BOUNDARY VALUE PROBLEMS OF STATICS OF THE LINEAR THERMOELASTICITY WITH MICROTEMPERATURES FOR A SPACE WITH SPHERICAL CEVITIES

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(Received: 15.03.2012; accepted: 16.11.2012)

Abstract

In this paper, systems of homogeneous equations of statics of the linear thermoelasticity with microtemperatures are solved in terms of four harmonic and three metaharmonic functions. These formulas are very convenient and useful in many particular problems for domains with concrete geometry. Here we demonstrate an application of these formulas to the Dirichlet and Neumann type boundary value problem for a space with spherical cavities. Uniqueness theorems are proved. We construct an explicit solutions in the form of absolutely and uniformly convergent series.

Key words and phrases: Microtemperature, Thermoelasticity, Fourier-Laplace series.

AMS subject classification: 74A15, 74B10, 74F20.

1 Introduction

Mathematical models describing the chiral properties of the linear thermoelasticity with microtemperatures materials have been proposed by Iesan [8], [9] and recently it has been extended to a more general case, when the material points admit micropolar stucture [10].

The Dirichlet, Neumann and mixed type boundary value problems corresponding to this model are well investigated for general domains of arbitrary shape and the uniqueness and existence theorems are proved, and regularity results for solutions are established by potential methods as by variational methods (see [1], [12], [15] and the references therein).

The main goal of this paper is to derive general representation formulas for the displacement, microtemperatures vectors and temperature function by means of harmonic and metaharmonic functions. That is, we can represent solutions to the very complicated coupled system of simultaneous differential equations of thermoelasticity with the help of solutions of a simpler canonical equations.

In particular, here we apply these representation formulas to construct explicit solutions to the Dirichlet and Neumann type boundary value problem for a ball. We represent the solution in the form of Fourier-Laplace series and show their absolute and uniform convergence along withtheir derivatives of the first order if the boundary data satisfy appropriate smoothness conditions. One of the approaches to fulfillment of the boundary conditions is given in A. Ulitko [18], F. Mors and G Feshbah [14], L. Giorgashvili [2], [3], L. Giorgashvili, K. Skhvitaridze [4], L. Giorgashvili, D. Natroshvili [5], L. Giorgashvili, G. Karseladze, G. Sadunishvili [6], L. Giorgashvili, A. Djagmaidze, K. Skhvitaridze [7] and other papers.

2 Basic equations and auxiliary theorems

The system of equations of statics in the linear theory of thermoelasticity with microtemperatures has the following form [9]

$$\mu\Delta u(x) + (\lambda+\mu)\operatorname{grad}\operatorname{div} u(x) - \gamma \operatorname{grad}\theta(x) = 0, \tag{2.1}$$

$$\varkappa_6\Delta w(x) + (\varkappa_5+\varkappa_4)\operatorname{grad}\operatorname{div} w(x) - \varkappa_3 \operatorname{grad}\theta(x) - \varkappa_2 w(x) = 0, \tag{2.2}$$

$$\varkappa\Delta\theta(x) + \varkappa_1 \operatorname{div} w(x) = 0, \tag{2.3}$$

where $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $w=(w_1,w_2,w_3)^\top$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0, $(T_0>0)$, Δ is three-dimensional Laplace operator, $\lambda,\ \mu,\ \gamma,\ \varkappa,\ \varkappa_j,\ \ j=1,2,\dots,6$ are constitutive coefficients, $\top$ is the transposition symbol.

We will suppose that the following assumptions on the constitutive coefficients hold [9]

$$\mu>0,\ \ 3\lambda+2\mu>0,\ \ \varkappa>0,\ \ 3\varkappa_4+\varkappa_5+\varkappa_6>0,\ \ \varkappa_6+\varkappa_5>0,$$

$$\varkappa_6-\varkappa_5>0,\ \ (\varkappa_1+T_0\varkappa_3)^2<4T_0\varkappa\varkappa_2,\ \ \gamma>0.$$

Assume that $U=(u,w,\theta)^\top$. The stress vector, which we denote by the symbol $P(\partial,n)U$, has the form

$$P(\partial,n)U=\Big(P^{(1)}(\partial,n)U',\ \ P^{(2)}(\partial,n)U'',\ \ P^{(3)}(\partial,n)U''\Big)^\top,$$

Where $U'=(u,\theta)^\top$, $U''=(w,\theta)^\top$, n is the unit vector,

$$
\begin{aligned}
&P^{(1)}(\partial,n)U' = T^{(1)}(\partial,n)u - \gamma n\theta, \quad P^{(2)}(\partial,n)U'' = T^{(2)}(\partial,n)w - \varkappa_3 n\theta,\\
&P^{(3)}(\partial,n)U'' = \varkappa\frac{\partial\theta}{\partial n} + (\varkappa_1+\varkappa_3)(n\cdot w),\\
&T^{(1)}(\partial,n)u = 2\mu\frac{\partial u}{\partial n} + \lambda n \operatorname{div} u + \mu[n\times \operatorname{rot} u],\\
&T^{(2)}(\partial,n)w = (\varkappa_6+\varkappa_5)\frac{\partial w}{\partial n} + \varkappa_4 n \operatorname{div} w + \varkappa_5[n\times \operatorname{rot} w].
\end{aligned}
\tag{2.4}
$$

The symbol (.) and $[\times]$ denotes the scalar and vector products of two vectors in $\mathbb{R}^3$.

