ON TRANSFORMATIONS OF A RANDOM MEASURE

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Abstract

In this paper random measures and their nonlinear transformations are considered. The conditions of absolute continuity for this measures are obtained in case of nonlinear and random transformation of a space. There is given explicit formula for Radon-Nikodym derivative. The notion of measurable functional is used and the logarithmic derivative technique of measures is developed.

Key words and phrases: Random measure; Nonlinear transformation; Radon-Nikodym theorem.

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Random measures were defined and studied in [1,2], where the conditions were obtained under which formulas of integration by parts are valid, an extended stochastic integral was defined and its properties were established. Random measures are of great practical importance because they surely appear as solutions of differential equations in measures (see [3]), with random coefficients and additive noise.

In the present paper we study the absolutely continuity of distributions of random measures in the case of their nonlinear transformation. The Radon-Nykodim density is calculated, the density formula containing an extended stochastic integral over a random measure.

The terms and notation used here are the same as in [4].

1. $\{\Omega, \mathcal{F}, P\}$ is a fixed probability space. Let $\mu(\Delta, \omega)$ be a real, random a.s. σ-additive function of sets on some measurable space $\{X, B(X)\}$. Assume that $E\mu(\Delta) = 0$ and there exists an σ-additive measure $\beta(\Delta_1 \times \Delta_2) = E\mu(\Delta_1)\mu(\Delta_2)$ on $\{X \times X, B(X) \times B(X)\}$. Let H_0 be the space of measurable function $\Delta\varphi$ on $\{X, B(X)\}$ with real values and scalar product

$$(\varphi, \psi)_0 = \int_X \int_X \varphi(x)\psi(y)\beta(dx \times dy) = E \int_X \varphi(x)\mu(dx) \int_X \psi(y)\mu(dy).$$

Following [2], we will construct the conjugate space to H_0 and realize it as a space of measures $\mu = \mu_\varphi = S\varphi$, where S is a unitary operator

$$S : H_0 \to H_0^*$$

and

$$\mu_\varphi(\Delta) = \int_X \varphi(x)\beta(dx \times \Delta)$$

so that the pairing of elements from H_0 and H_0^* can be written in the form

$$\langle\psi, \mu_\varphi\rangle = \langle\psi, S\varphi\rangle = (\psi, \varphi)_0.$$

It is obvious that H_0^* is also a Hilbert space with the scalar product

$$(\nu_\varphi, \nu_\psi)_* = (\varphi, \psi)_0.$$

As follows from [1], we can construct the embedding operator $\Im$ and the Hilbert space H_+ densely embedded in H_0 and being such that $\Im$ is the Hilbert-Shmidt operator. Thus we can construct a triple of Hilbert equipment

$$H_+ \overset{\Im}{\subset} H_0 \overset{S}{=} H_0^* \overset{\Im^*}{\subset} H_-, \tag{1}$$

observing that the distribution $\tilde{\mu}(A) = \tilde{\mu}_\Delta(A) = P\{\nu(\Delta) \in A\}$ is concentrated in H_-. Let $L(M, N)$ denote the space of real functionals defined on M and differentiated along constant elements $N \subset M$. In the sequel it will be assumed that the distribution $\tilde{\mu}$ is a smooth measure on H_- (see [3]). This means that there exists a measurable function $\lambda : H_- \to H_-$ called the logarithmic derivative of the measure $\tilde{\mu}$ and being such that the formula of integration by parts

$$\int_{H_-} \langle\varphi, f'(\mu)\rangle\tilde{\mu}(d\mu) = -\int_{H_-} f(\mu)\langle\varphi, \lambda(\mu)\rangle\tilde{\mu}(d\mu)$$

holds for functions $f \in L(H_-, H_-)$. Recall that the expression $\langle\varphi, \lambda(\mu)\rangle$ is called a stochastic integral. We denote it in a customary way by

$$\int_X \varphi(x)\lambda(\mu)(dx). \tag{2}$$

We can extend this expression to random functions of the form

$$\varphi(x, \mu) \in H_+$$

and write formally

$$\langle \lambda(\mu) + D, \varphi(x,\mu)\rangle \stackrel{def}{=} \int_X \varphi(x,\mu)\lambda(\mu)(dx) + \langle \lambda + D, \varphi\rangle.$$

