REPRESENTATION FORMULAS OF GENERAL SOLUTIONS OF THE TWO-TEMPERATURE THEORY OF TWO-COMPONENT ELASTIC MIXTURES

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Abstract

We consider the statics case of the two-temperature theory of two-component elastic mixtures. The representation formula of a general solution of the homogeneous system of differential equations obtained in the paper is expressed by means of four harmonic and four metaharmonic functions. These formulas are very convenient and useful in many particular problems for domains with concrete geometry. Here we demonstrate an application of these formulas to the Dirichlet and Neumann type boundary value problem for a ball. Uniqueness theorems are proved. We construct explicit solutions in the form of absolutely and uniformly convergent series.

Key words and phrases: Mixture theory, general representation of solutions, uniqueness theorem, Fourier-Laplace series.

AMS subject classification: 74A15, 74F20, 74H25, 74B10.

1 Introduction

Elastic composite materials with complex structures, as well as with structures composed of substantially differing materials are widely applied in the modern technological processes. Hemitropic elastic materials, mixtures produced from two or more elastic materials, etc., belong to the class of such composite materials and structures. The study of practical problems of mechanical properties of such materials naturally results in the necessity to develop mathematical models, which would allow to get more precise description of actual processes ongoing during the experiments. Mathematical modeling for such materials commenced as early as in the sixties of the past century. The first mathematical model of an elastic mixture (solid with solid), the so-called diffuse model, was developed by A. Green and T. Steel in 1966. In this model, the interaction force between components depend upon the difference of displacement vectors of components. In the same year they have developed the single-temperature thermoelasticity theory diffuse model of the elastic mixtures. Mathematical model of the linear theory of thermoelasticity of two-temperature elastic mixtures for the composites of granular, fibrous and layered structures was developed in 1984 by L. Khoroshun and N.Soltanov, Normally, the study of processes ongoing in the body is reduced in the relevant mathematical model described by the system of differential equations with partial derivatives to the study of boundary value problems (BVP$_s$), mixed type BVP$_s$ and boundary-contact problems, and also the fundamental matrix for solving the system of differential equations playing a substantial role. For the diffuse and displacement models of the two-component mixtures (single-temperature) thermoelasticity theory, the issue of steadiness and correctness, identification of the asymptotic behavior of problem solution, proving of the uniqueness and existence theorems, solution of the BVP$_s$ for the domains bounded by the specific surfaces, as absolutely and uniformly convergent series, are studied by many scientists, among them: Alves, Munoz Rivera, Quintanilla [2], Basheleishvili [3], Basheleishvili, Zazashvili [4], Burchuladze, Svanadze [6], Gales [8], Giorgashvili [10],Giorgashvili, Skhvitaridze [11], [12], Giorgashvili, Karseladze, Sadunishvili [13],Iesan [16], Natroshvili, Jaghmaidze, Svanadze [32], Svanadze [38], Quintanilla [37], Pompei [36] etc.

We treat here only the classical setting of basic boundary value problems for smooth domains, however applying the results obtained in the references Agranovich [1], Buchukuri, Chkadua, Duduchava, Natroshvili [5], Duduchava, Natroshvili [7], Gao [9], Jentsch, Natroshvili [17], [18], [19], Jentsch, Natroshvili, Wedland [20], [21], Kupradze, Gegelia, Basheleishvili, Burchuladze [23], Mitrea, Pipher [24], Natroshvili [26], [27], [38], Natroshvili, Giorgashvili, Stratis [29], Natroshvili, Giorgashvili, Zazashvili [30], Natroshvili, Kharibegashvili, Tediashvili [33], Natroshvili, Sadunishvili [34], Natroshvili, Stratis [35], and using the same type approaches and reasonings, one can analyze the generalized basic and mixed type boundary value problems, as well as crack type and interface problems in Sobolev-Slobodetskii and Bessel potential spaces for smooth and Lipschitz domains

In this paper we consider the statics case of the two-temperature theory of two-component elastic mixtures. The representation formula of a general solution of the homogeneous system of differential equations obtained in the paper is expressed by means of four harmonic and four metaharmonic functions. These formulas are very convenient and useful in many particular problems for domains with concrete geometry. Here we demonstrate an application of these formulas to the Dirichlet and Neumann type boundary value problem for a ball. Uniqueness theorems are proved. We construct an explicit solutions in the form of absolutely and uniformly convergent series.

2 Basic Equations and Auxiliary Theorems

The basic statical relationships for the two-component elastic mixtures, taking two-temperature thermal field into consideration, are mathematically described by the following system of partial differential equations [22], [15]

$$\begin{aligned}
&a_1\Delta u'(x) + b_1 \operatorname{grad}\operatorname{div} u'(x) + c\Delta u''(x) + d \operatorname{grad}\operatorname{div} u''(x)\\
&\qquad -\varkappa(u'(x) - u''(x)) - \operatorname{grad}(\eta_1\theta_1(x) + \eta_2\theta_2(x)) = 0, \qquad (2.1)\\
&c\Delta u'(x) + d \operatorname{grad}\operatorname{div} u'(x) + a_2\Delta u''(x) + b_2 \operatorname{grad}\operatorname{div} u''(x)\\
&\qquad +\varkappa(u'(x) - u''(x)) - \operatorname{grad}(\zeta_1\theta_1(x) + \zeta_2\theta_2(x)) = 0, \qquad (2.2)\\
&\qquad \varkappa_1\Delta\theta_1(x) + \varkappa_2\Delta\theta_2(x) - \alpha(\theta_1(x) - \theta_2(x)) = 0, \qquad (2.3)\\
&\qquad \varkappa_2\Delta\theta_1(x) + \varkappa_3\Delta\theta_2(x) - \alpha(\theta_1(x) - \theta_2(x)) = 0, \qquad (2.4)
\end{aligned}$$

where Δ is the three-dimensional Laplace operator, $u' = (u'_1, u'_2, u'_3)^\top$, $u'' = (u''_1, u''_2, u''_3)^\top$ are partial displacement vectors, θ_1 and θ_2 are temperatures of each component of the mixture, $a_j,\ b_j,\ c,\ d,\ j = 1,2$ are the elasticity coefficients, $\varkappa,\ \varkappa_3,\ \alpha,\ \eta_j,\ \ \zeta_j,\ \varkappa_j,\ j = 1,2,$ are the mechanical and thermal constants of the elastic mixture, $x = (x_1, x_2, x_3)$ is a point in the three-dimensional Cortesian space, $\top$ denotes transposition.