Definition. The vector $U=(u,w,\theta)^\top$, is assumed to be regular in a domain $\Omega\subset\mathbb{R}^3$ if $U\in C^2(\Omega)\cap C^1(\overline{\Omega})$.

The following theorem is valid.

Theorem 2.1. *For the vector $U=(u,w,\theta)^\top$ to a regular solution of system* (2.1)-(2.3) *in a domain $\Omega\subset\mathbb{R}^3$, it is necessary and sufficient that it be represented in the form*

$$
\begin{aligned}
u(x) &= \operatorname{grad}\Phi_1(x) - a\operatorname{grad} r^2\Big(r\frac{\partial}{\partial r}+1\Big)\Phi_2(x) + \operatorname{rot}\operatorname{rot}(xr^2\Phi_2(x)) +\\
&\quad + \operatorname{rot}(x\Phi_3(x)) + \gamma x\Phi_4(x) + \gamma\operatorname{grad}\Phi_5(x),\\
w(x) &= a_1\operatorname{grad}\Big[(\lambda+\mu)r\frac{\partial}{\partial r} + 3\lambda+5\mu\Big]\Phi_4(x) + a_2\operatorname{grad}\Phi_5(x) +\\
&\quad + \operatorname{rot}\operatorname{rot}(x\Phi_6(x)) + \operatorname{rot}(x\Phi_7(x))\\
\theta(x) &= \Big[(\lambda+\mu)r\frac{\partial}{\partial r} + 3\lambda+5\mu\Big]\Phi_4(x) + (\lambda+2\mu)\lambda_1^2\Phi_5(x),
\end{aligned}
\tag{2.5}
$$

where $\Delta\Phi_j(x)=0,\ \ j=1,2,3,4,\ (\Delta-\lambda_1^2)\Phi_5(x)=0,\ (\Delta-\lambda_2^2)\Phi_j(x)=0,\ \ j=6,7,$

$$
\lambda_1^2 = \frac{\varkappa\varkappa_2-\varkappa_1\varkappa_3}{\varkappa(\varkappa_4+\varkappa_5+\varkappa_6)}>0,\ \ \lambda_2^2=\frac{\varkappa_2}{\varkappa_6}>0,\ \ a=\frac{\mu}{\lambda+2\mu},\ \ a_1=-\frac{\varkappa_3}{\varkappa_2},
$$

$$
a_2 = -\frac{1}{\varkappa_1}\varkappa(\lambda+2\mu)\lambda_1^2,\ \ r=|x|,\ \ x=(x_1,x_2,x_3)^\top,\ \ r\frac{\partial}{\partial r}=x\cdot\operatorname{grad}.
$$

Assume that $r,\ \vartheta,\ \varphi\ (0\le r<+\infty,\ \ 0\le\vartheta\le\pi,\ \ 0\le\varphi<2\pi)$ are the spherical coordinates of the point $x\in R^3$. Denote by Σ_1 the sphere with unit radius and centre at the origin lying in the space R^3.

Let us consider, in the space $L_2(\Sigma_1)$, the following complete system of orthonormal vectors [2], [14], [18]

$$
\begin{aligned}
X_{mk}(\vartheta,\varphi) &= e_r Y_k^{(m)}(\vartheta,\varphi), \quad k \geq 0, \\
Y_{mk}(\vartheta,\varphi) &= \frac{1}{\sqrt{k(k+1)}}\Big(e_\vartheta \frac{\partial}{\partial\vartheta} + \frac{e_\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\Big) Y_k^{(m)}(\vartheta,\varphi), \quad k \geq 1, \\
Z_{mk}(\vartheta,\varphi) &= \frac{1}{\sqrt{k(k+1)}}\Big(\frac{e_\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi} - e_\varphi \frac{\partial}{\partial\vartheta}\Big) Y_k^{(m)}(\vartheta,\varphi), \quad k \geq 1,
\end{aligned} \tag{2.6}
$$

Where $|m| \leq k$, e_r, e_ϑ, e_φ are the orthonormal vectors in R^3,

$$
\begin{aligned}
e_r &= (\cos\varphi\sin\vartheta,\ \sin\varphi\sin\vartheta,\ \cos\vartheta)^\top, \\
e_\vartheta &= (\cos\varphi\cos\vartheta,\ \sin\varphi\sin\vartheta,\ -\sin\vartheta)^\top, \\
e_\varphi &= (-\sin\varphi,\ \cos\varphi,\ 0)^\top,
\end{aligned}
$$

$$
Y_k^{(m)}(\vartheta,\varphi) = \sqrt{\frac{2k+1}{4\pi}\frac{(k-m)!}{(k+m)!}} P_k^{(m)}(\cos\theta)e^{im\varphi},
$$

$P_k^{(m)}(\cos\vartheta)$ is the adjoint Legendre polynomial. Assume that $f^{(j)} = (f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)})^\top$, $j = 1,2$ is a vector-function and the function f_4, represent as the following Fourier-Laplace series

$$
f^{(j)}(\vartheta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{\alpha_{mk}X_{mk}(\vartheta,\varphi) + \sqrt{k(k+1)}\Big[\beta_{mk}^{(j)}Y_{mk}(\vartheta,\varphi) +
+\gamma_{mk}^{(j)}Z_{mk}(\vartheta,\varphi)\Big]\Big\}, \quad j = 1,2, \tag{2.7}
$$

$$
f_4(\vartheta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\alpha_{mk}Y_k^m(\vartheta,\varphi) \tag{2.8}
$$

Where α_{mk}, $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j = 1,2$, are Fourier-Laplace coefficients.

