ON THE NUMERICAL SOLUTIONS OF THE AXI-SYMMETRIC TURBULENT-DIFFUSION EQUATION

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(Received: 12.02.13; accepted: 17.06.13)

Abstract

In the paper the turbulent diffusion equation in the axi-symmetric case with the appropriate initial condition is considered.The approximate solution is obtained by means of the stable finite-difference schemes.The numerical example is given.

Key words and phrases: Turbulent diffusion equation; Finite-difference schemes
AMS subject classification: 30E25; 35M31; 35Q30; 76M20.

1 Introduction

The problem connected with the transport of some substance by the vortex in the infinite area is described by the turbulent diffusion equation with the appropriate initial condition [1]

$$\frac{\partial u}{\partial t} + V_x \frac{\partial u}{\partial x} + V_y \frac{\partial u}{\partial y} + V_z \frac{\partial u}{\partial z} = (D + D_1)\Delta u, \tag{1.1}$$

$$u(x, y, z, 0) = \psi(x, y, z) \tag{1.2}$$

where u is substance concentration, $\psi(x, y, z)$ is the given continues function, D_1 is the molecular diffusion coefficient, D is the turbulent diffusion coefficient depending on time t and given by

$$D(t) = \gamma\nu_*(t), \nu_*(t) = \lambda|V|R,$$

$\nu_*(t)$ is a turbulence viscosity, γ and λ are some constants, V is the velocity of the vortex and R is a radii. $\nu_*(t)$ and $D(t)$ could be calculated from the experiments.

2 The finite difference schemes for the axy-symmetric turbulent diffusion equation

We now consider the axi-symmetric case of the equation (1.1) (the axis of symmetry is $0x$) and suppose that V is known, D is the constant and is also known. From the experimental results is also known, that at the beginning of the turbulent process D_1 is negligible and far from the initial point the substance disappears very fast. Hence, instead of the infinite area we can consider some cylinder, at the surface of which the substance concentration is zero. The meridian cross-section of this cylinder will be denoted by $G = \{-l/2 < x < l/2, -h < r < h\}$, where $y = rcos\phi; z = rsin\phi$.

We also suppose that the vortex is axi-symmetric, at the initial time is located at the point $(0, 0, 0)$ and the substance concentration is given by the formula

$$\psi(x, y, z) = R_0 \sin \exp(-\alpha|x| - \beta r - N),$$

where α, β, N are definite positive constants, and $\exp(-5N)$ is negligible.

Let us consider the following problem

Problem 1. In the area $Q_T = G \times (0 \leq t \leq T)$ to find the solution u of the parabolic equation

$$\frac{\partial u}{\partial t} + V_x \frac{\partial u}{\partial x} + V_r \frac{\partial u}{\partial r} = D \left(\Delta u + \frac{1}{r} \frac{\partial u}{\partial r} \right), \tag{2.1}$$

satisfying the following initial- boundary conditions

$$u(x, r, 0) = \psi(x, r);\ u(x, -h, t) = u(x, h, t) = 0;$$
$$u(-l/2, r, t) = u(l/2, r, t) = 0; \tag{2.2}$$

where V_x, V_r are the velocity components, $V = (V_x, V_r)$.

A numerical treatment of the parabolic type equations by different finite difference schemes was considered by numerous authors [2-10]. We will construct the new type of finite difference schemes. Let us rewrite (2.1) and (2.2) in the form

$$\frac{\partial u}{\partial t} = a \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} \right) - b_1 \frac{\partial u}{\partial x} - b_2 \frac{\partial u}{\partial r} \tag{2.3}$$

$$(x, r, t) \in Q_T = G \times (0 \leq t \leq T), u(x, r, 0) = u_0(x, r),$$
$$u(x, r, t)\,|_\Gamma = \varphi,\ (x, r) \in G, \overline{G} = G + \Gamma, -l/2 \leq x \leq l/2,\ -h \leq y \leq h,$$

a, b_1, b_2, φ are definite functions, Γ is the contour of the rectangle.

