STATEMENT AND EFFECTIVE SOLUTION OF SOME NONCLASSICAL THREE-DIMENSIONAL PROBLEMS OF THERMOELASTICITY

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Abstract

Some special nonclassical problems of thermoelasticity are formulated and solved in the generalized cylindrical coordinate system (Cartesian, circular cylindrical, cylindrical elliptic, cylindrical parabolic and cylindrical bipolar coordinate systems). The nonclassical formulation of a thermoelastic problem means that, given zero stresses on the upper and lower plane boundaries of an elastic body and a temperature disturbance on the lower boundary, it is required to choose a temperature value on the upper boundary such that normal displacements to the plane boundaries would obey certain conditions within the body.

Key words and phrases: Nonclassical problems of thermoelasticity, Method of separation of variables, Analytical solutions.

AMS subject classification: 74B05, 74F05.

1 Introduction

In the elasticity theory there are quite a number of problems which we call nonclassical because the boundary conditions on a part of the boundary surface are either overdetermined or underdetermined [1], [2], [3], or the conditions on the boundary are related to the conditions within the body (the so-called nonlocal problems) [4], [5], [6], [7], [8], [9], [10].

In the present paper, we formulate and solve by the method of separation of variables some nonclassical three-dimensional problems of thermoelasticity.

The thermoelastic equilibrium of a homogeneous isotropic body occupying the domain $\Omega = \{\xi_0 < \xi < \xi_1,\, \varphi_0 < \varphi < \varphi_1,\, 0 < z < z_1\}$ is considered in the generalized cylindrical system of coordinates ξ, φ, z, where ξ and φ are orthogonal coordinates on the surface, and z is a linear coordinate. Homogeneous conditions of special type are given on the lateral surfaces of the domain Ω; the planes $z = 0$ and $z = z_1$ are assumed to be stress-free, and a thermal disturbance is given on the plane $z = 0$.

The problem consists in defining, on the plane $z = z_1$, a temperature such that a linear combination of normal displacements on the inner planes $z = z_2$ and $z = z_3$ of the body would take an a priori given value. A particular case of this problem is the problem of finding, on the surface $z = z_1$, a temperature disturbance that enables us to obtain the desired value of the function of normal displacements of points of some inner plane $z = z_2$. After solving the stated problems, it is quite an easy matter to define the stress-deformed state of the considered body.

The concrete examples of nonclassical problems of the above-mentioned type are considered in the Cartesian and circular cylindrical coordinate systems.

2 Statement of the problems

Assume that we are given a homogeneous isotropic elastic body occupying the domain $\Omega = \{\xi_0 < \xi < \xi_1,\ \varphi_0 < \varphi < \varphi_1,\ 0 < z < z_1\}$, where ξ, φ, z are generalized cylindrical coordinates (ξ, φ are orthogonal coordinates on the plane and z is a linear coordinate), ξ_0, ξ_1, φ_0, φ_1, z_1 are constants (see Fig. 1 a)). The considered elastic body is in the state of stationary thermoelastic equilibrium. On the lateral surfaces of the domain Ω, the following boundary conditions are given [11]:

$$\left.\begin{array}{ll} \text{For } \xi = \xi_j: & \text{a) } T = 0,\ \ D = 0,\ \ v = 0,\ \ w = 0 \\ & \qquad \text{or} \\ & \text{b) } \partial_\xi T = 0,\ \ u = 0,\ \ K = 0,\ \ \sigma_{\xi z} = 0; \end{array}\right\} \qquad (1)$$

$$\left.\begin{array}{ll} \text{For } \varphi = \varphi_j: & \text{a) } T = 0,\ \ D = 0,\ \ u = 0,\ \ w = 0 \\ & \qquad \text{or} \\ & \text{b) } \partial_\varphi T = 0,\ \ v = 0,\ \ K = 0,\ \ \sigma_{\varphi z} = 0. \end{array}\right\} \qquad (2)$$