Note that in formula (2.7) and in analogous series below the summation index k varies from 1 to $+\infty$ in the summands containing the vectors $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$.

Let us introduce a few important lemmas [3], [13]

Lemma 2.2. Let $f^{(j)} \in C^l(\Sigma_1)$, $l \geq 1$, then the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j = 1,2$, admit the following estimates

$$
\alpha_{mk}^{(j)} = O(k^{-l}),\ \ \beta_{mk}^{(j)} = O(k^{-l-1}),\ \ \gamma_{mk}^{(j)} = O(k^{-l-1}),\ \ j = 1,2.
$$

Lemma 2.3. If $f_4 \in C^l(\Sigma_1)$, $\ l \geq 1$ then the coefficients α_{mk}, admit the following estimates

$$\alpha_{mk} = O(k^{-l}).$$

Lemma 2.4. The vectors $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ defined by equalities (2.6) admit the following estimates

$$|X_{mk}(\vartheta,\varphi)| \leq \sqrt{\frac{2k+1}{4\pi}}, \quad k \geq 0,$$
$$|Y_{mk}(\vartheta,\varphi)| < \sqrt{\frac{k(k+1)}{2k+1}}, \quad |Z_{mk}(\vartheta,\varphi)| < \sqrt{\frac{k(k+1)}{2k+1}}, \quad k \geq 1.$$

Note that [17]

$$|Y_k^{(m)}(\vartheta,\varphi)| < \sqrt{\frac{2k+1}{4\pi}}, \quad k \geq 0,$$

Assume that $\Omega^+ = B(R) \subset R^3$ is the ball bounded by the spherical surface $\Sigma_R = \partial\Omega$, centered at the origin and having radius R. Further, let $\Omega^- = R^3 \setminus \overline{\Omega^+}$.

The equalities

$$\int\limits_{\partial\Omega} Y_k^{(m)}(\vartheta,\varphi)\,ds = \begin{cases} \sqrt{\pi}R^2, & k=0, \quad m=0, \\ 0, & \textit{in other cases}, \end{cases}$$

are valid too [17].

Theorem 2.5. *The vector $U = (u,w,\theta)^\top$ represented as (2.5) will be uniquely defined, in the domain Ω^+, by the functions $\Phi_j(x)$, $\ j=\overline{1,7}$, if the following conditions are fulfilled*

$$\int\limits_{\Sigma_r} \Phi_j(x) d\Sigma_r = 0, \ \ j=2,3,6,7, \ \ r=|x|\geq R, \tag{2.9}$$

Which means that to the zero value of the vector $U = (u,w,\theta)^\top$ there corresponds the zero value of the vector $(\Phi_1,\ \Phi_2,\ldots,\Phi_7)^\top$ and vice versa.

Proof. Assume that $U = (u,w,\theta)^\top = 0$, $x \in \Omega^-$. Then from (2.5) we obtain that $\Phi_4(x)=0$, $\Phi_5(x)=0$, $x\in\Omega^-$, and

$$\begin{gathered}\operatorname{grad}\Phi_1(x) - a\operatorname{grad} r^2\left(r\frac{\partial}{\partial r}+1\right)\Phi_2(x) + \operatorname{rot}\operatorname{rot}(xr^2\Phi_2(x)) \\ + \operatorname{rot}(x\Phi_3(x)) = 0,\end{gathered} \tag{2.10}$$

$$\operatorname{rot}\operatorname{rot}(x\Phi_6(x)) + \operatorname{rot}(x\Phi_7(x)) = 0. \tag{2.11}$$

From (2.10) we obtain $\Phi_j(x)=0,\quad j=1,2,3,\quad x\in\Omega^-$ [5]. From (2.11) taking into consideration the equalities $x\cdot \operatorname{rot}(x\Phi)=0$ we obtain

$$r^2\Big(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\lambda_2^2\Big)\Phi_j(x)=0,\quad j=6,7,\quad x\in\Omega^-. \tag{2.12}$$

Let us represent the function $\Phi_j(x),\quad j=6,7$ in the domain Ω^- as. Follows

$$\Phi_j(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}h_k(\lambda_2 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}=0,\quad j=6,7,$$

where $A_{mk}^{(j)}$ are unknown constants,

$$h_k(\lambda_2 r)=\sqrt{\frac{R}{r}}\,\frac{K_{k+1/2}(\lambda_2 r)}{K_{k+1/2}(\lambda_2 R)}$$

$K_{k+1/2}(x)$ is the Hankel function of complex (pure imaginary) argument [17].