In the system (2.1)-(2.4), $a_j,\ b_j,\ c,\ d,\ j = 1,2,$ are the constants given as follows [15]

$$\begin{aligned}
&a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_5 + \lambda_1 - \frac{\rho_2}{\rho}\alpha_0, \quad c = \mu_3 + \lambda_5\\
&a_2 = \mu_2 - \lambda_5, \quad b_2 = \mu_2 + \lambda_5 + \lambda_2 - \frac{\rho_1}{\rho}\alpha_0, \quad \rho = \rho_1 + \rho_2,\\
&d = \mu_3 - \lambda_5 + \lambda_3 - \frac{\rho_1}{\rho}\alpha_0, \quad \alpha_0 = \lambda_3 - \lambda_4,
\end{aligned}$$

where $\rho_1 > 0,\ \ \rho_0 > 0$ are the densities of mixture components, $\lambda_j,\ \ j = 1,2,\ldots,5,\ \mu_j,\ \ j = 1,2,3$ are elastic constants satisfying the conditions

$$\begin{aligned}
&\mu_1 > 0,\ \ \lambda_5 < 0,\ \ \mu_1\mu_2 - \mu_3^2 > 0,\ \ \lambda_1 + \frac{2}{3}\mu_1 - \frac{\rho_2}{\rho}\alpha_0 > 0,\\
&(\lambda_1 + \frac{2}{3}\mu_1 - \frac{\rho_2}{\rho}\alpha_0)(\lambda_2 + \frac{2}{3}\mu_2 - \frac{\rho_1}{\rho}\alpha_0) > (\lambda_3 + \frac{2}{3}\mu_3 - \frac{\rho_1}{\rho}\alpha_0)^2.
\end{aligned}$$

From these inequalities it follows that

$$a_1 > 0, \quad a_1 + b_1 > 0,$$
$$d_1 := a_1 a_2 - c^2 > 0, \quad d_2 := (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0.$$

In addition, from physical consideration it follows that

$$\varkappa > 0, \quad \alpha > 0, \quad , \varkappa_j > 0, \quad j = 1, 2, 3, \quad d_3 := \varkappa_1 \varkappa_3 - \varkappa_2^2 > 0.$$

Assume that $U = (u', u'', \theta_1, \theta_2)^\top$. The stress vector, which we denote by the symbol $P(\partial, n)U$, has the form

$$P(\partial, n)U = (P^{(1)}(\partial, n)U,\ P^{(2)}(\partial, n)U,\ P^{(3)}(\partial, n)U,\ P^{(4)}(\partial, n)U)^\top,$$

where $\theta = (\theta_1,\ \theta_2)^\top$, n is the unit vector,

$$\begin{aligned}
&P^{(1)}(\partial, n)U = T^{(1)}(\partial, n)U' - n(\eta_1 \theta_1 + \eta_2 \theta_2),\\
&P^{(2)}(\partial, n)U = T^{(2)}(\partial, n)U' - n(\xi_1 \theta_1 + \xi_2 \theta_2), \quad U = (u', u'')^\top,\\
&P^{(3)}(\partial, n)\theta = \varkappa_1 \partial_n \theta_1 + \varkappa_2 \partial_n \theta_2,\\
&P^{(4)}(\partial, n)\theta = \varkappa_2 \partial_n \theta_1 + \varkappa_3 \partial_n \theta_2,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
T^{(1)}(\partial, n)U' &= 2\partial_n(\mu_1 u' + \mu_3 u'')\\
&\quad + n\left[(\lambda_1 - \frac{\rho_2}{\rho}\alpha_0)\operatorname{div} u' + (\lambda_3 - \frac{\rho_1}{\rho}\alpha_0)\operatorname{div} u''\right]\\
&\quad + \left[n \times \left((\mu_1 + \lambda_5)\operatorname{rot} u' + (\mu_3 - \lambda_5)\operatorname{rot} u''\right)\right],\\
T^{(2)}(\partial, n)U' &= 2\partial_n(\mu_3 u' + \mu_2 u'')\\
&\quad + n\left[(\lambda_3 - \frac{\rho_1}{\rho}\alpha_0)\operatorname{div} u' + (\lambda_2 - \frac{\rho_1}{\rho}\alpha_0)\operatorname{div} u''\right]\\
&\quad + \left[n \times \left((\mu_3 - \lambda_5)\operatorname{rot} u' + (\mu_2 + \lambda_5)\operatorname{rot} u''\right)\right],
\end{aligned}$$

$\partial_n = \partial/\partial_n$ is the derivative along the normal, symbol $[\times]$ denotes the vector products of two vectors in R^3.

Definition. The vector $U = (u', u'', \theta_1, \theta_2)^\top$ is assumed to be regular in a domain $\Omega \subset R^3$ if $U \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

Theorem 2.1. *For the vector $U = (u', u'', \theta_1, \theta_2)^\top$ to a regular solution of system (2.1)–(2.4) in a domain $\Omega \subset R^3$ it is necessary and sufficient* *that it be represented in the form*

$$
\begin{aligned}
u'(x) &= \operatorname{grad}\Phi_1(x) + \beta_1 \operatorname{grad}\Phi_2(x) + \beta_3\Big[\operatorname{rot}\operatorname{rot}(x\Phi_3(x)) + \operatorname{rot}(x\Phi_4(x))\Big] \\
&\quad + \beta_5 \operatorname{grad} r^2\left(r\frac{\partial}{\partial r}+1\right)\Phi_5(x) - \alpha_1 \operatorname{rot}\operatorname{rot}(xr^2\Phi_5(x)) \\
&\quad + \beta_6 \operatorname{grad}\left(r\frac{\partial}{\partial r}+1\right)\left(2r\frac{\partial}{\partial r}+3\right)\Phi_5(x) + \operatorname{rot}(x\Phi_6(x)) \\
&\quad + \alpha_2 \operatorname{grad}\left(2r\frac{\partial}{\partial r}+3\right)\Phi_7(x) + \alpha_3 \operatorname{grad} r^2\Phi_7(x) + \alpha_4 \operatorname{grad}\Phi_8(x),
\end{aligned}
$$

$$
\begin{aligned}
u''(x) &= \operatorname{grad}\Phi_1(x) + \beta_2 \operatorname{grad}\Phi_2(x) \\
&\quad + \beta_4\Big[\operatorname{rot}\operatorname{rot}(x\Phi_3(x)) + \operatorname{rot}(x\Phi_4(x))\Big] \\
&\quad + \beta_5 \operatorname{grad} r^2\left(r\frac{\partial}{\partial r}+1\right)\Phi_5(x) - \alpha_1 \operatorname{rot}\operatorname{rot}(xr^2\Phi_5(x)) \\
&\quad + \beta_7 \operatorname{grad}\left(r\frac{\partial}{\partial r}+1\right)\left(2r\frac{\partial}{\partial r}+3\right)\Phi_5(x) + \operatorname{rot}(x\Phi_6(x)) \\
&\quad - \alpha_2 \operatorname{grad}\left(2r\frac{\partial}{\partial r}+3\right)\Phi_7(x) + \alpha_3 \operatorname{grad} r^2\Phi_7(x) \\
&\quad + \alpha_5 \operatorname{grad}\Phi_8(x),
\end{aligned} \tag{2.6}
$$

$$
\theta_1(x) = 2\alpha_1\left(2r\frac{\partial}{\partial r}+3\right)\Phi_7(x) + (\varkappa_2+\varkappa_3)\Phi_8(x),
$$