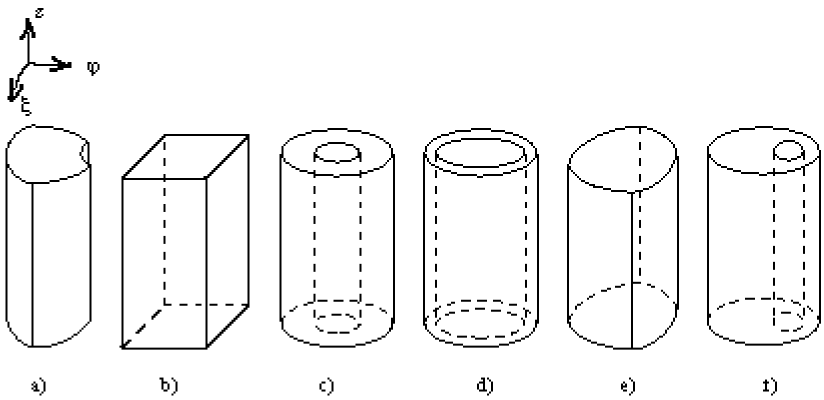

Fig. 2

Fig.1. Elastic cylindrical bodies in the generalized cylindrical (a), Cartesian (b), circular cylindrical (c), elliptical cylindrical (d), parabolic cylindrical (e) and bipolar cylindrical (f) coordinate systems.

On the upper and lower planes of the domain Ω, we have the following conditions:

$$\text{For } \; z = z_j: \qquad \sigma_{z\xi} = 0, \;\; \sigma_{z\varphi} = 0, \;\; \sigma_{zz} = 0; \tag{3}$$

$$\left.\begin{array}{l} \text{For } \; z = 0: \;\; \text{a)} \;\; T = \tau(\xi, \varphi) \quad \text{or} \quad \text{b)} \;\; \partial_z T = \widetilde{\tau}(\xi, \varphi) \\ \qquad\qquad\qquad \text{or} \\ \qquad\qquad\quad \text{c)} \;\; \partial_z T + \Theta T = \widetilde{\tau}(\xi, \varphi), \end{array}\right\} \tag{4}$$

where $j = 0, 1$ and $z_0 = 0$; u, v, w are the components of the displacement vector $\vec{U}$ along the tangents to the coordinate lines ξ, φ, z, respectively; T is a change of the elastic body temperature that satisfies the equation

$$\Delta T = 0, \tag{5}$$

where $\Delta \equiv \frac{1}{h^2}(\partial_{\xi\xi} + \partial_{\varphi\varphi}) + \partial_{zz}$; $\partial_\xi \equiv \frac{\partial}{\partial \xi}$; $\partial_\varphi \equiv \frac{\partial}{\partial \varphi}$, $\partial_z \equiv \frac{\partial}{\partial z}$; $h = h_\xi = h_\varphi$ are the Lamé coefficients of the system ξ, φ, z. In the case of circular cylindrical coordinates r, φ, z it is assumed that $h = r$ and the operation ∂_ξ is replaced by the operation $r\partial_r$;

$$D = \frac{E}{(1-2\nu)(1+\nu)} \left\{ \frac{1-\nu}{h^2} \left[\partial_\xi(hu) + \partial_\varphi(hv) \right] + \nu \partial_z w \right\} - kT;$$
$$K = \frac{E}{2(1+\nu)} \left[\partial_\xi(hv) - \partial_\varphi(hu) \right];$$

E is Young's modulus, ν is Poisson's ratio, k is the thermal expansion coefficient; $\sigma_{\xi\xi}$, $\sigma_{\varphi\varphi}$, σ_{zz} are normal stresses; $\sigma_{\xi\varphi}=\sigma_{\varphi\xi}$, $\sigma_{\xi z}=\sigma_{z\xi}$, $\sigma_{\varphi z}=\sigma_{z\varphi}$ are shear stresses; Θ is a given constant; $\tau(\xi,\varphi)$, $\widetilde{\tau}(\xi,\varphi)$ are given analytical functions of the variables ξ and φ in the closed domain $\overline{\omega}=\{\xi_0\leq\xi\leq\xi_1,\,\varphi_0\leq\varphi\leq\varphi_1\}$.

Note that the boundary conditions (1a) and (2a) can be called to be of antisymmetry type for the following reason: if the surfaces $\xi=\xi_j$ and $\varphi=\varphi_j$ are planes, then conditions (1a) and (2a) transform to antisymmetry conditions. Analogously, conditions (1b) and (2b) can be called to be of symmetry-type [12].

