PULSATING FLOW OF WEAKLY CONDUCTIVE LIQUID WITH HEAT TRANSFER

J. Sharikadze, V. Tsutskiridze, L. Jikidze

Georgian Technical University
77 Kostava Str., Tbilisi 0175, Tbilisi, Georgia

(Received: 15.03.2010; accepted: 16.11.2010)

Abstract

In this article we consider the unsteady flow of viscous incompressible electrically conducting fluid in an infinitely long pipe placed in an external uniform magnetic field perpendicular to the pipe axis. It is stated that the motion is created by applied at the initial time in constant longitudinal pressure fall. The exact general solution of problem is obtained.

Key words and phrases: Unsteady flow, conducting magnetic field, pipe, liquid.

AMS subject classification: 85A30, 76W05.

1 Introduction

In this section a formulation of the problem is given and the general considerations are stated, related with its solution for an arbitrary profile of transverse cross-section pipe. The next three sections of work (§§2-4) are devoted to the detailed study of flow in rectangular pipes. Finally in the last §5 the special case of motion in a circular pipe is considered.

Let's assume that the cross section of pipe coincides with the same coordinate plane xoy and the external magnetic field H_0 is parallel to the axisox.

It would be shown that all equations of magneto hydrodynamics will be satisfied, if we assume that the flow rate (v) and the induced magnetic field (H) has only one component along the axis oz, and the functions $v(x,y,t)$ and $H(x,y,t)$ satisfies the following system of equations [1-5]

$$\eta\Delta v+\frac{H_0}{4\pi}\frac{\partial H}{\partial x}=\rho\frac{\partial v}{\partial t}-P,\quad \frac{c^2}{4\pi\sigma}\Delta H+H_0\frac{\partial v}{\partial x}=\frac{\partial H}{\partial t},\tag{1}$$

Here ρ is the density; σ is the conductivity; η is the coefficient of fluid viscosity; c is the velocity of light; P is the applied to the axis pressure gradient.

The electric field and density of flow in fluid current are directed in plane xoy and are expressed through basic unknowns v and H formulae

$$E_x = \frac{c}{4\pi\sigma}\frac{\partial H}{\partial y}, \quad E_y = -\left(\frac{c}{4\pi\sigma}\frac{\partial H}{\partial x} + \frac{vH_0}{c}\right), \qquad (2)$$
$$j_x = \frac{c}{4\pi}\frac{\partial H}{\partial y}, \quad j_y = -\frac{c}{4\pi}\frac{\partial H}{\partial x}.$$

Finally, the pressure in fluid is defined by relation

$$p = -\left(Pz + \frac{H^2}{8\pi}\right) + const. \qquad (3)$$

Let's write down the system (1) in the dimensionless form [6]

$$\Delta u + M^2\frac{\partial h}{\partial \xi} = R\frac{\partial u}{\partial \tau} - Q, \quad \Delta h + \frac{\partial u}{\partial \xi} = R_m\frac{\partial h}{\partial \tau}, \qquad (4)$$

and introduce the designations

$$u = \frac{v}{v_0}, \quad h = \frac{H}{H_0R_m}, \quad \tau = \frac{v_0t}{l}, \quad Q = \frac{Pl^2}{v_0\eta}, \quad \xi = \frac{x}{l}, \quad \eta = \frac{y}{l}, \qquad (5)$$

where v_0 and l are the characteristic velocity and dimension; $M = \frac{H_0l}{c}\sqrt{\frac{\sigma}{\eta}}$ is the Hartmann number; $R = \frac{\rho}{\eta}v_0l$ and $R_m = \frac{4\pi\sigma}{c^2}v_0l$are the viscous and magnetic Reynolds numbers.

If the system of equations (4) will be subject of Laplace integral transformation [6] at zero initial condition, then for values

$$\overline{u}(\xi,\eta,p) = \int_0^\infty u(\xi,\eta,\tau)e^{-p\tau}d\tau, \overline{h}(\xi,\eta,p) = \int_0^\infty h(\xi,\eta,\tau)e^{-p\tau}d\tau \qquad (6)$$

we obtain the following equations

$$\Delta\overline{u} + M^2\frac{\partial\overline{h}}{\partial\xi} - Rp\overline{u} = -\frac{Q}{p}, \quad \Delta\overline{h} + \frac{\partial\overline{u}}{\partial\xi} - R_mp\overline{h} = 0. \qquad (7)$$

In the case of the fixed walls of pipe by one of boundary conditions of problem is the equality to zero the value $\bar{u}$ on the contour of pipe cross section. In addition, we would also put the requirement of continuity of tangential component of electric and magnetic fields at transition from liquid to the area of pipe walls.