$$
\theta_2(x) = 2\alpha_1\left(2r\frac{\partial}{\partial r}+3\right)\Phi_7(x) - (\varkappa_1+\varkappa_2)\Phi_8(x),
$$

where $\Delta\Phi_j(x) = 0$, $j = 1,5,6,7$, $(\Delta-\lambda_2^2)\Phi_2(x) = 0$, $(\Delta-\lambda_3^2)\Phi_j(x) = 0$, $j = 3,4$, $(\Delta-\lambda_1^2)\Phi_8(x) = 0$, $\alpha_1 = \varkappa(a_1+b_1+a_2+b_2+2(c+d))$, $\alpha_3 = \eta_1+\eta_2+\zeta_1+\zeta_2$, $\alpha_2 = (a_1+b_1+c+d)(\zeta_1+\zeta_2)-(a_2+b_2+c+d)(\eta_1+\eta_2)$,

$$
\begin{aligned}
\alpha_4 &= \frac{1}{\lambda_1^2 d_2(\lambda_1^2-\lambda_2^2)} \\
&\quad \times\left\{\left[\lambda_1^2(a_2+b_2)-\varkappa\right]\left[\eta_1(\varkappa_2+\varkappa_3)+\eta_2(\varkappa_1+\varkappa_2)\right]\right. \\
&\quad \left. -\left[\lambda_1^2(c+d)+\varkappa\right]\left[\zeta_1(\varkappa_2+\varkappa_3)+\zeta_2(\varkappa_1+\varkappa_2)\right]\right\}, \\
\alpha_5 &= \frac{1}{\lambda_1^2 d_2(\lambda_1^2-\lambda_2^2)}\left\{\left[\lambda_1^2(a_1+b_1)-\varkappa\right]\left[\zeta_1(\varkappa_2+\varkappa_3)+\zeta_2(\varkappa_1+\varkappa_2)\right]\right. \\
&\quad \left. -\left[\lambda_1^2(c+d)+\varkappa\right]\left[\eta_1(\varkappa_2+\varkappa_3)+\eta_2(\varkappa_1+\varkappa_2)\right]\right\},
\end{aligned} \tag{2.7}
$$

$$
\beta_1 = -(a_2+b_2+c+d), \quad \beta_2 = a_1+b_1+c+d, \quad \beta_3 = a_2+c,
$$
$$
\beta_4 = -(a_1+c), \quad \beta_5 = \varkappa(a_1+a_2+2c), \quad \beta_6 = 2(a_2b_1+cb_1+a_2d),
$$

$$\beta_7 = 2(a_1 b_2 + a_1 d + c b_2), \quad \lambda_1^2 = \alpha(\varkappa_1 + \varkappa_3 + 2\varkappa_2)/d_1 > 0,$$
$$\lambda_2^2 = \alpha_1/d_2 > 0, \quad \lambda_3^2 = \varkappa(a_1 + a_2 + 2c)/d_3 > 0.$$

Proof. Assume that the vector $U = (u', u'', \theta_1, \theta_2)^\top$ is a solution of system (2.1)–(2.4). From (2.3)–(2.4) we obtain

$$\Delta\left[(\varkappa_1 + \varkappa_2)\theta_1(x) + (\varkappa_2 + \varkappa_3)\theta_2(x)\right] = 0,$$
$$(\Delta - \lambda_1^2)[\theta_1(x) - \theta_2(x)] = 0,$$

where $\lambda_1^2 = \alpha(\varkappa_1 + \varkappa_3 + 2\varkappa_2)/d_1 > 0$.

From these system we get

$$\begin{aligned} \theta_1(x) &= 2\alpha_1\left(2r\frac{\partial}{\partial r} + 3\Phi_7(x) + (\varkappa_2 + \varkappa_3)\Phi_8(x)\right), \\ \theta_2(x) &= 2\alpha_1\left(2r\frac{\partial}{\partial r} + 3\right)\Phi_7(x) - (\varkappa_1 + \varkappa_2)\Phi_8(x), \end{aligned} \tag{2.8}$$

where $\Delta\Phi_7(x) = 0$, $(\Delta - \lambda_1^2)\Phi_8(x) = 0$, $\alpha_1 = \varkappa(a_1 + b_1 + a_2 + b_2 + 2(c + d))$.

Substituting the values of $\theta_1(x)$ and $\theta_2(x)$, from (2.8) into equations (2.1)–(2.2), we obtain

$$\begin{aligned} &a_1\Delta u'(x) + b_1 \operatorname{grad}\operatorname{div} u'(x) + c\Delta u''(x) + d \operatorname{grad}\operatorname{div} u''(x) \\ &-\varkappa(u'(x) - u''(x)) = 2\alpha_1(\eta_1 + \eta_2)\operatorname{grad}(2r\frac{\partial}{\partial r} + 3)\Phi_7(x) \\ &+(\eta_1(\varkappa_2 + \varkappa_3) + \eta_2(\varkappa_1 + \varkappa_2))\operatorname{grad}\Phi_8(x), \\ &c\Delta u'(x) + d \operatorname{grad}\operatorname{div} u'(x) + a_2\Delta u''(x) + b_2 \operatorname{grad}\operatorname{div} u''(x) \\ &+\varkappa(u'(x) - u''(x)) = 2\alpha_1(\zeta_1 + \zeta_2)\operatorname{grad}(2r\frac{\partial}{\partial r} + 3)\Phi_7(x), \\ &+(\zeta_1(\varkappa_2 + \varkappa_3) + \zeta_2(\varkappa_1 + \varkappa_2))\operatorname{grad}\Phi_8(x). \end{aligned} \tag{2.9}$$

A general solution of system (2.9) has the form

$$U'(x) = U_0'(x) + V(x), \tag{2.10}$$

where $U_0' = (u_0', u_0'')^\top$ is a general solution of the homogeneous system of differential equations

$$a_1\Delta u_0'(x) + b_1 \operatorname{grad}\operatorname{div} u_0'(x) + c\Delta u_0''(x) + d \operatorname{grad}\operatorname{div} u_0''(x)$$
$$-\varkappa(u_0'(x) - u_0''(x)) = 0,$$

$$c\Delta u_0'(x) + d\,\mathrm{grad\,div}\, u_0'(x) + a_2\Delta u_0''(x) + b_2\,\mathrm{grad\,div}\, u_0''(x) \\ +\varkappa(u_0'(x) - u_0''(x)) = 0, \tag{2.11}$$

and $V(x)$ is a particular solution of the nonhomogeneous system (2.9).