If we substitute the values of the functions $\Phi_j(x),\quad j=6,7$ into (2.12) and take into consideration the equalities

$$r^2\Big(\frac{d^2}{dr^2}+\frac{2}{r}\,\frac{d}{dr}-\lambda_2^2\Big)h_k(\lambda_2 r)=k(k+1)h_k(\lambda_2 r),$$

we obtain

$$\sum_{k=0}^{\infty}\sum_{m=-k}^{k}k(k+1)h_k(\lambda_2 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}=0,\quad j=6,7,\quad x\in\Omega^-.$$

This implies that $A_{mk}^{(j)}=0,\quad j=6,7,\quad k\geq 1$, i.e.

$$\Phi_j(x)=\frac{1}{\sqrt{4\pi}}h_0(\lambda_2 r)A_{00}^{(j)},\quad j=6,7,\quad x\in\Omega^-.$$

If this value is substituted into (2.9), then we obtain $A_{00}^{(j)}=0,\quad j=6,7$, i.e. $\Phi_j(x)=0,\quad x\in\Omega^-$. If $\Phi_j(x)=0,\quad j=\overline{1,7}$, when $x\in\Omega^-$, then from (2.5) it immediately follows that $U=(u,w,\theta)^\top=0,\quad x\in\Omega^-$.

Definition 2.6. Assume that in the domain Ω^-, the regular vector $U=(u,w,\theta)^\top$ has the property $Z(\Omega^-)$ if it satisfies the following conditions

$$u(x)=O(1),\quad w(x)=(|x|^{-2}),\quad \theta(x)=O(|x|^{-1}),\quad |x|\to\infty \tag{2.13}$$

$$\lim_{r\to\infty}\frac{1}{4\pi r}\int\limits_{\Sigma_r} n(x)\cdot u(x)d\Sigma_r = 0, \tag{2.14}$$

where $n(x)$ is the external normal unit vector passing at a point $x\in\Sigma_r$. Σ_r is a sphere centered at the origin and radius r.

3 Statement of the problem. The uniqueness theorem

Problem. Find, in the domain Ω^-, a vector $U=(u,w,\theta)^\top$ with property $Z(\Omega^-)$ that in this domain satisfies system (2.1)-(2.3) and, on the boundary $\partial\Omega$ the following boundary conditions

$(I)^-$ (the Dirichlet problem)

$$\{u(z)\}^- = f^{(1)}(z),\quad \{w(z)\}^- = f^{(2)}(z),\quad \{\theta(z)\}^- = f_4(z);$$

$(II)^-$ (the Neumann problem)

$$\begin{gathered}\{P^{(1)}(\partial,n)U'(z)\}^- = f^{(1)}(z),\quad \{P^{(2)}(\partial,n)U''(z)\}^- = f^{(2)}(z),\\ \{P^{(3)}(\partial,n)U''(z)\}^- = f_4(z),\end{gathered} \tag{3.1}$$

where the vector $f^{(j)}=(f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)})$, $j=1,2$ and the function f_4, are given on the boundary $\partial\Omega$, $n(z)$ is the external normal unit vector passing at the point $z\in\partial\Omega$ with respect to the domain Ω^+.

A solution of system (2.1)-(2.3) will be sought for in form (2.5), where the functions $\Phi_j(x)$, $j=1,2,\dots,7$, are represented as

$$\begin{aligned}\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\left(\frac{R}{r}\right)^{k+1}Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)},\quad j=1,2,3,4,\\ \Phi_5(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}h_k(\lambda_1 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(5)},\\ \Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}h_k(\lambda_2 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)},\quad j=6,7,\end{aligned} \tag{3.2}$$

where $A^{(j)}_{mk}$, $j=1,2,\dots,7$, are unknown constants

$$h_k(\lambda_j r)=\sqrt{\frac{R}{r}}\,\frac{K_{k+\frac{1}{2}}(\lambda_j r)}{K_{k+\frac{1}{2}}(\lambda_j R)},\quad j=1,2,$$

The substitution of the values of the functions $\Phi_j(x)$, $j=2,3,6,7$, from (3.2), into (2.9) yields $A^{(j)}_{00}=0$, $j=2,3,6,7$.

Let us substitute the expressions (3.2) into (2.5) and apply the following identities [3]

$$\begin{aligned}
\operatorname{grad}[a(r)Y_k^{(m)}(\vartheta,\varphi)] &= \frac{da(r)}{dr}X_{mk}(\vartheta,\varphi)+\frac{\sqrt{k(k+1)}}{r}a(r)Y_{mk}(\vartheta,\varphi),\\
\operatorname{rot}[xa(r)Y_k^{(m)}(\vartheta,\varphi)] &= \sqrt{k(k+1)}a(r)Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\operatorname{rot}[xa(r)Y_k^{(m)}(\vartheta,\varphi)] &= \frac{k(k+1)}{r}a(r)X_{mk}(\vartheta,\varphi)+\\
&+\sqrt{k(k+1)}\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Y_{mk}(\vartheta,\varphi),\\
xa(r)Y_k^{(m)}(\vartheta,\varphi) &= ra(r)X_{mk}(\vartheta,\varphi),
\end{aligned}$$

