\partial_{\bar z}\overset{(k)}{\theta} = \overset{(k)}{X}_+\left(\overset{(0)}{u}_i, ..., \overset{(k-1)}{u}_i\right), \\ &\mu\nabla^2 \overset{(k)}{u}_3 = \overset{(k)}{X}_3\left(\overset{(0)}{u}_i, ..., \overset{(k-1)}{u}_i\right), \end{aligned} \tag{12}$$

where

$$x^1 = \tan\frac{\theta}{2}\cos\varphi, \quad x^2 = \tan\frac{\theta}{2}\sin\varphi,$$

$\left(z = x^1 + ix^2, \Lambda = \frac{4\rho^2}{(1+z\bar z)^2}, \nabla^2 = \frac{4}{\Lambda}\partial^2_{z\bar z}\right)$, are the isometric coordinates on the shell midsurface of spherical shell,

$$\overset{(k)}{u}_+ = \overset{(k)}{u}_1 + i\overset{(k)}{u}_2,$$

$$\overset{(k)}{\theta}=\frac{1}{\Lambda}\left(\partial_z\overset{(k)}{u}_{+}+\partial_{\bar z}\overset{(k)}{\overline{u}}_{+}\right).$$

Introducing the well-known differential operators

$$\partial_z=\frac{1}{2}\left(\partial x^1-i\partial x^2\right),\quad \partial_{\bar z}=\frac{1}{2}\left(\partial x^1+i\partial x^2\right).$$

$\overset{(k)}{X}_{+}$ and $\overset{(k)}{X}_3$ are expressed by $\overset{(0)}{u}_{+}, \overset{(0)}{u}_3, ..., \overset{(k-1)}{u}_{+}, \overset{(k-1)}{u}_3$ and it is assumed that they are already found.

Simple calculations show that general solutions of the system (12) can be represented by means of three analytic functions of z in the form

$$\overset{(k)}{u}_{+}=-\frac{\varkappa}{\pi}\int\int_D\frac{\Lambda(\zeta,\bar\zeta)\varphi'(\zeta)d\xi d\eta}{\bar\zeta-\bar z}+\left(\frac{1}{\pi}\int\int_D\frac{\Lambda(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}\right)\overline{\varphi'(z)}\tag{13}$$

$$-\overline{\psi(z)}+\frac{1}{8\mu h^2}\frac{\lambda+\mu}{\lambda+2\mu}\frac{1}{\pi}\int\int_D\frac{\overset{(k)}{F}_{+}(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}$$

$$\overset{(k)}{u}_3=f(z)+\overline{f(z)}-\frac{2}{\pi}\int\int_D\overset{(k)}{X}_3\ln|\zeta-z|d\xi d\eta.\tag{14}$$

where $\varphi'(z), f(z)$ and $\psi(z)$ are analytic functions of $z=x_1+ix_2\in D$, and $\zeta=\xi+i\eta$.

Further,

$$\overset{(k)}{F}_{+}(z,\bar z)=-\frac{1}{\pi}\int\int_D\left(\frac{\overline{\overset{(k)}{X}}}{\bar\zeta-\bar z}-\frac{\varkappa\overset{(k)}{X}}{\zeta-z}\right)d\xi d\eta,\quad\left(\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}\right).$$

D is the domain of the plane Ox_1x_2 onto which the midsurface S of the shell Ω is mapped topologically.

Here we present a general scheme of solution of boundary problems when the domain D is the circular ring with radius R_1 and R_2.

The second boundary problem (in displacements) for any k takes the form

$$\overset{(k)}{u}_{+}=-\frac{\varkappa}{\pi}\iint_D\frac{\Lambda(\zeta,\bar\zeta)\varphi'(\zeta)d\xi d\eta}{\bar\zeta-\bar z}+\left(\frac{1}{\pi}\iint_D\frac{\Lambda(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}\right)\overline{\varphi'(z)}$$

$$-\overline{\psi(z)}=\begin{cases}\overset{(k)}{G}{}'_{+}, & |z|=R_1,\\ \overset{(k)}{G}{}''_{+}, & |z|=R_2.\end{cases}\tag{15}$$

$$\overset{(k)}{u}_3 = f(z) + \overline{f(z)}|_{r_0} = \begin{cases} \overset{(k)}{G}{}'_3, & |z| = R_1, \\ \overset{(k)}{G}{}''_3, & |z| = R_2. \end{cases} \tag{16}$$

where $\overset{(k)}{G}_+$ and $\overset{(k)}{G}_3$ are the known values containing solutions $\overset{(0)}{u}_i$, $\overset{(1)}{u}_i, ...,$ $\overset{(k-1)}{u}_i$, $(i = 1, 2, 3)$ of the previous approximations.

Next $\varphi'(z)$, $\psi(z)$ and $f(z)$ are expanded in power series of the type

$$\varphi'(z) = \sum_{-\infty}^{\infty} a_n z^n, \quad \psi(z) = \sum_{-\infty}^{\infty} b_n z^n, \quad f(z) = \gamma \ln z + \sum_{-\infty}^{\infty} c_n z^n, \tag{17}$$

and the expression $\overset{(k)}{G}_+$ and $\overset{(k)}{G}_3$ in the form of a complex Fourier series

$$\overset{(k)}{G}_+ = \sum_{-\infty}^{\infty} A_k e^{ik\theta}, \quad \overset{(k)}{G}_3 = \sum_{-\infty}^{\infty} B_k e^{ik\theta}.$$

By substituting (17) into (15) we obtain the system of algebraic equations

$$\varkappa\alpha_{-n+1}a_{-n} - 2\bar{a}_n - \bar{b}_{n-1} = \frac{A'_{-n+1}}{R_1^{n-1}}, \quad n \geq 1,$$

$$\varkappa\alpha_n a_n - 2\bar{a}_{-n} - \bar{b}_{-n-1} = R_2^{n+1}A''_{n+1}, \quad n \geq 0,$$

$$-2\bar{a}_n - \bar{b}_{n-1} = \frac{A''_{-n+1}}{R_2^{n-1}}, \quad n \geq 1,$$

$$-2\bar{a}_{-n} - \bar{b}_{-n-1} = R_1^{n+1}A'_{n+1}, \quad n \geq 0,$$

where $\alpha_n = 8R^2 \int\limits_{R_1}^{R_2} \frac{\rho^{2n+1}}{(1+\rho^2)^2} d\rho$.

For coefficients a_n and b_n we have:

$$a_{-n} = \frac{R_2^{n-1}A'_{-n+1} - R_1^{n-1}A''_{-n+1}}{\varkappa R_1^{n-1}R_2^{n-1}\alpha_{-n+1}}, \quad n \geq 1,$$

$$a_n = \frac{R_2^{n+1}A''_{n+1} - R_1^{n+1}A'_{n+1}}{\varkappa\alpha_n}, \quad n \geq 0,$$

$$b_{n-1} = -2a_n - \frac{\bar{A}''_{-n+1}}{R_2^{n-1}}, \quad n \geq 1,$$

$$b_{-n-1} = -2a_{-n} - R_1^{n+1}\bar{A}'_{n+1}, \quad n \geq 0.$$

For coefficients c_n and γ we have:

$$\gamma = \frac{B_0' - B_0''}{\ln R_1/R_2}, \quad c_0 + \bar{c}_0 = \frac{B_0' \ln R_2 - B_0'' \ln R_1}{\ln R_2 - \ln R_1},$$

$$c_n = \frac{R_2^n B_n'' - R_1^n B_n'}{R_2^{2n} - R_1^{2n}}.$$

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