The solution $U_0'(x)$ has the form

$$\begin{aligned} u_0'(x) &= \mathrm{grad}\,\Phi_1(x) + \beta_1\,\mathrm{grad}\,\Phi_2(x) + \beta_3\Big[\mathrm{rot\,rot}(x\Phi_3(x)) + \mathrm{rot}(x\Phi_4(x))\Big] \\ &+ \beta_5\,\mathrm{grad}\, r^2\left(r\frac{\partial}{\partial r}+1\right)\Phi_5(x) + \beta_6\,\mathrm{grad}\left(r\frac{\partial}{\partial r}+1\right) \\ &\times\left(2r\frac{\partial}{\partial r}+3\right)\Phi_5(x) - \alpha_1\,\mathrm{rot\,rot}\left(xr^2\Phi_5(x)\right) + \mathrm{rot}\left(xr^2\Phi_6(x)\right), \end{aligned}$$

$$\begin{aligned} u_0''(x) &= \mathrm{grad}\,\Phi_1(x) + \beta_2\,\mathrm{grad}\,\Phi_2(x) + \beta_4\Big[\mathrm{rot\,rot}(x\Phi_3(x)) + \mathrm{rot}(x\Phi_4(x))\Big] \\ &+ \beta_5\,\mathrm{grad}\, r^2\left(r\frac{\partial}{\partial r}+1\right)\Phi_5(x) + \beta_6\,\mathrm{grad}\left(r\frac{\partial}{\partial r}+1\right) \\ &\times\left(2r\frac{\partial}{\partial r}+3\right)\Phi_5(x) - \alpha_1\,\mathrm{rot\,rot}\left(xr^2\Phi_5(x)\right) + \mathrm{rot}\left(x\Phi_6(x)\right), \end{aligned}$$

where $\Delta\Phi_j(x) = 0,\ \ j = 1,5,6,\ (\Delta-\lambda_2^2)\Phi_2(x) = 0,\ (\Delta-\lambda_3^2)\Phi_j(x) = 0,\ \ j = 3,4$, the constant $\alpha_1,\ \lambda_2^2,\ \lambda_3^2,\ \beta_j,\quad j = 1,2,...,7$ are defined by formulas (2.7).

The particular solution of the system (2.9) will be written as

$$\begin{aligned} v'(x) &= \alpha_2\,\mathrm{grad}\left(2r\frac{\partial}{\partial r}+3\right)\Phi_7(x) + \alpha_4\,\mathrm{grad}\,\Phi_8(x), \\ v''(x) &= -\alpha_2\,\mathrm{grad}\left(2r\frac{\partial}{\partial r}+3\right)\Phi_7(x) + \alpha_5\,\mathrm{grad}\,\Phi_8(x), \end{aligned}$$

where $\Delta\Phi_7(x) = 0$, $(\Delta-\lambda_1^2)\Phi_8(x) = 0$, the constants $\lambda_1^2,\ \ \alpha_j,\ \ j = 2,3,4,5$ are defined by (2.7).

Substituting the values of the vectors $U_0'(x)$ and $V(x)$ into (2.10), we there by prove the first part of the theorem. As to the second part, it is proved by a straight forward verification that the vector $U = (u', u'', \theta_1, \theta_2)^\top$ represented in form (2.6) is solution of system (2.1)–(2.4). □

Assume that $r,\ \vartheta,\ \varphi,\ (0 \le r < +\infty,\ \ 0 \le \vartheta \le,\ \ 0 \le \varphi < 2\pi)$ are the spherical coordinates of the point $x \in R^3$. Denote by Σ_1 the sphere with unit radius and centre at the origin lying in the space R^3.

Let us consider, in the space $L_2(\Sigma_1)$, the following complete system of orthonormal vectors [10],

$$
\begin{aligned}
X_{mk}(\vartheta,\varphi) &= e_r Y_k^{(m)}(\vartheta,\varphi), \quad k\geq 0,\\
Y_{mk}(\vartheta,\varphi) &= \frac{1}{\sqrt{k(k+1)}}\left(e_\vartheta \frac{\partial}{\partial\vartheta}+\frac{e_\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)Y_k^{(m)}(\vartheta,\varphi), \quad k\geq 0,\\
Z_{mk}(\vartheta,\varphi) &= \frac{1}{\sqrt{k(k+1)}}\left(\frac{e_\vartheta}{\sin\vartheta} \frac{\partial}{\partial\vartheta}-e_\varphi\frac{\partial}{\partial\varphi}\right)Y_k^{(m)}(\vartheta,\varphi), \quad k\geq 1,
\end{aligned}
$$

where $|m|\leq k$, e_r, e_ϑ, e_φ are the orthonormal vectors in R^3,

$$
\begin{aligned}
e_r &= (\cos\varphi\sin\vartheta,\ \sin\varphi\sin\vartheta,\ \cos\vartheta)^\top,\\
e_\vartheta &= (\cos\varphi\cos\vartheta,\ \sin\varphi\sin\vartheta,\ -\sin\vartheta)^\top,\\
e_\varphi &= (-\sin\varphi,\ \cos\varphi,\ 0)^\top,
\end{aligned}
$$

$$
Y_k^{(m)}(\vartheta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\frac{(k-m)!}{(k+m)!}}P_k^{(m)}(\cos\theta)e^{im\varphi},
$$

$P_k^{(m)}(\cos\vartheta)$ is the adjoint Legendre polynomial.

Assume that $f^{(j)}=(f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)})^\top$, $j=1,2$ is a vector-function and the function f_4 and f_5, represent as the following Fourier-Laplace series

$$
\begin{aligned}
f^{(j)}(\vartheta,\varphi) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\{\alpha_{mk}X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\\
&\times[\beta_{mk}^{(j)}Y_{mk}(\vartheta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\vartheta,\varphi)]\}, \quad j=1,2,\\
f_j(\vartheta,\varphi) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\alpha_{mk}^{(j)}Y_k^m(\vartheta,\varphi), \quad j=4,5, \qquad (2.12)
\end{aligned}
$$

where $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j=1,2$, $\alpha_{mk}^{(j)}$, $l=4,5$ are Fourier-Laplace coefficients.

Note that in formula (2.12) and in further analogous series containing the vectors $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ the summation index k varies from 1 to $+\infty$.

Let us formulate several important lemmas [10], [25]

Lemma 2.2. *If $f^j\in C^l(\Sigma_1)$, $j=1,2$, $f_j\in C^l(\Sigma_1)$, $j=4,5$, $l\geq 1$ then the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j=4,5$ admit the following estimates*

$$
\begin{aligned}
\alpha_{mk}^{(j)} &= O(k^{-l}), \quad \beta_{mk}^{(j)}=O(k^{-l-1}),\\
\gamma_{mk}^{(j)} &= O(k^{-l-1}), \quad \alpha_{mk}^{(3+j)}=O(k^{-l}), \quad j=1,2.
\end{aligned}
$$

Lemma 2.3. *The vectors $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ defined by equalities* (2.11) *admit the following estimates*

$$|X_{mk}(\vartheta,\varphi)| \leq \sqrt{\frac{2k+1}{4\pi}}, \quad k \geq 0,$$
$$|Y_{mk}(\vartheta,\varphi)| < \sqrt{\frac{k(k+1)}{2k+1}}, \quad |Z_{mk}(\vartheta,\varphi)| < \sqrt{\frac{k(k+1)}{2k+1}}, \quad k \geq 1.$$

Note that [39]

$$|Y_k^{(m)}(\vartheta,\varphi)| < \sqrt{\frac{2k+1}{4\pi}}, \quad k \geq 0,$$

The equalities

$$\int_0^{2\pi} d\varphi \int_0^{\pi} Y_k^{(m)}(\vartheta,\varphi)\sin\vartheta\, d\vartheta = \begin{cases} \sqrt{\pi}, & k=0, \quad m=0, \\ 0, & \textit{in other cases}, \end{cases} \tag{2.13}$$

are also true.