$a(r)$ is the function of r, we obtain

$$\begin{aligned}
u(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u^{(1)}_{mk}(r)X_{mk}(\vartheta,\varphi)+\\
&+\sqrt{k(k+1)}[v^{(1)}_{mk}(r)Y_{mk}(\vartheta,\varphi)+w^{(1)}_{mk}(r)Z_{mk}(\vartheta,\varphi)]\Big\},\\
w(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u^{(2)}_{mk}(r)X_{mk}(\vartheta,\varphi)+\\
&+\sqrt{k(k+1)}[v^{(2)}_{mk}(r)Y_{mk}(\vartheta,\varphi)+w^{(2)}_{mk}(r)Z_{mk}(\vartheta,\varphi)]\Big\},\\
\theta(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}u_{mk}(r)Y_k^{(m)}(\vartheta,\varphi),
\end{aligned}\tag{3.3}$$

where

$$u^{(1)}_{mk}(r) = -\frac{k+1}{R}\left(\frac{R}{r}\right)^{k+2} A^{(1)}_{mk} + Rk(bk+a+1)\left(\frac{R}{r}\right)^{k} A^{(2)}_{mk} +$$
$$+ \gamma R\left(\frac{R}{r}\right)^{k} A^{(4)}_{mk} + \gamma\frac{d}{dr}h_k(\lambda_1 r)A^{(5)}_{mk}, \quad k \geq 0,$$
$$v^{(1)}_{mk}(r) = \frac{1}{R}\left(\frac{R}{r}\right)^{k+2} A^{(1)}_{mk} - R(bk-2)\left(\frac{R}{r}\right)^{k} A^{(2)}_{mk} +$$
$$+ \gamma\frac{1}{r}h_k(\lambda_1 r)A^{(5)}_{mk}, \quad k \geq 1,$$
$$w^{(1)}_{mk}(r) = \left(\frac{R}{r}\right)^{k+1} A^{(3)}_{mk}, \quad k \geq 1,$$

$$u^{(2)}_{mk}(r) = \alpha_k a_1(k+1)\left(\frac{R}{r}\right)^{k+2} A^{(4)}_{mk} + a_2\frac{d}{dr}h_k(\lambda_1 r)A^{(5)}_{mk} +$$
$$+ \frac{k(k+1)}{r}h_k(\lambda_2 r)A^{(6)}_{mk}, \quad k \geq 0,$$
$$v^{(2)}_{mk}(r) = -\alpha_k a_1\left(\frac{R}{r}\right)^{k+2} A^{(4)}_{mk} + a_2\frac{1}{r}h_k(\lambda_1 r)A^{(5)}_{mk} +$$
$$+ \left(\frac{d}{dr} + \frac{1}{r}\right)h_k(\lambda_2 r)A^{(6)}_{mk}, \quad k \geq 1,$$

$$w^{(2)}_{mk}(r) = h_k(\lambda_2 r)A^{(7)}_{mk}, \quad k \geq 1,$$
$$u_{mk}(r) = -R\alpha_k\left(\frac{R}{r}\right)^{k+1} A^{(4)}_{mk} + (\lambda+2\mu)\lambda_1^2 h_k(\lambda_1 r)A^{(5)}_{mk}, \quad k \geq 0,$$
$$\alpha_k = \frac{1}{R}\left[(\lambda+\mu)(k-2) - 2\mu\right], \quad b = 1 - a = (\lambda+\mu)(\lambda+2\mu)^{-1}.$$

If we substitute the values of the vectors $u(x)$, $w(x)$ and the function $\theta(x)$, into (2.4) and use the equality [3]

$$e_r \cdot X_{mk}(\vartheta,\varphi) = Y^{(m)}_k(\vartheta,\varphi), \quad e_r \cdot Y_{mk}(\vartheta,\varphi) = 0, \quad e_r \cdot Z_{mk}(\vartheta,\varphi) = 0,$$
$$e_r \times X_{mk}(\vartheta,\varphi) = 0, \quad e_r \times Y_{mk}(\vartheta,\varphi) = -Z_{mk}(\vartheta,\varphi),$$
$$e_r \times Z_{mk}(\vartheta,\varphi) = Y_{mk}(\vartheta,\varphi);$$

$$
\begin{aligned}
\operatorname{div}[a(r)X_{mk}(\vartheta,\varphi)] &= \left(\frac{d}{dr}+\frac{2}{r}\right)a(r)Y_k^{(m)}(\vartheta,\varphi),\\
\operatorname{div}[a(r)Y_{mk}(\vartheta,\varphi)] &= -\frac{\sqrt{k(k+1)}}{r}a(r)Y_k^{(m)}(\vartheta,\varphi),\\
\operatorname{div}[a(r)Z_{mk}(\vartheta,\varphi)] &= 0,\\
\operatorname{rot}[a(r)X_{mk}(\vartheta,\varphi)] &= \frac{\sqrt{k(k+1)}}{r}a(r)Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}[a(r)Y_{mk}(\vartheta,\varphi)] &= -\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}[a(r)Z_{mk}(\vartheta,\varphi)] &= \frac{\sqrt{k(k+1)}}{r}a(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Y_{mk}(\vartheta,\varphi),
\end{aligned}
$$