The system of equations (7) allows the dividing of variables in Cartesian co-ordinates that gives the possibility to get exact solutions of assigned task in the case of a rectangular pipe. However, the boundary value problems of considered type have such singularity that the Fourier method is applicable to all possible types of boundary conditions on the rectangle contour. As it is indicated by investigation, in the case of rectangular system (7) would be solved by the method of particular solutions in the following three cases: 1) the pipe walls are non-conducting (§2) – the solution is expanded in series on coordinate y, and 2) the pipe wall are perfectly conductive (§3) – the solution is presented by Fourier series on the variable, and 3) the walls $x = \pm a$, perpendicular to the external field, are ideally conductive and wall $y = \pm b$, are non-conductive in this case the solution is obtained in two different forms (§4).

In the particular case of equal to each other viscous and magnetic Reynolds numbers the problem solution for rectangular section would be simplified. In addition in this especial case, when$R = R_m$, the basic system (7) is reduced to the form that would be applied to obtain exact solutions in the case of other profiles pipes. In fact, if at $R = R_m$ make a substitution

$$F = e^{\frac{M}{2}\xi}\left(\overline{u} + M\overline{h} - \frac{Q}{Rp^2}\right), \quad \Phi = e^{-\frac{M}{2}\xi}\left(\overline{u} - M\overline{h} - \frac{Q}{Rp^2}\right), \tag{8}$$

it turns out that the functions F and Φ will satisfy Helmholtz equation

$$\Delta f - \omega^2 f = 0, \quad \omega = \sqrt{Rp + \frac{1}{4}M^2}. \tag{9}$$

However, the boundary conditions for the unknown functions F and Φ would be directly formulated only for the case of non-conducting walls, when $\overline{u} = \overline{h} = 0$ on the contour of pipe cross section.

It follows that in the case of $R = R_m$ for pipe with non-conducting walls the exact solution would be found by the Fourier method for cross-sections, limiting by the coordinate lines of those curvilinear coordinate systems, in which the Helmholtz equation allows the separation of variables.

2 Rectangular Pipe with Non-Conductive Walls

If the pipe walls are considered as non-conducting, then in the area of encircling fluid, the induced magnetic field is absent, as in the considered case the total current flowing in the fluid is equal to zero. From the condition of continuity of tangential component of the magnetic field, we find that the value h (and consequently $\overline{h}$) must becomes a zero on pipe walls. Thus, the problem is reduced to the solution of system (7) with the boundary conditions

$$\overline{u}|_{\eta=\pm 1} = \overline{u}|_{\xi=\pm k} = \overline{h}|_{\eta=\pm 1} = \overline{h}|_{\xi=\pm k} = 0; \quad \xi = \frac{x}{b}, \quad \eta = \frac{y}{b}, \quad k = \frac{a}{b}, \tag{10}$$

with as characteristic dimension l is accepted value b.

As in considered problems the distribution of velocity and magnetic field is symmetric on coordinate y, then the solution of problem would be look out in the form of trigonometric series

$$\overline{u} = \sum_{n=0}^{\infty} u_n(\xi) \cos \lambda_n \eta, \quad \overline{h} = \sum_{n=0}^{\infty} h_n(\xi) \sin \lambda_n \eta, \quad \lambda_n = \frac{2n+1}{2}\pi, \tag{11}$$

satisfying the boundary conditions on the sides $\eta = \pm 1$.

By substituting (11) into (7), we obtain for values $u_n(\xi)$, $h_n(\xi)$ the following system of ordinary differential equations

$$\left.\begin{array}{c} u_n^{''}(\xi) + M^2 h_n^{'}(\xi) - \left(\lambda_n^2 + Rp\right) u_n(\xi) = -q_n, \\ h_n^{''}(\xi) + u_n^{'}(\xi) - \left(\lambda_n^2 + R_m p\right) h_n(\xi) = 0, \end{array}\right\} q_n = \frac{2Q(-1)^n}{\lambda_n p}. \tag{12}$$

The solution of system (12) would be written as

$$u_n(\xi) = \frac{q_n}{\lambda_n^2 + Rp}\left[1 - \omega_n^{(0)}(\xi)\right], \quad h_n(\xi) = \frac{q_n}{\lambda_n^2 + Rp} g_n^{(0)}(\xi), \tag{13}$$

where the functions $\omega_n^{(0)}(\xi)$ and $g_n^{(0)}(\xi)$ are general solutions of according homogeneous system.