Let $\Omega^+ = B(R) \subset R^3$ be the ball bounded by the spherical surface $\Sigma_R = \partial\Omega^+$ of radius R and centre at the origin. We introduce the notation $\Omega^- = R^3 \setminus \overline{\Omega^+}$.

Theorem 2.4. *The vector $U = (u', u'', \theta_1, \theta_2)^\top$ represented in form* (2.6) *is uniquely defined in the domain Ω^+ by the functions $\Phi_j(x)$, $j = \overline{1,8}$, if the following conditions are fulfilled*

$$\int_{\Sigma_r} \Phi_j(x) d\Sigma_r = 0, \quad j = 1,3,4,6, \quad r = |x| \geq R, \tag{2.14}$$

i.e. to the zero value of the vector $U = (u', u'', \theta_1, \theta_2)^\top$ there corresponds the zero value of the vector $(\Phi_1,\ \Phi_2, \ldots, \Phi_8)^\top$ and vice versa.

The proof of this theorem is similar to that of theorem in [12].

Hereinafter we make use the following equalities [10]

$$\begin{gathered} e_r \cdot X_{mk}(\vartheta,\varphi) = Y_k^{(m)}(\vartheta,\varphi), \quad e_r \cdot Y_{mk}(\vartheta,\varphi) = 0, \\ e_r \cdot Z_{mk}(\vartheta,\varphi) = 0, \quad e_r \times X_{mk}(\vartheta,\varphi) = 0, \\ e_r \times Y_{mk}(\vartheta,\varphi) = -Z_{mk}(\vartheta,\varphi), \quad e_r \times Z_{mk}(\vartheta,\varphi) = Y_{mk}(\vartheta,\varphi); \end{gathered} \tag{2.15}$$

$$
\begin{aligned}
\operatorname{grad}[a(r)Y_k^{(m)}(\vartheta,\varphi)] &= \frac{da(r)}{dr}X_{mk}(\vartheta,\varphi) \\
&+ \frac{\sqrt{k(k+1)}}{r}a(r)Y_{mk}(\vartheta,\varphi), \\
\operatorname{rot}[xa(r)Y_k^{(m)}(\vartheta,\varphi)] &= \sqrt{k(k+1)}a(r)Z_{mk}(\vartheta,\varphi), \\
\operatorname{rot}\operatorname{rot}[xa(r)Y_k^{(m)}(\vartheta,\varphi)] &= \frac{k(k+1)}{r}a(r)X_{mk}(\vartheta,\varphi)+ \\
&+ \sqrt{k(k+1)}\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Y_{mk}(\vartheta,\varphi),
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
\operatorname{div}[a(r)X_{mk}(\vartheta,\varphi)] &= \left(\frac{d}{dr}+\frac{2}{r}\right)a(r)Y_k^{(m)}(\vartheta,\varphi), \\
\operatorname{div}[a(r)Y_{mk}(\vartheta,\varphi)] &= -\frac{\sqrt{k(k+1)}}{r}a(r)Y_k^{(m)}(\vartheta,\varphi), \\
\operatorname{div}[a(r)Z_{mk}(\vartheta,\varphi)] &= 0, \\
\operatorname{rot}[a(r)X_{mk}(\vartheta,\varphi)] &= \frac{\sqrt{k(k+1)}}{r}a(r)Z_{mk}(\vartheta,\varphi), \\
\operatorname{rot}[a(r)Y_{mk}(\vartheta,\varphi)] &= -\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Z_{mk}(\vartheta,\varphi), \\
\operatorname{rot}[a(r)Z_{mk}(\vartheta,\varphi)] &= \frac{\sqrt{k(k+1)}}{r}a(r)X_{mk}(\vartheta,\varphi)+ \\
&+ \left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Y_{mk}(\vartheta,\varphi).
\end{aligned} \tag{2.17}
$$

3 Statement of the Problem. The Uniqueness Theorem

Problem. Find, in the domain Ω^+, such a regular vector $U = (u', u'', \theta_1, \theta_2)^\top$ that satisfies in this domain the system of differential equations (2.1)-(2.4) and, on the boundary $\partial\Omega^+$, one of the following boundary conditions:

$(I)^+$ (the Dirichlet problem)

$$
\{u'(z)\}^+ = f^{(1)}(z), \quad \{u''(z)\}^+ = f^{(2)}(z),
$$
$$
\{\theta_1(z)\}^+ = f_4(z), \quad \{\theta_2(z)\}^+ = f_5(z);
$$

$(II)^+$ (the Neumann problem)

$$\{P^{(1)}(\partial,n)U(z)\}^{+}=f^{(1)}(z),\quad \{P^{(2)}(\partial,n)U(z)\}^{+}=f^{(2)}(z),$$
$$\{P^{(3)}(\partial,n)U''(z)\}^{+}=f_4(z),\quad \{P^{(4)}(\partial,n)\theta(z)\}^{+}=f_5(z),$$

where the vector $f^{(j)}=(f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)})$, $j=1,2$ and the functions f_4, f_5, are given on the boundary $\partial\Omega^{+}$, $n(z)$ is the external normal unit vector passing at the point $z\in\partial\Omega^{+}$ with respect to the domain Ω^{+}.

Theorem 3.1. *Homogeneous* ($f^{(j)}=0,\quad j=1,2,\quad f_j=0,\quad j=4,5$) *problem* $(I)_0^{+}$ *has the trivial solution only, whereas any solution of problem* $(II)^{+}$ *is defined to within a summand*

$$u'(x)=a+[b\times x]+\left[B_0+\beta_1B_1\frac{1}{r}\frac{d}{dr}g_0(\lambda_2r)\right]cx,$$
$$u''(x)=a+[b\times x]+\left[B_0+\beta_3B_1\frac{1}{r}\frac{d}{dr}g_0(\lambda_2r)\right]cx,$$
$$\theta_1(x)=\theta_2(x)=c,\quad x\in\Omega^{+},$$

where a and b are any three-dimensional constant vectors, $c=const$, $\beta_j,\ j=1,3$ *has a* (2.7) *from,* B_0 *and* B_1 *are the solutions of system*

$$\begin{aligned}[3(b_1+d)+\beta_4]B_0&=-\left[4(\mu_1\beta_1+\mu_3\beta_3)\frac{1}{R}\frac{d}{dR}g_0(\lambda_2R)+\alpha_1\right]B_1\\&=\eta_1+\eta_2,\\ [3(b_2+d)-\beta_2]B_0&=-\left[4(\mu_3\beta_1+\mu_2\beta_3)\frac{1}{R}\frac{d}{dR}g_0(\lambda_2R)-\alpha_1\right]B_1\\&=\zeta_1+\zeta_2,\end{aligned}\tag{3.1}$$

$$\frac{d}{dR}g_0(\lambda_2R)=\lim_{r\to R}\frac{d}{dr}g_0(\lambda_2r),\quad g_0(\lambda_2r)=\sqrt{\frac{R}{r}}\frac{I_{1/2}(\lambda_2r)}{I_{1/2}(\lambda_2R)},$$