The extension of integral (2) to smooth functions $\varphi \in H_0$ is called an extended stochastic integral and follows:

$$\int_X \varphi(x,\mu)\lambda(\mu)(dx) = \langle \lambda + D, \varphi\rangle.$$

2. Let us consider the equipped Hilbert space of measures

$$H_+ \subset H_0 \subset H_-. \tag{3}$$

Let the distribution $\tilde{\mu}$ of a random measure μ be concentrated in H_-. We reformulate the theorem from [5] in terms of measures for triple (3). Assume that μ is a smooth measure in a sense as explained in paragraph 1. Therefore there exists a logarithmic derivative $\lambda(\mu)$ along constant directions of elements H_+. Then such $\tilde{\mu}$ has a logarithmic derivative along vector fields $z(\mu) : H_- \to H_+$ of the form

$$\rho_{\tilde{\mu}}(z,\mu) = \langle \lambda(\mu), z(\mu)\rangle + tr z'(\mu).$$

Let $U_{ts}, t, s \in (\alpha, \beta)$ be an integral flux consistent with the vector field $z(\beta, t)$, i.e.

$$\frac{du_{ts}\mu}{dt} = z(u_{ts}\mu, t), ; \quad u_{ss} = \mu$$

and $\tilde{\mu}_t = \tilde{\mu} u_{tt_0}^{-1}$. The following statement is valid

Theorem 1 ([5]). If $z(\mu, t)$ is differentiable with respect to μ and $z'_\mu(\mu, t) : H_0 \to H_0, z(\mu, t) \in H_0$ for all t and μ and $\tilde{\mu}$ has the logarithmic derivative along constant fields H_+ of the form $\rho_{\tilde{\mu}}(\varphi, \mu) = \langle \lambda(\mu), \varphi\rangle$, then all measures μ_t are equivalent and the Radon-Nikodym density can be written in the form

$$\frac{d\tilde{\mu}_t}{d\tilde{\mu}_\tau}(\mu) = \exp\left\{-\int_\tau^t \left[\langle \lambda(u_{st_0}^{-1}\mu), v_s(u_{st_0}^{-1}\mu)\rangle + tr\frac{dv_s}{d\mu}\right] ds\right\} \tag{4}$$

where $v_t(\mu) = -(u'_{tt_0})^{-1} z(t, \mu)$.

3. In the space H_- we consider the equation

$$\Im(t, \nu, \mu_t) = 0, \quad t \in [\alpha, \beta] \tag{5}$$

where ν is a constant element (measure) in H_- and $\mu = \mu_t$ is a curve. Assuming that $\Im$ in (5) has the smoothness property, we obtain

$$\frac{\partial\Im}{\partial t}+\frac{\partial\Im}{\partial\mu}\mu_t'=0,\ \ \mu_t'=-\left(\frac{\partial\Im}{\partial\mu}\right)^{-1}\frac{d\Im}{dt}$$

and if we take

$$z(t,\mu_t)=-\left(\frac{\partial\Im}{\partial\mu}\right)^{-1}\frac{d\Im}{dt}$$

as a vector field, then we have

$$\frac{d\mu_t}{dt}=z(t,\mu).$$

In that case, $\mu_t=u(t,0,\nu)$ is an integral flux for z. It is also assumed that (5) is solvable with respect to ν. Then we write

$$\nu=\Phi(t,\mu_t)$$

and calculate

$$u^{-1}(t,0,\mu_t)=\Phi(t,\mu),\quad (u^{-1})'=\frac{\partial\Phi}{\partial\mu}(t,\mu_t),$$

$$v(\nu)=-(u)_t^{-1}=-\Phi_\mu^{-1}(t,\mu_t)\Im_\mu'(t,\nu,\mu_t)^{-1}\Im_t'(t,\nu,\mu_t).$$

But

$$\Im(t,\Phi(t,\mu_t),\mu_t)\equiv 0$$

and therefore

$$\frac{\partial\Im}{\partial\nu}\cdot\frac{\partial\Phi}{\partial\mu}+\frac{\partial\Im}{\partial\mu}=0.$$

Hence we obtain

$$\frac{\partial\Im}{\partial\nu}(t,\nu,\mu_t)^{-1}=-\frac{\partial\Phi}{\partial\mu}(t,\mu_t)\frac{\partial\Im}{\partial\mu}(t,\nu,\mu_t)^{-1},$$

$$v(\nu)=\left(\frac{\partial\Im}{\partial\nu}\right)^{-1}\frac{\partial\Im}{\partial t},\ \ \frac{\partial\Phi}{\partial\mu}(t,\mu_t)=-\left(\frac{\partial\Im}{\partial\nu}\right)^{-1}\frac{\partial\Im}{\partial\mu}.$$