Our aim is to define a change of the temperature T on the plane $z=z_1$ for which the following condition is fulfilled:

$$w(\xi,\varphi,z_2)-aw(\xi,\varphi,z_3)=g(\xi,\varphi), \tag{6}$$

where z_2 and z_3 are constants; without loss of generality it is assumed that $0<z_3<z_2<z_1$; a is a constant; $g(\xi,\varphi)$ is a given analytical function of the variables ξ and φ in the domain $\overline{\omega}$.

If in condition (6) we put $a=0$, then the problem will consist in satisfying a given value of the function of normal displacements $w(\xi,\varphi,z_2)$ of points of the plane $z=z_2$. If in condition (6) we put $a=1$ ($a=-1$) and $g(\xi,\varphi)=0$, then it will imply that the normal displacements of the planes $z=z_2$ and $z=z_3$ are equal (are equal with respect to a modulus and opposite in the direction) and so on. The validity of these conditions is obtained by an appropriate choice of a temperature value on the upper boundary of the considered body.

As is well known, in the absence of mass forces, the thermoelastic equilibrium of an isotropic homogeneous elastic body is described by means of the differential equation

$$\operatorname{grad}\left[2(1-\nu)\operatorname{div}\vec{U}-2k(1+\nu)T\right]-(1-2\nu)\operatorname{rot}\operatorname{rot}\vec{U}=0. \tag{7}$$

Stresses and displacements are related by the well-known Duhamel–Neumann formula (see e.g. [11]).

3 Solution of the stated problems

The construction of solutions of the stated problems is based on the method of separation of variables, taking into account the results of [11]. In that paper, a general solution of the system of equilibrium equations (7) is represented through three arbitrary harmonic functions and the function $\widetilde{T}$ which is also a solution of the Laplace equation and is related to the change in the temperature T by

$$T=\partial_{zz}\widetilde{T}. \tag{8}$$

It is shown in [6] that when the boundary conditions (1)–(3) are fullfilled, the above-mentioned harmonic functions, except for $\widetilde{T}$, are equal to zero, while the displacements and stresses are expressed through the function $\widetilde{T}$ as follows

$$w = k(1+\nu)\partial_z \widetilde{T}; \tag{9}$$

$$hu = -k(1+\nu)\partial_\xi \widetilde{T}, \quad hv = -k(1+\nu)\partial_\varphi \widetilde{T}; \tag{10}$$

$$\sigma_{zz} = \sigma_{z\varphi} = \sigma_{z\xi} = 0; \tag{11}$$

$$\left.\begin{aligned} \sigma_{\xi\xi} &= -\frac{E}{1+\nu}\left(\frac{1}{h}\,\partial_\varphi v + \frac{1}{h^2}\,\partial_\xi hu\right), \\ \sigma_{\varphi\varphi} &= -\frac{E}{1+\nu}\left(\frac{1}{h}\,\partial_\xi u + \frac{1}{h^2}\,\partial_\varphi hv\right), \\ \sigma_{\xi\varphi} &= \frac{E}{2(1+\nu)}\left[\partial_\xi\left(\frac{v}{h}\right) + \partial_\varphi\left(\frac{u}{h}\right)\right]. \end{aligned}\right\} \tag{12}$$

Problems analogous to the ones studied here are considered in classical terms in [13].

Since the solutions of all the stated problems are constructed here by one and the same method, we give a detailed description of the solution of only one of the problems, namely of problem (7), (5), (1a), (2a), (3), (4a), (6).

Using the method of separation of variables and taking into account relation (8) and the boundary conditions (1a), (2a), we can write the harmonic function $\widetilde{T}$ in the form

$$\widetilde{T} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{\gamma_{mn}^2}\left(A_{mn}e^{-\gamma_{mn}z} + B_{mn}e^{\gamma_{mn}z}\right)\psi_{mn}(\xi,\varphi), \tag{13}$$

where $\gamma_{mn} > 0$ are the constants defined in the process of separation of variables; A_{mn}, B_{mn} are the variables which depend on m and n; $\psi_{mn}(\xi,\varphi)$ is a non-trivial solution of the following Sturm–Liuville problem

$$\Delta_2\psi_{mn} + \gamma_{mn}^2\psi_{mn} = 0; \quad \Delta_2 \equiv \frac{1}{h^2}(\partial_{\xi\xi} + \partial_{\varphi\varphi});$$
$$\text{for } \xi = \xi_j: \ \ \psi_{mn} = 0; \quad \text{for } \varphi = \varphi_j: \ \ \psi_{mn} = 0; \ \ j = 0, 1.$$

ψ_{mn} is the product of trigonometric functions in the Cartesian system; ψ_{mn} is the product of a trigonometric function and a Bessel function in the case of circular cylindrical coordinates; ψ_{mn} is the product of Mathieu functions in the cylindrical-elliptical system of coordinates, and so on.