$I_{1/2}(\lambda_2r)$ *is the Bessel function with an imaginary argument.*

Proof. Assume that the vector $U=(u',u'',\theta_1,\theta_2)^{\top}$ is a regular solution of system (2.1)–(2.4). Multiplying both sides of the equation (2.3) by θ_1, and of the equation (2.4) by θ_2 and applying the Stokes' formula, we obtain

$$\int\limits_{\partial\Omega^{+}}\left[\theta_1(z)P^{(3)}(\partial,n)\theta(z)+\theta_2(z)P^{(4)}(\partial,n)\theta(z)\right]^{+}ds-\int\limits_{\Omega^{+}}[\varkappa_1(\operatorname{grad}\theta_1(x))^2$$
$$+2\varkappa_2\operatorname{grad}\theta_1(x)\cdot\operatorname{grad}\theta_2(x)+\varkappa_3(\operatorname{grad}\theta_2(x))^2]dx=0.$$

By the boundary conditions of homogeneous problems $(I)_0^+$ and $(II)_0^+$ we obtain

$$\int\limits_{\Omega^+} [\varkappa_1(\operatorname{grad}\theta_1(x))^2 + 2\varkappa_2 \operatorname{grad}\theta_1(x)\cdot\operatorname{grad}\theta_2(x) + \varkappa_3(\operatorname{grad}\theta_2(x))^2]dx = 0.$$

From this it follows that $\theta_1(x) = \theta_2(x) = c = const, \quad x \in$ then for the vectors $u'(x)$ and $u''(x)$, we obtain

$$\begin{aligned} a_1\Delta u'(x) + b_1 \operatorname{grad}\operatorname{div} u'(x) + c\Delta u''(x) + d \operatorname{grad}\operatorname{div} u''(x) \\ - \varkappa(u'(x) - u''(x)) = 0, \\ c\Delta u'(x) + d \operatorname{grad}\operatorname{div} u'(x) + a_2\Delta u''(x) + b_2 \operatorname{grad}\operatorname{div} u''(x) \\ - \varkappa(u'(x) - u''(x)) = 0. \end{aligned} \tag{3.2}$$

1) In the case of problem $(I)_0^+$ vector $U' = (u', u'')^\top$ satisfying the boundary conditions

$$\{u'(z)\}^+ = 0, \quad \{u''(z)\}^+ = 0, \quad z \in \partial\Omega^+. \tag{3.3}$$

2) In the case of problem $(II)_0^+$ vector $U' = (u', u'')^\top$ satisfying the boundary conditions

$$\begin{aligned} \Big\{T^{(1)}(\partial, n)U'(z)\Big\}^+ &= C(\eta_1 + \eta_2)n(z), \\ \Big\{T^{(2)}(\partial, n)U'(z)\Big\}^+ &= C(\zeta_1 + \zeta_2)n(z), \quad z \in \Omega^+. \end{aligned} \tag{3.4}$$

Let us multiply the first equation of (3.2) by the vector u', the second one by the vector u'', sum up and applying the Stokes formula, we obtain

$$\begin{aligned} \int\limits_{\partial\Omega^+} [u'(z)\cdot T^{(1)}(\partial, n)U'(z) + u''(z)\cdot T^{(2)}(\partial, n)U'(z)]^+ \, ds \\ - \int\limits_{\Omega^+} [E(U', U') + \varkappa(u'(x) - u''(x))^2] \, dx = 0, \end{aligned} \tag{3.5}$$

where

$$
\begin{gathered}
E(U', U') = (a_1 + b_1)(\operatorname{div} u')^2 + (a_2 + b_2)(\operatorname{div} u'')^2 \\
+2(c+d) \operatorname{div} u' \operatorname{div} u'' + \frac{\mu_1}{2} \sum_{k \neq j = 1}^{3} \Big(\frac{\partial u'_k}{\partial x_j} + \frac{\partial u'_j}{\partial x_k} \Big)^2 \\
+\frac{\mu_2}{2} \sum_{k \neq j = 1}^{3} \Big(\frac{\partial u''_k}{\partial x_j} + \frac{\partial u''_j}{\partial x_k} \Big)^2 + \mu_3 \sum_{k \neq j = 1}^{3} \Big(\frac{\partial u'_k}{\partial x_j} + \frac{\partial u'_j}{\partial x_k} \Big)\Big(\frac{\partial u''_k}{\partial x_j} + \frac{\partial u''_j}{\partial x_k} \Big) \\
-\frac{\lambda_5}{2} \sum_{k,j=1}^{3} \Big(\frac{\partial u'_k}{\partial x_j} - \frac{\partial u'_j}{\partial x_k} + \frac{\partial u''_k}{\partial x_j} - \frac{\partial u''_j}{\partial x_k} \Big)^2.
\end{gathered} \tag{3.6}
$$

From (3.3) and (3.5) we have

$$
\int_{\Omega^+} [E(U', U') + \varkappa (u'(x) - u''(x))^2] dx = 0.
$$

Since $E(U', U') \geq 0, \ \ \varkappa > 0$, from (3.6) we get

$$
u'(x) = u''(x), \ \ E(U', U') = 0, \quad x \in \Omega^+,
$$

A solution of this equation has the form

$$
u'(x) = u''(x) = a + [b \times x], \quad x \in \Omega^+, \tag{3.7}
$$

where a and b are any three-dimensional constant vectors.

In the case of problem $(I)_0^+$ we have

$$
\big\{u'(z)\big\}^+ = 0, \ \ \big\{u''(z)\big\}^+ = 0, \ \ z \in \partial\Omega^+,
$$

from (3.7) we obtained $a = 0, \ \ b = 0$, i.e. $u'(x) = 0, \ \ u''(x) = 0, \ \ x \in \Omega^+$, also $\{\theta_1(z)\}^+ = \{\theta_2(z)\}^+ = c = 0$, it follows that $\theta_1(x) = \theta_2(x) = 0, \ \ x \in \Omega^+$. Thus we conclude that $U(x) = 0, \ \ x \in \Omega^+$.