In the non-dimensional variables $x = x_1,\ r = x_2 h,\ (-1 \leq x_1, x_2 \leq 1)$, the equation (2.3) becomes

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x_1^2} + \frac{a}{h^2}\frac{\partial^2 u}{\partial x_2^2} - b_1\frac{\partial u}{\partial x_1} - \frac{b_2'}{h}\frac{\partial u}{\partial x_2}, \tag{2.4}$$

where

$$b_2' = \frac{h^2(x_2+1)^2 - a}{h(x_2+1)}.$$

Let us construct a finite difference scheme. We divide the area of integration $Q_T = \overline{G} \times [0, T]$ by the planes $x_{1i} = -1 + ih_1$, $x_{2j} = jh_2$, $t_n = n\tau$, $h_1, h_2, \tau > 0$ into cells, where $i = 1, 2, \ldots, (N-1), j = 1, 2, \ldots, (N-1), n = 0, 1, 2, \ldots L$; $h_1 = \frac{2}{M}, h_2 = \frac{h}{N}, \tau = \frac{1}{L}$. $\overline{w_k} = \{x_k = (k_1h_1, k_2h_2) \in G\}$ is a squared net with the steps h_1 and h_2. $\overline{w_\tau} = \{t_n = n\tau\}$ is the net with the step $\tau = \frac{1}{L}$; $0 \le t \le 1$.

For the equation (2.4) we introduce following alternating direction finite difference schemes

$$\frac{y^{n+\frac{1}{2}} - y^n}{\frac{1}{2}\tau} = 0,5\sigma\left\{(2a\Delta_{11} - b_1\Delta_1)(y^{n+\frac{1}{2}} - y^n)\right\}$$

$$+a(\Delta_{11} + \frac{1}{h^2}\Delta_{22})y^n - (b_1\Delta_1 + \frac{b_2'}{h}\Delta_2)y^n,$$

$$\frac{y^{n+1} - y^{n+\frac{1}{2}}}{\frac{1}{2}\tau} = 0,5\sigma\left\{(2a\cdot\frac{1}{h^2}\Delta_{22} - \frac{b_2'}{h}\Delta_2)\,(y^{n+1} - y^{n+\frac{1}{2}})\right\}$$

$$+a(\Delta_{11} + \frac{1}{h^2}\Delta_{22})y^{n+\frac{1}{2}} - (b_1\Delta_1 + \frac{b_2'}{h}\Delta_2)y^{n+\frac{1}{2}},$$

$$\frac{y^{n+\frac{1}{2}} - y^n}{\frac{1}{2}\tau} = \sigma a[\Delta_{11}(y^{n+\frac{1}{2}} - y^n)] + a(\Delta_{11} + \frac{1}{h^2}\Delta_{22})y^n - (b_1\Delta_1 + \frac{b_2^1}{h}\Delta_2)y^n,$$

$$\frac{y^{n+1} - y^{n+\frac{1}{2}}}{\frac{1}{2}\tau} = \sigma a[\frac{1}{h^2}\Delta_{22}(y^{n+1} - y^{n+\frac{1}{2}})]$$

$$+a(\Delta_{ii} + \frac{1}{h^2}\Delta_{22})y^{n+\frac{1}{2}} - (b_1\Delta_1 + \frac{b_2^1}{h}\Delta_2)y^{n+\frac{1}{2}}.$$

where

$$\Delta_i y = y_{\overset{0}{x_i}} = (y_{x_i} + y_{\bar{x}_i});\quad \Delta_{ii}y = y_{x_i\bar{x}_i};\quad y[\big(k + \frac{p}{n}\big)\tau] = y^{k+\frac{p}{n}};$$

$$y_{x_1} = \frac{1}{h_1}[u(x_1 + h_1, x_2) - u(x_1, x_2)];$$

$$y_{x_2} = \frac{1}{h_2}[u(x_1, x_2 + h_2) - u(x_1, x_2)];$$

$$y_{\bar{x}_1} = \frac{1}{h_1}[u(x_1, x_2) - u(x_1 - h_1, x_2)];$$

$$y_{\bar{x}_2} = \frac{1}{h_2}[u(x_1, x_2) - u(x_1, x_2 - h_2)];$$

The parameter σ is chosen from the condition [9]

$$\sigma \geq \frac{c_2 n^2}{\nu(n-1)}.$$

The stability and complete approximation of this schemes were proved in [9, 10].