we get

$$
\begin{aligned}
P^{(1)}(\partial,n)U'(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{a_{mk}^{(1)}(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}[b_{mk}^{(1)}(r)Y_{mk}(\vartheta,\varphi)+c_{mk}^{(1)}(r)Z_{mk}(\vartheta,\varphi)]\Big\},\\
P^{(2)}(\partial,n)U''(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{a_{mk}^{(2)}(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}[b_{mk}^{(2)}(r)Y_{mk}(\vartheta,\varphi)+c_{mk}^{(2)}(r)Z_{mk}(\vartheta,\varphi)]\Big\},\\
P^{(3)}(\partial,n)U''(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}a_{mk}(r)Y_k^{(m)}(\vartheta,\varphi),
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
a_{mk}^{(1)}(r) &= \frac{2\mu(k+1)(k+2)}{R^2}\left(\frac{R}{r}\right)^{k+3}A_{mk}^{(1)}\\
&\quad-2\mu k\Big[b(k+1)(k+2)+1-4b\Big]\left(\frac{R}{r}\right)^{k+1}A_{mk}^{(2)}\\
&\quad-\mu\gamma(k+4)\left(\frac{R}{r}\right)^{k+1}A_{mk}^{(4)}+2\mu\gamma\left(\frac{d^2}{dr^2}-\lambda_1^2\right)h_k(\lambda_1 r)A_{mk}^{(5)},\quad k\geq 0,
\end{aligned}
$$

$$b_{mk}^{(1)}(r) = -\frac{2\mu(k+2)}{R^2}\left(\frac{R}{r}\right)^{k+3} A_{mk}^{(1)} + 2\mu\left(bk^2-1\right)\left(\frac{R}{r}\right)^{k+1} A_{mk}^{(2)} +$$
$$+ \mu\gamma\left(\frac{R}{r}\right)^{k+1} A_{mk}^{(4)} + 2\mu\gamma\frac{1}{r}\left(\frac{d}{dr}-\frac{1}{r}\right)h_k(\lambda_1 r)A_{mk}^{(5)}, \quad k \geq 1,$$
$$c_{mk}^{(1)}(r) = -\frac{\mu(k+2)}{R}\left(\frac{R}{r}\right)^{k+2} A_{mk}^{(3)},$$

$$a_{mk}^{(2)}(r) = \frac{\alpha_k}{R}\left[-(\varkappa_6+\varkappa_5)a_1(k+1)(k+2)+\varkappa_3 r^2\right]\left(\frac{R}{r}\right)^{k+3} A_{mk}^{(4)} +$$
$$+ a_2(\varkappa_6+\varkappa_5)\left(\frac{d^2}{dr^2}+a_3\right)h_k(\lambda_1 r)A_{mk}^{(5)} +$$
$$+ (\varkappa_6+\varkappa_5)k(k+1)\frac{1}{r}\left(\frac{d}{dr}-\frac{1}{r}\right)h_k(\lambda_2 r)A_{mk}^{(6)}, \quad k \geq 0,$$
$$b_{mk}^{(2)}(r) = \frac{\varkappa_6+\varkappa_5}{R}a_1\alpha_k(k+2)\left(\frac{R}{r}\right)^{k+3} A_{mk}^{(4)} +$$
$$+ a_2(\varkappa_6+\varkappa_5)\frac{1}{r}\left(\frac{d}{dr}-\frac{1}{r}\right)h_k(\lambda_1 r)A_{mk}^{(5)} +$$
$$+ \left[(\varkappa_6+\varkappa_5)\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-\lambda_2^2\varkappa_5\right]h_k(\lambda_2 r)A_{mk}^{(6)}, \quad k \geq 0,$$
$$c_{mk}^{(2)}(r) = \left(\varkappa_6\frac{d}{dr}-\frac{\varkappa_5}{r}\right)h_k(\lambda_1 r)A_{mk}^{(7)}, \quad k \geq 1,$$

$$a_{mk}(r) = \alpha_k(k+1)[\varkappa+(\varkappa_1+\varkappa_3)a_1]\left(\frac{R}{r}\right)^{k+2} A_{mk}^{(4)} + a_2\varkappa_3\frac{d}{dr}h_k(\lambda_1 r)A_{mk}^{(5)} +$$
$$+ (\varkappa_1+\varkappa_3)\frac{k(k+1)}{r}h_k(\lambda_2 r)A_{mk}^{(6)} \quad k \geq 0,$$
$$a_3 = \frac{\lambda_1^2(a_2\varkappa_4-(\lambda+2\mu)\varkappa_3)}{a_2(\varkappa_6+\varkappa_5)}.$$

From the limit equality (2.14) and (2.13) it follows that

$$\lim_{r\to\infty}\int_{\Sigma_r} u(x)\cdot P^{(1)}(\partial,n)U'(x)\,dx = 0,$$
$$\lim_{r\to\infty}\int_{\Sigma_r} w(x)\cdot P^{(2)}(\partial,n)U''(x)\,dx = 0, \tag{3.5}$$
$$\lim_{r\to\infty}\int_{\Sigma_r} \theta(x)P^{(3)}(\partial,n)U''(x)\,dx = 0.$$

Theorem 3.1. *The Homogeneous Problem $(I)_0^-$ and $(II)_0^-$ where $(f^{(j)} = 0,\ j = 1,2,\ f_4 = 0)$, have only a trivial solution in the class of regular functions.*