In the case of problem $(II)_0^+$ we have $\theta_1(x) = \theta_2(x) = c, \ \ x \in \Omega^+$, and obtain for the vector $U'(x)$ the problem (3.2), (3.4). A general solution of this problem has the form

$$
u'(x) = a + [b \times x] + \left[B_0 + \beta_1 B_1 \frac{1}{r} \frac{d}{dr} g_0(\lambda_2 r) \right] cx, \quad x \in \Omega^+
$$

$$
u''(x) = a + [b \times x] + \left[B_0 + \beta_3 B_1 \frac{1}{r} \frac{d}{dr} g_0(\lambda_2 r) \right] cx, \quad x \in \Omega^+,
$$

where a and b are arbitrary three-dimensional constant vectors, $c = const$, β_j, $j = 1, 3$ has (2.7) form, B_0 and B_1 are the solution of system (3.1). □

4 Solution of the Dirichlet and Neumann Boundary Value Problems

A solution of these problems is sought for in the form (2.6), where the functions $\Phi_j(x), \quad j=\overline{1,8}$ are written as

$$\begin{aligned}
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\left(\frac{r}{R}\right)^k Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}, \quad j=1,5,6,7,\\
\Phi_2(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(\lambda_2 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(2)},\\
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(\lambda_3 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}, \quad j=3,4,\\
\Phi_8(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(\lambda_1 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(8)}.
\end{aligned} \tag{4.1}$$

Here $A_{mk}^{(j)}, \quad j=\overline{1,8}$, are the constants to be defined, and

$$g_k(\lambda_j r) = \sqrt{\frac{R}{r}}\frac{I_{k+1/2}(\lambda_j r)}{I_{k+1/2}(\lambda_j R)}, \quad j=1,2,3,$$

$I_{k+1/2}(x)$ is the Bessel function with an imaginary argument.

Substituting the values of $\Phi_j(x), \quad j=1,3,4,6$ from (4.1), into (2.14) and taking into account the equalities (2.13), we get that $A_{00}^{(j)}=0, \quad j=1,3,4,6$.

Substituting the values of the function $\Phi_j(x), \quad j=\overline{1,8}$, defined by (4.1) into (2.6) and taking into consideration the equalities (2.16), we obtain

$$\begin{aligned}
u'(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(1)}(r)X_{mk}(\vartheta,\varphi)\\
&\quad + \sqrt{k(k+1)}[v_{mk}^{(1)}(r)Y_{mk}(\vartheta,\varphi)+w_{mk}^{(1)}(r)Z_{mk}(\vartheta,\varphi)]\Big\},\\
u''(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(2)}(r)X_{mk}(\vartheta,\varphi)\\
&\quad + \sqrt{k(k+1)}[v_{mk}^{(2)}(r)Y_{mk}(\vartheta,\varphi)+w_{mk}^{(2)}(r)Z_{mk}(\vartheta,\varphi)]\Big\},
\end{aligned}$$

$$\theta_j(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} \theta_{mk}^{(j)}(r) Y_k^{(m)}(\vartheta,\varphi), \quad j=1,2, \tag{4.2}$$

where

$$\begin{aligned}
u_{mk}^{(j)}(r) &= \frac{k}{R}\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(1)} + \beta_j \frac{d}{dr} g_k(\lambda_2 r) A_{mk}^{(2)} + \beta_{2+j}\frac{k(k+1)}{r} g_k(\lambda_3 r) A_{mk}^{(3)} \\
&+ \frac{k+1}{R}\left[(\beta_5(k+2)-\alpha_1 k)r^2 + \beta_{5+j}k(2k+3)\right]\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(5)} \\
&+ \frac{1}{R}\left[\alpha_3(k+2)r^2 - (-1)^j \alpha_2 k(2k+3)\right]\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(7)} \\
&+ \alpha_{3+j}\frac{d}{dr} g_k(\lambda_1 r) A_{mk}^{(8)}, \quad k \geq 0, \\
v_{mk}^{(j)}(r) &= \frac{1}{R}\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(1)} + \beta_j \frac{1}{r} g_k(\lambda_2 r) A_{mk}^{(2)} + \beta_{2+j}\left(\frac{d}{dr}+\frac{1}{r}\right) g_k(\lambda_3 r) A_{mk}^{(3)} \\
&+ \frac{1}{R}\left[(\beta_5(k+1)-\alpha_1 k)r^2 + \beta_{5+j}(k+1)(2k+3)\right]\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(5)} \\
&+ \frac{1}{R}\left[\alpha_3 r^2 - (-1)^j \alpha_2 (2k+3)\right]\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(7)} \\
&+ \alpha_{3+j}\frac{1}{r} g_k(\lambda_1 r) A_{mk}^{(8)}, \quad k \geq 1, \\
w_{mk}^{(j)}(r) &= \beta_{2+j} g_k(\lambda_2 r) A_{mk}^{(4)} + \left(\frac{r}{R}\right)^{k} A_{mk}^{(6)}, \quad k \geq 1, \\
\theta_{mk}^{(j)}(r) &= 2\alpha_1(2k+3)\left(\frac{r}{R}\right)^{k} A_{mk}^{(7)} + \left[(\varkappa_2+\varkappa_3)\delta_{1j} - (\varkappa_1+\varkappa_2)\delta_{2j}\right] \\
&\times g_k(\lambda_1 r) A_{mk}^{(8)}, \qquad k \geq 0, \quad j=1,2.
\end{aligned}$$

If we substitute the values of the vectors $u'(x)$, $u''(x)$ and the functions $\theta_j(x)$, $\quad j=1,2$, into (2.5) and use the equality (2.15), (2.17), we get

$$\begin{aligned}
P^{(j)}(\partial,n)U(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{ a_{mk}^{(j)}(r) X_{mk}(\vartheta,\varphi) + \sqrt{k(k+1)} \\
&\times [b_{mk}^{(j)}(r) Y_{mk}(\vartheta,\varphi) + c_{mk}^{(j)}(r) Z_{mk}(\vartheta,\varphi)]\Big\}, \quad j=1,2, \quad (4.3) \\
P^{(j)}(\partial,n)\theta(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} a_{mk}^{(j)}(r) Y_k^{(m)}(\vartheta,\varphi), \quad j=3,4,
\end{aligned}$$