By taking into account circumstances that the rate is even, and the magnetic field is an odd function of the coordinate ξ, we find

$$g_n^{(0)}(\xi) = Ash\nu\xi, \quad \omega_n^{(0)}(\xi) = Bch\nu\xi, \tag{14}$$

and the values A and B would be obtained from the homogeneous algebraic system

$$\left.\begin{array}{c} \left(\nu^2 - \lambda_n^2 - R_m p\right) A + \nu B = 0, \\ M^2 \nu A + \left(\nu^2 - \lambda_n^2 - Rp\right) B = 0. \end{array}\right\} \tag{15}$$

Due equating to zero the determinant of this system, we find the value of the parameter ν

$$\nu_{1,2} = \left(\lambda_n^2 + \frac{M^2}{2} + \frac{p}{2}(R+R_m) \pm \right.$$
$$\left. \pm \sqrt{M^2\lambda_n^2 + \frac{M^4}{2} + M^2 p(R+R_m) + \frac{\mu^2}{4}(R-R_m)^2}\right)^{1/2}. \tag{16}$$

If we find now the arbitrary constants from remaining boundary conditions $h_n(k) = u_n(k) = 0$ and applying the Riemann-Mellin conversion, we obtain the solution of problem in the form of the following complex integrals

$$\begin{aligned} u(\xi,\eta,\tau) &= \frac{Q}{\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{p\tau}\sum_{n=0}^{\infty}\frac{(-1)^n\cos\lambda_n\eta}{\lambda_n(\lambda_n^2+Rp)}\left[1-\frac{\omega_n(\xi)}{D_n(p)}\right]\frac{dp}{p}\,; \\ h(\xi,\eta,\tau) &= \frac{Q}{\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{p\tau}\sum_{n=0}^{\infty}\frac{(-1)^n\cos\lambda_n\eta}{\lambda_n(\lambda_n^2+Rp)}\cdot\frac{g_n(\xi)}{D_n(p)}\frac{dp}{p}\,, \end{aligned} \tag{17}$$

where

$$\left.\begin{aligned} \omega_n(\xi) &= \nu_1(\lambda_n^2+R_mp-\nu_2^2)sh\nu_1k\cdot ch\nu_2\xi- \\ &\quad -\nu_2(\lambda_n^2+R_mp-\nu_1^2)sh\nu_2k\cdot ch\nu_1\xi, \\ g_n(\xi) &= \nu_1\nu_2(sh\nu_1k\cdot sh\nu_2\xi-sh\nu_2k\cdot sh\nu_1\xi), \\ D_n(p) &= \nu_1(\lambda_n^2+R_mp-\nu_2^2)sh\nu_1k\cdot ch\nu_2k- \\ &\quad -\nu_2(\lambda_n^2+R_mp-\nu_1^2)sh\nu_2k\cdot ch\nu_1k. \end{aligned}\right\} \tag{18}$$

In the limiting case $M\to 0$ of ordinary hydrodynamics we find

$$u=\frac{Q}{\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{p\tau}\sum_{n=0}^{\infty}\frac{(-1)^n\cos\lambda_n\eta}{\lambda_n(\lambda_n^2+Rp)}\left[1-\frac{ch\xi\sqrt{\lambda_n^2+Rp}}{chk\sqrt{\lambda_n^2+Rp}}\right]\frac{dp}{p},\quad H=0. \tag{19}$$

The summation of the residues at the poles $p=0$ and $p=-\frac{1}{R}\left[\lambda_n^2+\left(\frac{\lambda_n}{k}\right)^2\right]$. leads to the following form of the solution

$$\begin{gathered} u=2Q\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^2}\left(1-\frac{ch\lambda_n\xi}{ch\lambda_nk}\right)\cos\lambda_n\eta- \\ -4Q\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{n+s}\cos\lambda_n\eta\cos\lambda_n\frac{\xi}{k}}{\lambda_n\lambda_s\left[\lambda_n^2+\left(\frac{\lambda_s}{k}\right)^2\right]}e^{-\frac{\tau}{R}\left[\lambda_n^2+\left(\frac{\lambda_s}{k}\right)^2\right]}. \end{gathered} \tag{20}$$

Also is of interest to carry out the passage to the limit $b\to\infty$, giving a generalization of the Hartmann problem in the case of an unsteady flow.