Proof. Let Σ_r be the sphere with center at the origin and radius r $(r = |x| > R)$. Denote by Ω_r the domain bounded by the spheres $\partial\Omega = \Sigma_R$ and Σ_r. For the domain Ω_r we write Green's formulas

$$-\int\limits_{\partial\Omega} [u(z)\cdot P^{(1)}(\partial,n)U'(z)]^- \, ds + \int\limits_{\Sigma_r} u(x)\cdot P^{(1)}(\partial,n)U'(x)\, d\Sigma_r - \\ -\int\limits_{\Omega_r} [E^{(1)}(u,u) - \gamma\theta(x)\operatorname{div} u(x)]dx = 0, \quad (3.6)$$

$$-\int\limits_{\partial\Omega} [w(z)\cdot P^{(2)}(\partial,n)U''(z)]^- \, ds + \int\limits_{\Sigma_r} w(x)\cdot P^{(2)}(\partial,n)U''(x)\, d\Sigma_r - \\ -\int\limits_{\Omega_r} \Big[E^{(2)}(w,w) - \varkappa_3\theta(x)\operatorname{div} w(x) + \varkappa_2 w^2(x)\Big]\, dx = 0, \quad (3.7)$$

$$-\int\limits_{\partial\Omega} [\theta(z)\cdot P^{(3)}(\partial,n)U''(z)]^- \, ds + \int\limits_{\Sigma_r} \theta(x)\cdot P^{(3)}(\partial,n)U''(z)\, d\Sigma_r - \\ -\int\limits_{\Omega_r} \big[\varkappa \operatorname{grad}^2\theta(x) + (\varkappa_1+\varkappa_3)w(x)\cdot \operatorname{grad}\theta(x) + \varkappa_3\theta(x)\operatorname{div} w(x)\big]\, dx = 0, \quad (3.8)$$

where $U' = (u,\theta)^\top$, $U'' = (w,\theta)^\top$, the vectors $P^{(1)}(\partial,n)U'$, $P^{(2)}(\partial,n)U''$ and function $P^{(3)}(\partial,n)U''$ have form (2.4), and [11], [**?**]

$$E^{(1)}(u,u) = \frac{3\lambda+2\mu}{3}(\operatorname{div} u)^2 + \frac{\mu}{2}\sum_{k\neq j=1}^{3}\Big(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\Big)^2 \\ + \frac{\mu}{3}\sum_{k,j=1}^{3}\Big(\frac{\partial u_k}{\partial x_k} - \frac{\partial u_j}{\partial x_j}\Big)^2, \\ E^{(2)}(w,w) = \frac{3\varkappa_4+\varkappa_5+\varkappa_6}{3}(\operatorname{div} w)^2 + \frac{\varkappa_6-\varkappa_5}{2}(\operatorname{rot} w)^2 + \\ + \frac{\varkappa_5+\varkappa_6}{4}\sum_{k\neq j=1}^{3}\Big(\frac{\partial w_k}{\partial x_j} + \frac{\partial w_j}{\partial x_k}\Big)^2 + \frac{\varkappa_5+\varkappa_6}{6}\sum_{k,j=1}^{3}\Big(\frac{\partial w_k}{\partial x_k} - \frac{\partial w_j}{\partial x_j}\Big)^2. \quad (3.9)$$

Passing to the limit on both sides of equalitys (3.6)-(3.8) as $r \to +\infty$ and taking into consideration the boundary conditions of the homogeneous problems $(I)_0^-$ and $(II)_0^-$ as well as the asymptotic representations (3.5), we obtain

$$\int\limits_{\Omega^-} [E^{(1)}(u,u) - \gamma\theta(x)\,\mathrm{div}\,u(x)]dx = 0, \tag{3.10}$$

$$\int\limits_{\Omega^-} [E^{(2)}(w,w) - \varkappa_3\theta(x)\,\mathrm{div}\,w(x) + \varkappa_2 w^2(x)]dx = 0, \tag{3.11}$$

$$\begin{gathered}\int\limits_{\Omega^-} [\varkappa grad^2\theta(x) + (\varkappa_1+\varkappa_3)w(x)\cdot\mathrm{grad}\,\theta(x)+ \\ +\varkappa_3\theta(x)\,\mathrm{div}\,w(x)]dx = 0,\end{gathered} \tag{3.12}$$

From the equalities (3.11) and (3.12) , we obtained

$$\begin{gathered}\int\limits_{\Omega^+} \Big\{E^{(2)}(w,w) + \frac{4\varkappa\varkappa_2 - (\varkappa_1+\varkappa_3)^2}{4\varkappa}w^2(x) \\ +\frac{1}{4\varkappa}\Big[(\varkappa_1+\varkappa_3)w(x) + 2\varkappa\,\mathrm{grad}\,\theta(x)\Big]^2\Big\}dx = 0.\end{gathered} \tag{3.13}$$

Since $E^{(2)}(w,w) \geq 0$, $4\varkappa\varkappa_2 - (\varkappa_1+\varkappa_3)^2 > 0$, $\varkappa > 0$, from (3.13) we obtained that $w(x) = 0$, $\theta(x) = c' = const$, $x \in \Omega^-$.

From the equalities $\{\theta(z)\}^- = 0$,, and $\{P^{(2)}(\partial,n)U''(z)\}^- = 0$, we obtained $c' = 0$,, i.e. $\theta(x) = 0$, $x \in \Omega^-$.