Moreover

$$-v'(\nu)\left(\frac{\partial\Im}{\partial\nu}\right)^{-1}\frac{\partial\Im}{\partial\mu}=\left(-\frac{\partial\Im}{\partial\nu}\right)^{-1}\frac{\partial^2\Im}{\partial\nu\partial\mu}\left(\frac{\partial\Im}{\partial\nu}\right)^{-1}\frac{\partial\Im}{\partial t}+\left(\frac{\partial\Im}{\partial\nu}\right)^{-1}\frac{\partial^2\Im}{\partial t\partial\mu}$$

and therefore

$$-v'(\nu) = \left(\frac{\partial\Im}{\partial\nu}\right)^{-1} \frac{\partial^2\Im}{\partial\nu\partial\mu}\frac{\partial\Im}{\partial t}\left(\frac{\partial\Im}{\partial\mu}\right)^{-1} - \frac{\partial^2\Im}{\partial t\partial\mu}\left(\frac{\partial\Im}{\partial\mu}\right)^{-1}.$$

Now Theorem 1 gives rise to

Theorem 2. If in equation (5) the function $\Im(t,\nu,\mu_t)$ has the derivatives $\frac{\partial\Im}{\partial\nu}, \frac{\partial\Im}{\partial t}, \frac{\partial\Im}{\partial\mu}, \frac{\partial^2\Im}{\partial t\partial\mu}$ and $\frac{\partial^2\Im}{\partial\nu\partial\mu}$, while $\left(\frac{\partial\Im}{\partial\nu}\frac{\partial^2\Im}{\partial\nu\partial\mu}\frac{\partial\Im}{\partial t} - \frac{\partial^2\Im}{\partial t\partial\mu}\right)^{-1}\left(\frac{\partial\Im}{\partial\mu}\right)^{-1}$ is a kernel operator, then measure $\tilde{\mu}_t$ are equivalent and

$$\frac{d\tilde{\mu}_t}{d\tilde{\mu}_\tau}(\mu) = \exp\{-\int_\tau^t [\langle\lambda(\Phi(s,\mu)), \left(\frac{\partial\Im}{\partial\nu}\right)^{-1}\frac{\partial\Im}{\partial t}\rangle$$
$$-tr\left(\frac{\partial\Im}{\partial\nu}\frac{\partial^2\Im}{\partial\nu\partial\mu}\frac{\partial\Im}{\partial t} - \frac{\partial^2\Im}{\partial t\partial\mu}\right)\left(\frac{\partial\Im}{\partial\mu}\right)^{-1}]ds\}.$$

Remark 1. This formula can be simplified by writing it in the terms of Φ. In that case

$$\Im(t,\nu,\mu) = \nu - \Phi(t,\mu),\ \frac{\partial\Im}{\partial\nu} = 1,\ \frac{\partial\Im}{\partial t} = -\frac{\partial\Phi}{\partial t}(t,\mu),$$
$$\frac{\partial^2\Im}{\partial\nu\partial\mu} = 0,\ \frac{\partial^2\Im}{\partial t\partial\mu} = -\frac{\partial^2\Phi}{\partial t\partial\mu},\ \frac{\partial\Im}{\partial\nu} = -\frac{\partial\Phi}{\partial\nu}$$

and we obtain

$$\frac{d\tilde{\mu}_t}{d\tilde{\mu}_\tau}(\mu) = \exp\left\{-\int_\tau^t [\langle\lambda(\Phi(s,\mu)), \Phi'_t(s,\mu)\rangle + tr\Phi''_{t\mu}(\Phi'_\mu)^{-1}]ds\right\}. \qquad (6)$$

4. In the triple of spaces

$$H_+ \subset H_0 \subset H_-.$$

Let us consider the measure $\tilde{\mu}$ on H_- which has the logarithmic derivative along constant directions from H_+ of the form $\langle\lambda(\mu),\varphi\rangle, \lambda(\mu): H_- \to H_-, \varphi \in H_+$, and investigate the problem of nonlinear transformations of $\tilde{\mu}$ in H_-.