In our case

$$a = D;\ b_1 = V_x;\ b_2 = V_r - D/r.$$

Below the numerical example is given in non-dimensional variables in a simple case, when the velocity V is a constant $V = (0, 1)$, $\psi(x, y, z) = \sin\exp(-|x| - r - 3)$, and $D = 10^{-3}$ (Fig.1; Fig.2). The graphs are constructed by using "Maple".

Note. In more general case,for low Reynolds number for the definition of velocity components the Stokes linear axi-symmetric system is valid [1, 11]. In this case the velocity components are given by the formulas [11]

$$V_x = -\frac{2q}{(r^2 + (x+c)^2)^{\frac{3}{2}}} + \frac{2q}{(r^2 + (x-c)^2)^{\frac{3}{2}}} + \frac{3qr^2}{(r^2 + (x+c)^2)^{\frac{5}{2}}}$$

$$-\frac{3qr^2}{(r^2 + (x-c)^2)^{\frac{5}{2}}} + C_1 r^2 - A,$$

$$V_r = -\frac{3qr(x+c)}{(r^2 + (x+c)^2)^{\frac{5}{2}}} + \frac{3qr(x-c)}{(r^2 + (x-c)^2)^{\frac{5}{2}}},$$

where q, c, C_1, A are definite constants.

For this case the numerical results are in preparation.

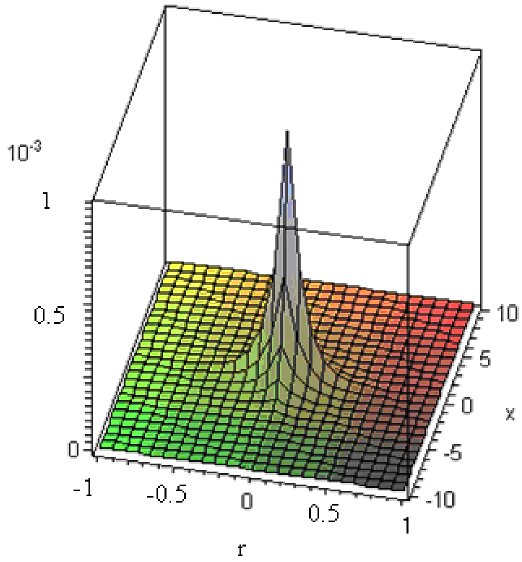

Fig. 1. Initial distribution of substance

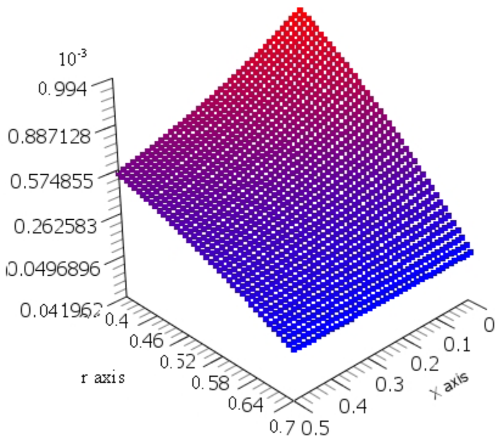

Fig. 2. Distribution of substance at the moment $t = 1$

Acknowledgement

The designated project has been fulfilled by financial support of the Georgia National Rustaveli Science Foundation (Grant #GNSF/ST08/3-395). Any idea in this publication is possessed by the author and may not represent

the opinion of the GNSF itself.

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