where

$$
\begin{aligned}
a_{mk}^{(j)}(r) &= 2\frac{d}{dr}\left[(\mu_1\delta_{1j}+\mu_3\delta_{2j})\,u_{mk}^{(1)}(r)+(\mu_3\delta_{1j}+\mu_2\delta_{2j})\,u_{mk}^{(2)}(r)\right]\\
&+\left[\left(\lambda_1-\frac{\rho_2}{\rho}\alpha_0\right)\delta_{1j}+\left(\lambda_3-\frac{\rho_1}{\rho}\alpha_0\right)\delta_{2j}\right]\\
&\times\left[\left(\frac{d}{dr}+\frac{2}{r}\right)u_{mk}^{(1)}(r)-\frac{k(k+1)}{r}v_{mk}^{(1)}(r)\right]\\
&+\left[\left(\lambda_3-\frac{\rho_1}{\rho}\alpha_0\right)\delta_{1j}+\left(\lambda_2-\frac{\rho_1}{\rho}\alpha_0\right)\delta_{2j}\right]\\
&\times\left[\left(\frac{d}{dr}+\frac{2}{r}\right)u_{mk}^{(2)}(r)-\frac{k(k+1)}{r}v_{mk}^{(2)}(r)\right]\\
&-\left[(\eta_1\delta_{1j}+\zeta_1\delta_{2j})\,\vartheta_{mk}^{(1)}(r)+(\eta_2\delta_{1j}+\zeta_2\delta_{2j})\,\vartheta_{mk}^{(2)}\right],\quad k\geq 0,\\
b_{mk}^{(j)}(r) &= 2\frac{d}{dr}\left[(\mu_1\delta_{1j}+\mu_3\delta_{2j})\,v_{mk}^{(1)}(r)+(\mu_3\delta_{1j}+\mu_2\delta_{2j})\,v_{mk}^{(2)}(r)\right]\\
&+\left[(\mu_1+\lambda_5)\,\delta_{1j}+(\mu_3-\lambda_5)\,\delta_{2j}\right]\left[\frac{1}{r}u_{mk}^{(1)}(r)-\left(\frac{d}{dr}+\frac{1}{r}\right)v_{mk}^{(1)}(r)\right]\\
&\times\left[(\mu_3-\lambda_5)\,\delta_{1j}+(\mu_2+\lambda_5)\,\delta_{2j}\right]\left[\frac{1}{r}u_{mk}^{(2)}(r)-\left(\frac{d}{dr}+\frac{1}{r}\right)v_{mk}^{(2)}(r)\right],\\
&k\geq 1,\\
c_{mk}^{(j)}(r) &= 2\frac{d}{dr}\left[(\mu_1\delta_{1j}+\mu_3\delta_{2j})\,w_{mk}^{(1)}(r)+(\mu_3\delta_{1j}+\mu_2\delta_{2j})\,w_{mk}^{(2)}(r)\right]\\
&-\left[(\mu_1+\lambda_5)\,\delta_{1j}+(\mu_3-\lambda_5)\,\delta_{2j}\right]\left(\frac{d}{dr}+\frac{1}{r}\right)w_{mk}^{(1)}(r)\\
&-\left[(\mu_3-\lambda_5)\,\delta_{1j}+(\mu_2+\lambda_5)\,\delta_{2j}\right]\left(\frac{d}{dr}+\frac{1}{r}\right)w_{mk}^{(2)}(r),\\
&k\geq 1,\ j=1,2,\\
a_{mk}^{(j)}(r) &= (\varkappa_1\delta_{3j}+\varkappa_2\delta_{4j})\frac{d\theta_{mk}^{(1)}(r)}{dr}+(\varkappa_2\delta_{3j}+\varkappa_3\delta_{4j})\frac{d\theta_{mk}^{(2)}(r)}{dr},\quad j=3,4,
\end{aligned}
$$

δ_{lj} is the Kroneker symbol.

Assume that the vectors $f^{(j)}(z),\ \ j=1,2$ and the function $f^{(j)}(z),\ \ j=4,5$ satisfy the sufficient smoothness conditions. Then they can be expanded into (2.12) Fourier-Laplace series.

Let us now consider problem $(I)^+$. If in both parts of the equalities (4.2) we pass to the limit as $x\to z\in\partial\Omega^+$ $(r\to R)$, take into account the boundary condition (4.1) and the formulas (2.12) that for the desired constants $A_{mk}^{(j)},\ \ j=\overline{1,8}$ we obtain the system of linear algebraic equation

$$\begin{aligned}&\beta_j \frac{d}{dR} g_k(\lambda_2 R) A_{00}^{(2)} - 2R\beta_5 A_{00}^{(5)} + 2R\alpha_3 A_{00}^{(7)} \\ &+ \alpha_{3+j} \frac{d}{dR} g_k(\lambda_1 R) A_{00}^{(8)} = \alpha_{00}^{(j)}, \\ &6\alpha_1 A_{00}^{(7)} + [(\varkappa_2 + \varkappa_3)\delta_{1j} - (\varkappa_1 + \varkappa_3)\delta_{2j}]\, A_{00}^{(8)} = \alpha_{00}^{(3+j)}, \quad j = 1,2;\end{aligned} \tag{4.4}$$

$$\begin{aligned}&u_{mk}^{(j)}(R) = \alpha_{mk}^{(j)}, \quad v_{mk}^{(j)}(R) = \beta_{mk}^{(j)}, \\ &w_{mk}^{(j)}(R) = \gamma_{mk}^{(j)}, \quad \theta_{mk}^{(j)}(R) = \alpha_{mk}^{(3+j)}, \quad j = 1,2, \ \ k \geq 1.\end{aligned} \tag{4.5}$$

By virtue of theorem 2.4 and theorem 3.1, the systems (4.4)–(4.5) have a unique solution. If we substitute the solutions of these systems into (4.2), then the vector $U = (u', u'', \theta_1, \theta_2)^\top$ given by (4.2) will be a formal solution of problem $(I)^+$. T_0 justify the existent of such a solution we must show the convergence of the series (4.2)–(4.3).

Note that for $k \to +\infty$ the Bessel function admit the following asymptotic estimates [39]

$$g_k(\lambda_j r) \approx \left(\frac{r}{R}\right)^k, \quad g_k'(\lambda_j r) \approx \frac{k}{r}\left(\frac{r}{R}\right)^k, \quad r < R. \tag{4.6}$$

According to the asymptotic estimates (4.6), the series (4.2)–(4.3) will be absolutely and uniformly convergent, if the following majorant series is convergent

$$\alpha_0 \sum_{k=k_0}^{\infty} k^{3/2} \sum_{j=1}^{2} \left(|\alpha_{mk}^{(j)}| + k|\beta_{mk}^{(j)}| + |\gamma_{mk}^{(j)}| + k|\alpha_{mk}^{(3+j)}| \right), \tag{4.7}$$

where $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$ $\gamma_{mk}^{(j)}$ $\alpha_{mk}^{(3+j)}$, $j = 1,2$, are Fourier-Laplace coefficients. The series (4.7) will be convergent if these coefficients admit the following estimates

$$\begin{aligned}&\alpha_{mk}^{(j)} = O(k^{-3}), \quad \beta_{mk}^{(j)} = O(k^{-4}), \\ &\gamma_{mk}^{(j)} = O(k^{-3}), \quad \alpha_{mk}^{(3+j)} = O(k^{-4}) \ \ j = 1,2.\end{aligned} \tag{4.8}$$

According to Lemma 2.2, the estimates (4.8) will hold if we require of the boundary vector-functions to satisfy the following smoothness conditions

$$f^{(j)}(z) \in C^3(\partial\Omega^+), \quad j = 1,2, \quad f_j(z) \in C^4(\partial\Omega^+), \ \ j = 4,5. \tag{4.9}$$

Therefor if the boundary vector-functions satisfy the conditions (4.9), then the vector $U = (u', u'', \theta_1, \theta_2)^\top$ represented by the equalities (4.2) will be a regular solution of Problem $(I)^+$.

The solving of the Problem $(II)^+$ is analogous.

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