By introducing the notation

$$\begin{gathered} \tau'=\frac{v_0t}{a},\ \ Q'=\frac{Pa^2}{v_0\eta},\ \ R'=\frac{\rho}{\eta}v_0a,\ \ R_m'=\frac{4\pi\sigma}{c^2}v_0a, \\ M'=\frac{H_0a}{c}\sqrt{\frac{\sigma}{\eta}},\ \xi'=\frac{x}{a}, \end{gathered} \tag{21}$$

$$2\mu_{1,2}=\sqrt{M'^2+q\left(\sqrt{R'}+\sqrt{R_m'}\right)^2}\pm\sqrt{M'^2+q\left(\sqrt{R'}-\sqrt{R_m'}\right)^2}, \tag{22}$$

is possible the formula (17) for velocity at $b \to \infty$ to reduce for following form

$$\left.\begin{aligned} \frac{R'}{Q'}u = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left[1-\frac{f(q,\xi)}{F(q)}\right]e^{q\tau'}\frac{dq}{q}, \\ f(q,\xi) = \mu_1(R_m'q-\mu_2^2)sh\mu_1 ch\mu_2\xi') - \mu_2(R_m'q-\mu_1^2)sh\mu_2 ch\mu_1\xi', \\ F(q) = \mu(R_m'q-\mu_2^2)sh\mu_1 ch\mu_2 - \mu_2(R_m'q-\mu_1^2)sh\mu_2 ch\mu_1, \end{aligned}\right\} \quad (23)$$

coincides with the solution of the unsteady flow between two parallel non-conductive plates obtained, in the work [?, ?, ?, ?]. A similar passage to the limit would be carried out also for magnetic field.

Turning now to the general formula (17), we note first of all that as special points of the integrands would be only the poles, so the solution of problem by using the residue theorem would be expressed in the form of double series. The residue in $p = 0$ point gives steady mode of considered flow. In this case from formula (8) we find the values

$$\nu_{1,2}^{(0)} = \sqrt{\frac{M^2}{4}+\lambda_n^2} \pm \frac{M}{2}, () \quad (24)$$

and the resulting solution

$$\begin{aligned} u_{st.} &= 2a\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\left[1-\frac{f_n(\xi)}{D_n}\right]\cos\lambda_n\eta\,; \\ h_{st.} &= 2Q\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\frac{\varphi_n(\xi)}{D_n}\cos\lambda_n\eta\,, \end{aligned} \quad (25)$$

where

$$\left.\begin{aligned} f_n(\xi) = sh\nu_1^{(0)}k\cdot ch\nu_2^{(0)}\xi + sh\nu_2^{(0)}k\cdot ch\nu_2^{(0)}\xi, \\ \varphi_n(\xi) = sh\nu_1^{(0)}k\cdot sh\nu_2^{(0)}\xi - sh\nu_2^{(0)}k\cdot sh\nu_1^{(0)}\xi, \end{aligned}\right\}$$
$$D_n = shk\sqrt{M^2+4\lambda_n^2}, \quad (26)$$

coincides with the solution found in [?, ?, ?].

Studying the asymptotic expressions for large on modulus roots of equation $D_n(p) = 0$, it is possible to show that this equation has an infinite set of both the real and complex roots (last-at $R \neq R_m$). Direct examination shows that all real roots are negative. Relating to complex roots it is possible to prove that they have a negative real part. Therefore, the transitional regime for sufficiently large time in the general case is presented by damping oscillations.

Without stopping, due to the complexity of calculations, on transformation of general solution of task to real form, let's turn to the calculation of velocity for that special case when the viscous and magnetic Reynolds numbers are equal to each other ($R = R_m$).