The given analytical functions $\tau(\xi,\varphi)$ and $g(\xi,\varphi)$ satisfy the following consistency conditions

$$\tau(\xi_0,\varphi)=\tau(\xi_1,\varphi)=0,\quad g(\xi_0,\varphi)=g(\xi_1,\varphi)=0,$$
$$\tau(\xi,\varphi_0)=\tau(\xi,\varphi_1)=0,\quad g(\xi,\varphi_0)=g(\xi,\varphi_1)=0.$$

We expand these functions into Fourier series in terms of basic functions $\psi_{mn}(\xi,\varphi)$

$$\tau(\xi,\varphi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\tau_{mn}\psi_{mn}(\xi,\varphi), \tag{14}$$

$$g(\xi,\varphi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}g_{mn}\psi_{mn}(\xi,\varphi). \tag{15}$$

Let us assume that the Fourier coefficients τ_{mn} and g_{mn} satisfy the conditions

$$\tau_{mn}=O\left(\frac{1}{e^{\gamma_{mn}(z_1-z_3)}}\right),\quad g_{mn}=O\left(\frac{1}{e^{\gamma_{mn}z_1}}\right). \tag{16}$$

Conditions (16) guarantee the convergence of the series in the domain $\overline{\Omega}$ (naturally, for $z_3<z<z_2$, too).

From (13) with formula (8) taken into account we obtain the following expression for a change in the temperature T

$$T=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(A_{mn}e^{-\gamma_{mn}z}+B_{mn}e^{\gamma_{mn}z}\right)\psi_{mn}(\xi,\varphi) \tag{17}$$

Substituting expression (13) in formula (9), for the displacement w we have

$$w=k(1+\nu)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\gamma_{mn}}\left(-A_{mn}e^{-\gamma_{mn}z}+B_{mn}e^{\gamma_{mn}z}\right)\psi_{mn}(\xi,\varphi). \tag{18}$$

Substituting series (17) and (14) in the boundary condition (4a), and series (18) and (15) in condition (6) and equating the coefficients of the identical basic functions, for the sought coefficients A_{mn} and B_{mn} we obtain the system of two linear algebraic equations with two unknowns, which will be investigated below.

4 Discussion of the obtained results

The system of equations mentioned at the end of the preceding section has the form

$$\begin{cases} A_{mn}+B_{mn}=\tau_{mn}, \\ -\left(e^{-\gamma_{mn}z_2}-ae^{-\gamma_{mn}z_3}\right)A_{mn}+\left(e^{\gamma_{mn}z_2}-ae^{\gamma_{mn}z_3}\right)B_{mn} \\ \qquad =\dfrac{\gamma_{mn}}{k(1+\nu)}\,g_{mn}\,. \end{cases} \tag{19}$$

Assuming that the determinant of the obtained system is different from zero, we can impose the following conditions on the coefficient a

$$a \neq \frac{\operatorname{ch}(\gamma_{mn} z_2)}{\operatorname{ch}(\gamma_{mn} z_3)}, \quad m = 0, 1, \dots; \quad n = 0, 1, \dots. \tag{20}$$

If conditions (20) are fulfilled, then the sought coefficients A_{mn} and B_{mn} are uniquely defined from (19) as follows:

$$\begin{aligned} A_{mn} = {} & \frac{1}{e^{\gamma_{mn} z_2} + e^{-\gamma_{mn} z_2} - a\big(e^{\gamma_{mn} z_3} + e^{-\gamma_{mn} z_3}\big)} \\ & \times \left\{ \big(e^{\gamma_{mn} z_2} - a e^{\gamma_{mn} z_3}\big)\, \tau_{mn} - \frac{\gamma_{mn}}{k(1+\nu)}\, g_{mn} \right\}, \end{aligned} \tag{21}$$

$$\begin{aligned} B_{mn} = {} & \frac{1}{e^{\gamma_{mn} z_2} + e^{-\gamma_{mn} z_2} - a\big(e^{\gamma_{mn} z_3} + e^{-\gamma_{mn} z_3}\big)} \\ & \times \left\{ \big(e^{-\gamma_{mn} z_2} - a e^{-\gamma_{mn} z_3}\big)\, \tau_{mn} + \frac{\gamma_{mn}}{k(1+\nu)}\, g_{mn} \right\}. \end{aligned} \tag{22}$$