Theorem 3. Let us have the Hilbert-Schmidt triple (3) of the space of measures. Let $\tilde{\mu}$ be a smooth measure from H_- with the logarithmic derivative

$$\rho_{\tilde{\mu}}(\varphi,\mu) = \langle\lambda(\mu),\varphi\rangle.$$

Let $f: H_- \to H_-$ be an invertible transform, the inverse of which is given b the formula

$$f^{(-1)} : \mu \to \nu = \mu + F(\mu)$$

when the following conditions are fulfilled:

1) $F : H_- \to H_+$ is a continuously differentiable mapping;

2) The inverse of the linear operator $I + tF'(\mu)$ is bounded for $0 \leq t \leq 1, \mu \in H_0$.

Then the measures $\tilde{\mu}$ and $\tilde{\tilde{\mu}} = \tilde{\mu} f^(- 1)$ are equivalent and the Radon-Nykodim derivative has the form

$$\frac{d\tilde{\tilde{\mu}}}{d\tilde{\mu}}(\mu) = \det(I + F'(\mu))) \exp \left\{ \left\langle \int_0^1 \lambda(\mu + tF(\mu))dt, F(\mu) \right\rangle \right\}. \quad (7)$$

Proof. Consider the homotopy

$$\Phi(t, \mu) = \mu + tF(\mu) \quad 0 \leq t \leq 1,$$

which connects μ with $\mu + F(\mu)$. Applying Theorem 2 and formula (6), we obtain

$$\frac{\partial \Phi}{\partial t} = F(\mu), \ \frac{\partial^2 \Phi}{\partial t \partial \mu} = F'(\mu), \ \left(\frac{\partial \Phi}{\partial \mu} \right)^{-1} = [I + tF'(\mu)]^{-1}.$$

In the conditions of the theorem $\tilde{\tilde{\mu}} \sim \tilde{\mu}$ and (6) implies

$$\frac{d\tilde{\tilde{\mu}}}{d\tilde{\mu}}(\mu) = \exp \left\{ \left\langle \int_0^1 \lambda(\mu + tF(\mu))dt, F(\mu) \right\rangle + \int_0^1 trF'(\mu)[I + tF'(\mu)]^{-1} dt \right\}.$$

Using well-known Goursat-Wronski formula by which

$$\int_0^1 tr(I + tR)^{-1} R dt = \ln \det(I + R).$$

we obtain (7). The theorem is proved.

Example 1. If $\tilde{\mu}$ is a canonical Gauss measure H_-, then $\lambda(\mu) = -\mu$ and from (7) we obtain

$$\frac{d\tilde{\tilde{\mu}}}{d\tilde{\mu}}(\mu) = \det(I + F'(\mu))) \exp \left\{ -\langle \mu, F(\mu) \rangle - \frac{1}{2} \|F(u)\|_0^2 \right\}. \quad (8)$$

Remark 2. Let $\tilde{\mu}$ be a Gauss measure with zero mean and correlation kernel operator R in H_0,

$$R = \int_X \int_X E\mu(t)\mu(z)\beta(dt \times dz).$$

Then $\lambda(\mu) = -R^{-1}\mu$. Now the expression

$$(f(\mu), S(\mu))_0 - TrS^* f'(\mu)R,$$

where S is a linear bounded operator, can be written also when the operator $f'(\mu)$ is bounded and RS^* is the Hilbert-Schmidt type operator. More specifically, taking $S = R^{-\frac{1}{2}}$, we denote

$$l^{(R)}(f)(\mu) = (f(\mu), R^{-\frac{1}{2}}\mu) - trf'(\mu)R^{\frac{1}{2}}.$$

As has been shown in Subsection 1, this expression can be continued up to an extended stochastic integral. Thus in the conditions of the theorem we can write

$$\frac{d\tilde{\tilde{\mu}}}{d\tilde{\mu}}(\mu) = \tilde{\det}(I + F'(\mu)) \exp\left\{ l^R(f)(\mu) - \frac{1}{2}\|F(u)\|_0^2 \right\}, \tag{9}$$

where $\tilde{\det}(I+T)$ is the regularized determinant (see [6]) defined by the relation

$$\tilde{\det}(I+T) = \prod_{k=1}^{\infty}(1+\lambda_k)e^{-\lambda_k},$$

and T is a Hilbert-Schmidt operator with eigenvalues $\{\lambda_k\}$.

Theorem 3 and formulas (7),(8),(9) can be used when studying distributions of measures of solutions of differential equations in measures with random summands.

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