If we find from formula (8) the value ν

$$\nu_{1,2} = \sqrt{\frac{M^2}{4} + \lambda_n^2 + Rp} \pm \frac{M}{2}, \tag{27}$$

would convert the solution (17) to the following form

$$u = \frac{Q}{\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{p\tau} \sum_{n=0}^{\infty} \frac{(-1)^n \cos \lambda_n \eta}{\lambda_n (\lambda_n^2 + Rp)} \times$$
$$\times \left[1 - \frac{ch\nu_2 \xi \cdot sh\nu_1 k + ch\nu_1 \xi \cdot sh\nu_2 k}{shk(\nu_1 + \nu_2)} \right] \frac{dp}{p}. \tag{28}$$

Calculating the residues in the poles$p = 0$ and $p = p_{n,s}$, where

$$p_{n,s} = -\frac{1}{R} \left[\frac{M^2}{4} + \lambda_n^2 + \left(\frac{\pi s}{2k} \right)^2 \right], \tag{29}$$

obtain for a flow rate the following expression

$$u = u_{st.} + 32\pi Q k^2 \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \frac{(-1)^{n+s} s \cos \lambda_n \eta \cdot \psi_n(\xi) e^{p_{n,s}\tau}}{\lambda_n (k^2 M^2 + \pi^2 s^2)(k^2 M^2 + \pi^2 s^2 + 4k^2 \lambda_n^2)}, \tag{30}$$

where the is introducing the notation

$$\psi_s(\xi) = \sin \frac{s\pi}{2} ch \frac{Mk}{2} ch \frac{M\xi}{2} \cos \frac{s\pi\xi}{2k} - \cos \frac{s\pi}{2} sh \frac{Mk}{2} sh \frac{M\xi}{2} \sin \frac{s\pi\xi}{2k}. \tag{31}$$

It is interesting to note that in the considered particular case $R = R_m$ transition magneto hydrodynamic mode is purely aperiodic, as well as in ordinary hydrodynamics.

3 Rectangular Pipe with Perfectly Conducting Walls

In that limiting case, when the conductivity of walls is assumed as infinitely large, on pipe cross-section contour would be assumed as equal to zero tangent component of electric field that leads to the following boundary condition for transformed on Laplace induced magnetic field

$$\left. \frac{\partial \overline{h}}{\partial \xi} \right|_{\xi = \pm 1} = \left. \frac{\partial \overline{h}}{\partial \eta} \right|_{\eta = \pm \frac{b}{a}} = 0, \xi = \frac{x}{a}, \eta = \frac{y}{a} \tag{32}$$

(as characteristic size l is accepted value a). The solution of basic system (7) in this case would be found as series on coordinate x

$$\overline{u}=\sum_{n=0}^{\infty}u_n(\eta)\cos\lambda_n\xi,\quad \overline{h}=\sum_{n=0}^{\infty}h_n(\eta)\sin\lambda_n\xi,\quad \lambda_n=\frac{2n+1}{2}\pi, \tag{33}$$

satisfies the boundary conditions at the walls $x=\pm a$.

Substituting (33) into (7) gives for the functions $u_n(\eta)$ and $h_n(\eta)$ the system of equations

$$\left.\begin{array}{r} u_n''(\eta)-(\lambda_n^2+Rp)u_n(\eta)+M^2\lambda_n h_n'(\eta)=-q_n,\\ h_n''(\eta)-\left(\lambda_n^2+R_mp\right)h_n(\eta)-\lambda_n u_n'(\eta)=0, \end{array}\right\}\quad q_n=\frac{2Q(-1)^n}{\lambda_n p}, \tag{34}$$

The solution would be written in the following form

$$\begin{aligned} u_n(\eta)&=\frac{(\lambda_n^2+R_mp)q_n\left[1-\omega_n^{(0)}(\eta)\right]}{(\lambda_n^2+Rp)(\lambda_n^2+R_mp)+M^2\lambda_n^2},\\ h_n(\eta)&=\frac{-\lambda_n q_n\left[1-g_n^{(0)}(\eta)\right]}{(\lambda_n^2+Rp)(\lambda_n^2+R_mp)+M^2\lambda_n^2}, \end{aligned} \tag{35}$$

where $\omega_n^{(0)}(\eta)$ and $g_n^{(0)}(\eta)$ are the general solution of homogeneous system (34).