The substitution of (21) and (22) in formula (17) gives the following expression for a change in the temperature T

$$\begin{aligned} T = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} & \frac{1}{\operatorname{ch}(\gamma_{mn} z_2) - a \operatorname{ch}(\gamma_{mn} z_3)} \\ & \times \Big\{ \big[\operatorname{ch}(\gamma_{mn}(z - z_2)) - a \operatorname{ch}(\gamma_{mn}(z - z_3))\big]\, \tau_{mn} \\ & \qquad + \frac{\gamma_{mn}}{k(1+\nu)} \operatorname{sh}(\gamma_{mn} z) g_{mn} \Big\} \psi_{mn}(\xi, \varphi). \end{aligned} \tag{23}$$

It can be easily verified that if the coefficients τ_{mn} and g_{mn} satisfy conditions (16), then series (23) converges absolutely and uniformly in the domain $\overline{\Omega}$. Moreover, the obtained function T will be the analytical function of the variables ξ, φ, z in the domain $\overline{\Omega}$.

After replacing z in formula (23) by z_1, we obtain the desired value of a temperature change on the boundary $z = z_1$. This value is a unique solution of the considered problem. It can be easily proved that the obtained solution depends continuously on the initial data provided that the Fourier coefficients of the functions $\tau^*(\xi, \varphi)$, $g^*(\xi, \varphi)$ which are some disturbances of the functions $\tau(\xi, \varphi)$, $g(\xi, \varphi)$ also satisfy conditions (16).

When the function $\widetilde{T}$ is known, by virtue of formulas (9)–(12) it is easy to find displacements and stresses in the considered body.

Thus, if conditions (16) and (20) are fulfilled, then the problem we have formulated and solved is well-posed.

Let us now assume that conditions (20) are not fulfilled for some m_k and n_l (m_k, n_l are some values of the indices m, n), and only for them. Then

1) if the conditions

$$\tau_{m_k n_l} + \frac{\gamma_{m_k n_l}}{k(1+\nu)(e^{-\gamma_{m_k n_l} z_2} - a e^{-\gamma_{m_k n_l} z_3})}\, g_{m_k n_l} = 0 \tag{24}$$

are fulfilled, then the stated problem has infinitely many analytical solutions in $\overline{\Omega}$;

2) if condition (24) is not fulfilled for at least one pair of values m_k, n_l, then the stated problem has no solution.

Note that conditions (20) are satisfied for $a \leq 1$ and thereby for $a = 0$, $a = 1$, $a = -1$, too.

If $a = 0$ in conditions (6), this means that the normal displacement value is assumed to be given on the inner surface of the body $z = z_2$.

If $a = 1$ ($a = -1$), this means that the difference (the sum) of normal displacements of the corresponding points of the planes $z = z_2$ and $z = z_3$ is assumed to be given. Also, if $g(\xi, \varphi) = 0$, then normal displacements are equal (for $a = -1$, normal displacements are equal with respect to a modulus and opposite in the direction.)

Note that we could not give any geometrical, physical or any other interpretation of condition (20) perhaps because of the non-classical formulation of the problem, which is not so obvious as the classical one.

In the next two sections, some concrete examples are given in the Cartesian and circular cylindrical coordinate systems.

5 Solution of the problem for a rectangular parallelepiped

Before we proceed to considering an elastic parallelepiped, it must be said that the problems stated and solved in Sections 1 and 2 represent quite a large class of nonclassical problems since the shapes of the elastic bodies considered in these problems are absolutely different. In particular, an elastic body can be a rectangular parallelepiped (Fig.1 b), a circular cylinder (Fig.1 c), an elliptical cylinder (Fig.1 d), a parabolic cylinder (Fig.1 e) or a body bounded by the coordinate surfaces of bipolar cylindrical coordinates (Fig.1 f), and so on.