Substituting the value of the function $\theta(x) = 0$, $x \in \Omega$, into (3.10), we obtain

$$\int\limits_{\Omega^-} E^{(1)}(u,u)dx = 0. \tag{3.14}$$

Taking into account that $3\lambda + 2\mu > 0$, $\mu > 0$, from (3.9) it follows that $E^{(1)}(u,u) \geq 0$. By virtue of this fact, (3.14) implies

$$E^{(1)}(u,u) = 0, \quad x \in \Omega^-.$$

A solution of this equation has the form

$$u(x) = [c \times x] + d, \quad x \in \Omega^-,$$

where c and d are any three-dimensional constant vectors. By asymptotics (2.13)-(2.14) we have $c = d = 0$, i.e. $u(x) = 0$, $x \in \Omega^-$.

4 Solution of the Dirichlet and Neumann boundary value problems

Let us first investigate the Neumann problem.

Assume that the vectors $f^{(j)}(\vartheta,\varphi)$, $j=1,2$ and the function $f_4(\vartheta,\varphi)$, satisfy the sufficient smoothness conditions by means of which they can be represented in form (2.7)-(2.8).

Passing to the limit on both sides of (3.4) as $x\to z\in\partial\Omega$ and using both the Neumann boundary conditions (3.1) and equalities (2.7)-(2.8), for the sought constants $A^{(j)}_{mk}$, $j=\overline{1,7}$, we obtain the following system of linear algebraic equations:

a) $k=0$,

$$\begin{aligned}
&\frac{4\mu}{R^2}A^{(1)}_{00}-4\mu\gamma A^{(4)}_{00}+2\mu\gamma\left(\frac{d^2}{dR^2}-\lambda_1^2\right)h_0(\lambda_1 R)A^{(5)}_{00}=\alpha^{(1)}_{00},\\
&\frac{2(\lambda+2\mu)}{R^2}\left[2(\varkappa_6+\varkappa_5)a_1-\varkappa_3R^2\right]A^{(4)}_{00}\\
&+a_2(\varkappa_6+\varkappa_5)\left(\frac{d^2}{dR^2}+a_3\right)h_0(\lambda_1 R)A^{(5)}_{00}=\alpha^{(2)}_{00},\\
&-\frac{2(\lambda+2\mu)}{R}(\varkappa+(\varkappa_1+\varkappa_3)a_1)A^{(4)}_{00}+a_2\varkappa_3\frac{d}{dR}h_0(\lambda_1 R)A^{(5)}_{00}=\alpha_{00};
\end{aligned}\tag{4.1}$$

b) $k\geq 1$,

$$\begin{aligned}
&a^{(j)}_{mk}(R)=\alpha^{(j)}_{mk},\quad b^{(j)}_{mk}(R)=\beta^{(j)}_{mk},\\
&c^{(j)}_{mk}(R)=\gamma^{(j)}_{mk},\quad a_{mk}(R)=\alpha_{mk},\quad j=1,2.
\end{aligned}\tag{4.2}$$

It is assumed here that

$$\frac{d}{dR}h_k(\lambda_j R)=\lim_{r\to R}\frac{d}{dr}g_k(\lambda_j r),\quad j=1,2.$$

Systems (4.1)–(4.2) are compatible by Theorem 2.5 and Theorem 3.1.If the solutions of these system are substituted into (3.3), then we obtain a formal solution of the Neumann problem. We need to show that series (3.3)–(3.4) are absolutely and uniformly convergent in the domain $\overline{\Omega^-}$.

The following asymtotic representations are true as $k\to+\infty$ [16]

$$h_k(\lambda_j r)\sim\left(\frac{R}{r}\right)^{k+1},\quad h'_k(\lambda_j r)\sim-\frac{k}{r}\left(\frac{R}{r}\right)^{k+1},\quad j=1,2.\tag{4.3}$$

If $x\in\Omega^-$ $(r>R)$, then by the asimptotic representation (4.3) the above-mentioned series are convergent.

If $x \in \partial\Omega \quad (r = R)$, then by Lemma 2.4 and asymptotic representation (4.3), series (3.3)–(3.4) are absolutely and uniformly convergent provided that the majorant series

$$\sum_{k=k_0}^{\infty} k^{3/2} \sum_{j=1}^{2} \left(|\alpha_{mk}^{(j)}| + k|\beta_{mk}^{(j)}| + k|\gamma_{mk}^{(j)}| + |\alpha_{mk}| \right), \tag{4.4}$$

is convergent. The series (4.4) will be convergent if these coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, α_{mk}, $j = 1, 2$, admit the following estimates

$$\alpha_{mk}^{(j)} = O(k^{-3}), \quad \beta_{mk}^{(j)} = O(k^{-4}), \quad \gamma_{mk}^{(j)} = O(k^{-4}), \quad \alpha_{mk} = O(k^{-3}) \;\; j = 1, 2. \tag{4.5}$$

According to Lemma 2.2 and Lemma 2.3, the estimates (4.5) will hold if we require of the boundary vector-functions to satisfy the following smoothness conditions

$$f^{(j)}(z) \in C^3(\partial\Omega), \quad j = 1, 2, \quad f_4(z) \in C^3(\partial\Omega). \tag{4.6}$$

Therefor if the boundary vector-functions satisfy the conditions (4.6), then the vector $U = (u, w, \theta)^\top$ represented by the equalities (3.3) will be a regular solution of Problem $(II)^-$.

The solving of the Problem $(I)^-$ is analogous.

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