If we assume that

$$g_n^{(0)}(\eta)=Ach\nu\eta,\quad \omega_n^{(0)}(\eta)=Bch\nu\eta, \tag{36}$$

Obtain for A and B the homogeneous system

$$\left.\begin{array}{r} \left(\nu^2-\lambda_n^2-R_mp\right)A-\lambda_nB=0,\\ M^2\lambda_nA-\left(\nu^2-\lambda_n^2-Rp\right)B=0. \end{array}\right\} \tag{37}$$

And the value ν would be found by equating to zero of its determinant.

$$\nu_{1,2}=\sqrt{\lambda_n^2+\frac{p}{2}(R+R_m)\pm\sqrt{\frac{p^2}{4}(R-R_m)^2-M^2\lambda_n^2}}. \tag{38}$$

If we define the including in solution constants from the boundary conditions $u_n\left(\frac{b}{a}\right)=h_n'\left(\frac{b}{a}\right)=0$, then the solution of task takes the following form

$$\begin{aligned} u&=\frac{Q}{\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{p\tau}\sum_{n=0}^{\infty}\frac{(-1)^n(\lambda_n^2+R_mp)\cos\lambda_n\xi}{(\lambda_n^2+Rp)(\lambda_n^2+R_mp)+M^2\lambda_n^2}\left[1-\frac{\omega_n(\eta)}{\Delta_n(p)}\right]\frac{dp}{p};\\ h&=-\frac{Q}{\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{p\tau}\sum_{n=0}^{\infty}\frac{(-1)^n\sin\lambda_n\xi}{(\lambda_n^2+Rp)(\lambda_n^2+R_mp)+M^2\lambda_n^2}\left[1-\frac{\lambda_n g_n(\eta)}{\Delta_n(p)}\right]\frac{dp}{p}, \end{aligned} \tag{39}$$

where

$$\left.\begin{aligned}\omega_n(\eta) &= \nu_1(\lambda_n^2 + R_m p - \nu_2^2) sh\nu_1 \frac{b}{a} \cdot ch\nu_2\eta - \\ &\quad - \nu_2(\lambda_n^2 + R_m p - \nu_1^2) sh\nu_2 \frac{b}{a} \cdot ch\nu_1\eta, \\ g_n(\eta) &= \nu_2 ch\nu_2 \frac{b}{a} \cdot ch\nu_1\eta - \nu_1 sh\nu_1 \frac{b}{a} \cdot ch\nu_2\eta, \\ \Delta_n(p) &= \nu_1(\lambda_n^2 + R_m p - \nu_2^2) sh\nu_1 \frac{b}{a} \cdot ch\nu_2 \frac{b}{a} - \\ &\quad - \nu_2(\lambda_n^2 + R_m p - \nu_1^2) sh\nu_2 \frac{b}{a} \cdot ch\nu_1 \frac{b}{a}.\end{aligned}\right\} \tag{40}$$

Implementing in the obtained solution passage to the limit $b \to \infty$, we obtain the velocity distribution in a plane-parallel layer with perfectly conducting walls [1, 4]

$$\left.\begin{aligned}&\frac{u}{2Q} = \frac{chM - chM\xi}{2M^2 chM} + \\ &\quad + \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n} \left[\frac{(\lambda_n^2 + R_m p_1) e^{p_1\tau}}{p_1 \psi_n'(p_1)} + \frac{(\lambda_n^2 + R_m p_2) e^{p_2\tau}}{p_2 \psi_n'(p_2)} \right] \cos\lambda_n\xi, \\ &\psi_n(p) = (\lambda_n^2 + Rp)(\lambda_n^2 + R_m p) + M^2\lambda_n^2, \\ &2RR_m p_{1,2} = -\lambda_n^2(R + R_m) \pm \lambda_n \sqrt{\lambda_n^2 (R - R_m)^2 - 4M^2 RR_m}.\end{aligned}\right\} \tag{41}$$

The obtained solution (39) in general case is represented as a double series, including aperiodic members, as well as vibrations, damping in the course of time. Without any calculations, we present only the formula for steady mode of given task that is given as residue in simple pole $p = 0$. Equation (38) leads to the values

$$\begin{aligned}\nu_{1,2} = r_n \pm is_n, \quad r_n &= \sqrt{\frac{\lambda_n}{2}\left(\sqrt{M^2 + \lambda_n^2} + \lambda_n\right)}, \\ s_n &= \sqrt{\frac{\lambda_n}{2}\left(\sqrt{M^2 + \lambda_n^2} - \lambda_n\right)},\end{aligned} \tag{42}$$

and for velocity, after some calculations, we find the expression

$$\frac{u}{Q} = \frac{chM - chM\xi}{M_2 chM} +$$
$$+2\sum_{n=0}^{\infty} \frac{(-1)^n \cos\lambda_n\xi}{\lambda_n(\lambda_n^2 + M^2)\Delta_n} \left[\delta_n ch\eta r_n \cos\eta s_n + \gamma_n sh\eta r_n \sin\eta s_n\right], \tag{43}$$

$$\left.\begin{aligned} \Delta_n &= \frac{1}{2}(s_n \sin 2\frac{b}{a}s_n - r_n sh2\frac{b}{a}r_n), \\ \delta_n &= r_n sh\frac{b}{a}r_n \cos\frac{b}{a}s_n - s_n ch\frac{b}{a}r_n \sin\frac{b}{a}s_n, \\ \gamma_n &= s_n sh\frac{b}{a}r_n \cos\frac{b}{a}s_n + r_n ch\frac{b}{a}r_n \sin\frac{b}{a}s_n, \end{aligned}\right\} \tag{44}$$

found in [1-5].

4 Rectangular Pipe at Mixed Boundary

When pipe walls $x = \pm a$- are perfect conductive and the walls $y = \pm b$- are non-conductive, the boundary conditions of problem have the following form

$$\overline{u}|_{x=\pm a} = \overline{u}|_{y=\pm b} = \left.\frac{\partial\overline{h}}{\partial x}\right|_{x=\pm a} = \overline{h}\big|_{y=\pm b} = 0. \tag{45}$$

As it is indicated in §1, in this case, the solution would be presented in two forms.

The first of these forms would be obtained, if the solution is found in the form (11). In this case all of the formulae (12) - (17) retain their form, and changes only the values (18) $g_n(\xi)$, $\omega_n(\xi)$ and $D_n(p)$that in this case are equal.

$$\left.\begin{aligned} \omega_n(\xi) &= \nu_1^2(\lambda_n^2 + R_m p - \nu_2^2) sh\nu_1 k \cdot ch\nu_2\xi - \\ &\quad -\nu_2^2(\lambda_n^2 + R_m p - \nu_1^2) ch\nu_2 k \cdot ch\nu_1\xi, \\ g_n(\xi) &= \nu_1\nu_2(\nu_1 ch\nu_1 k \cdot sh\nu_2\xi - \nu_2 ch\nu_2 k \cdot sh\nu_1\xi), \\ D_n(p) &= (\nu_1^2 - \nu_2^2)(\lambda_n^2 + R_m p) ch\nu_1 k ch\nu_2 k. \end{aligned}\right\} \tag{46}$$

The second form of solutions would be found in the form (33), and remain valid the formulae (34)-(39), and formulae (40) should be replaced by the following ones

$$\left.\begin{aligned}\lambda_n\omega_n(\eta) &= (\nu_1^2-\lambda_n^2-R_mp)(\nu_2^2-\lambda_n^2-R_mp-\lambda_n)ch\nu_2\frac{b}{a}\cdot ch\nu_1\eta-\\ &\quad -(\nu_2^2-\lambda_n^2-R_mp)(\nu_1^2-\lambda_n^2-R_mp-\lambda_n)ch\nu_1\frac{b}{a}\cdot ch\nu_2\eta,\\ \lambda_n g_n(\eta) &= (\nu_2^2-\lambda_n^2-R_mp-\lambda_n)ch\nu_2\frac{b}{a}\cdot ch\nu_1\eta-\\ &\quad -(\nu_1^2-\lambda_n^2-R_mp-\lambda_n)ch\nu_1\frac{b}{a}\cdot ch\nu_2\eta,\\ \Delta_n(p) &= (\nu_2^2-\nu_1^2)ch\nu_1\frac{b}{a}\cdot ch\nu_2\frac{b}{a}.\end{aligned}\right\} \quad (47)$$

In the particular case $R = R_m$, obtained in §§3-4 solutions, would be, also as in §2, simplified, however transient behavior, in contrary of considered in §2 case of all non-conducting walls, is no longer be purely aperiodic.

5 The Flow in a Circular Tube at $R = R_m$

As indicated in the introduction, the problem of unsteady flow in a round pipe with non-conducting walls at $R = R_m$ is reduced to the solution of the equation

$$\Delta F - \omega^2 F = 0, \omega = \sqrt{\mu^2 + Rp}, \mu = \frac{M}{2} \quad (48)$$

at boundary conditions

$$F|_{\rho=1} = -\frac{Q}{Rp^2}e^{\mu\cos\vartheta}, \rho = \frac{r}{l} \quad (49)$$

(l is the radius if pipe; r, ϑ are the polar coordinates).

After the variables separation we will find

$$F(\rho,\vartheta) = -\frac{Q}{Rp^2}\left[\frac{a_0}{2}\frac{I_0(\omega\rho)}{I_0(\omega)} + 2\sum_{n=1}^{\infty} a_n\frac{I_n(\omega\rho)}{I_n(\omega)}\cos n\vartheta\right]. \quad (50)$$

The condition (49) gives

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{\mu\cos\vartheta}\cos n\vartheta d\vartheta = 2I_n(\mu),$$

after that grounding on (8) we obtain

$$\begin{aligned}\overline{u} = \frac{Q}{Rp^2} - \frac{1}{2}e^{-\mu\xi}\left[\frac{I_0(\mu)}{I_0(\omega)}I_0(\omega\rho) + 2\sum_{n=1}^{\infty}\frac{I_n(\mu)}{I_n(\omega)}I_n(\omega\rho)\cos n\vartheta\right] -\\ -\frac{1}{2}e^{\mu\xi}\left[\frac{I_0(\mu)}{I_0(\omega)}I_0(\omega\rho) + 2\sum_{n=1}^{\infty}(-1)^n\frac{I_n(\mu)}{I_n(\omega)}I_n(\omega\rho)\cos n\vartheta\right]. \quad (51)\end{aligned}$$

Application of the residue theorem leads to the solution of problem in the following form (by the same method a formula for induced magnetic field would be obtained)

$$u = u_{st.} + Qe^{-\mu\xi} \sum_{m=1}^{\infty} \left[\frac{I_0(\mu)J_0(\rho\gamma_{m0})}{J_0'(\gamma_{mo})} \cdot \frac{\gamma_{m0} e^{\alpha_{m0}\frac{\tau}{R}}}{\alpha_{m0}^2} + \right.$$

$$\left. +2\sum_{n=1}^{\infty} \frac{I_n(\mu)J_n(\rho\gamma_{mn})}{J_n'(\gamma_{mn})} \cdot \frac{\gamma_{mn} e^{\alpha_{mn}\frac{\tau}{R}}}{\alpha_{mn}^2} \cos n\vartheta \right] +$$

$$+Qe^{\mu\xi} \sum_{m=1}^{\infty} \left[\frac{I_0(\mu)J_0(\rho\gamma_{m0})}{J_0'(\gamma_{mo})} \cdot \frac{\gamma_{m0} e^{\alpha_{m0}\frac{\tau}{R}}}{\alpha_{m0}^2} + \right.$$

$$\left. +2\sum_{n=1}^{\infty} (-1)^n \frac{I_n(\mu)J_n(\rho\gamma_{mn})}{J_n'(\gamma_{mn})} \cdot \frac{\gamma_{mn} e^{\alpha_{mn}\frac{\tau}{R}}}{\alpha_{mn}^2} \cos n\vartheta \right], \tag{52}$$

$$\alpha_{mn} = -(\gamma_{mn}^2 + \mu^2), \tag{53}$$

where γ_{mn} are the roots of the equation $J_n(\gamma) = 0$.

The value $u_{st.}$ represents the rate in the corresponding steady mode.

At $\mu = 0$ the formula (52) becomes in the well-known solution of problem on non-conductive fluid flow in cylindrical pipe.

6 Conclusions

In conclusion let's note that at $R = R_m$ in the same way would be given exact solution of external problem of circular cylinder longitudinal flow, as well as the more general problem on flow in the annular pipe with non-conductive walls.

References

1. L.G. Loitsiansky, *Mechanic of a fluid and gas* (in Russian). Nauka, Moscow, 1987, 840p.

2. A.B. Vatazin, G.A. Lubimov, C.A. Regirer, *Magnetohydrodynamic Flow in Channels* (In Russian). Nauka, Moscow, 1970, 672 p.

3. L.A. Landau, E.M. Lifshits, *Electrodynamics of continua* (In Russian). GITTL, Moscow, 1982, 620p.

4. V. Tsutskiridze, L. Jikidze, Heat transfer in tubes with porous walls under internal heat sources. *Proc. A. Razmadze Math. Inst.* **154** (2010), 127-133.

5. J. Sharikadze, V. Tsutskiridze, L. Jikidze, The flow conducting liquid in magnetohydrodynamics tubes with heat transfer. *International scientific journal IFToMM "Problems of Mechanics"*, Tbilisi, **44** (2011), No. 3, 63–70.

6. V.A. Ditkin, A.P. Prudnikov, *Integrated transformations and operational calculus* (In Russian). Nauka, Moscow, 1974, 542 p.