In the Cartesian system of coordinates x, y, z, for the rectangular parallelepiped $\Omega = \{0 < x < x_1,\, 0 < y < y_1,\, 0 < z < z_1\}$, which is in the state of thermal equilibrium, conditions (1)–(4) take the form

$$\left.\begin{array}{l} \text{For } x = x_j : \quad \text{a) } T = 0, \ \ \sigma_{xx} = 0, \ \ v = 0, \ \ w = 0 \\ \qquad\qquad\qquad\qquad \text{are antisymmetry conditions} \\ \qquad\qquad\qquad \text{or} \\ \qquad\qquad\quad \text{b) } \partial_x T = 0, \ \ u = 0, \ \ \sigma_{xy} = 0, \ \ \sigma_{xz} = 0 \\ \qquad\qquad\qquad\qquad \text{are symmetry conditions.} \end{array}\right\} \tag{25}$$

$$\left.\begin{array}{l} \text{For } y = y_j : \quad \text{a) } T = 0, \ \ \sigma_{yy} = 0, \ \ u = 0, \ \ w = 0 \\ \qquad\qquad\qquad\qquad \text{are antisymmetry conditions} \\ \qquad\qquad\qquad \text{or} \\ \qquad\qquad\quad \text{b) } \partial_y T = 0, \ \ v = 0, \ \ \sigma_{xy} = 0, \ \ \sigma_{yz} = 0 \\ \qquad\qquad\qquad\qquad \text{are symmetry conditions.} \end{array}\right\} \tag{26}$$

$$\text{For } z = z_j : \qquad \sigma_{zx} = 0, \ \ \sigma_{zy} = 0, \ \ \sigma_{zz} = 0. \tag{27}$$

$$\left.\begin{array}{l} \text{For } z = 0 : \quad \text{a) } T = \tau(x,y) \ \text{ or } \ \text{b) } \partial_z T = \widetilde{\tau}(x,y) \\ \qquad\qquad\qquad \text{or} \\ \qquad\qquad \text{c) } \partial_z T + \Theta T = \widetilde{\tau}(x,y), \end{array}\right\} \tag{28}$$

where $j = 0, 1$ and $x_0 = y_0 = z_0 = 0$.

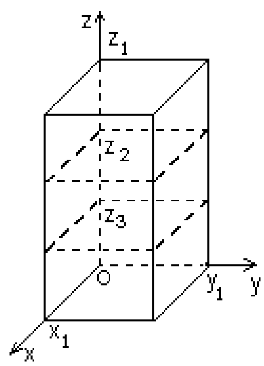

Fig. 2

Fig.2. The investigated thermoelastic rectangular parallelepiped.

The problem is to define a change in the temperature T on the face $z = z_1$ such that the following condition be fulfilled (see Fig.2)

$$w(x, y, z_2) - a w(x, y, z_3) = g(x, y). \tag{29}$$

Bearing in mind that the basic functions ψ_{mn} in the Cartesian system are the products of trigonometric functions, the solution of problem (7), (5), (25a), (26a), (27), (28a), (29) immediately follows from formula (23) (it is assumed that conditions (20) are fulfilled). In particular we obtain

$$T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\operatorname{ch}(\gamma_{mn} z_2) - a \operatorname{ch}(\gamma_{mn} z_3)} \times \Big\{ [\operatorname{ch}(\gamma_{mn}(z - z_2)) - a \operatorname{ch}(\gamma_{mn}(z - z_3))] \tau_{mn} + \frac{\gamma_{mn}}{k(1+\nu)} \operatorname{sh}(\gamma_{mn} z) g_{mn} \Big\} \sin \frac{\pi m x}{x_1} \sin \frac{\pi n y}{y_1}, \quad (30)$$

where

$$\gamma_{mn} = \sqrt{\left(\frac{\pi m}{x_1}\right)^2 + \left(\frac{\pi n}{y_1}\right)^2}.$$

Replacing z in in formula (30) by z_1, we obtain the desired value for a change in the temperature T on the face $z = z_1$, which is a unique solution of the problem.

Let us consider a concrete practical example in which functions $\tau(x, y)$ and $g(x, y)$ have the form

$$\tau(x, y) = \tau_{22} \sin \frac{2\pi x}{x_1} \sin \frac{2\pi y}{y_1},$$
$$g(x, y) = g_{11} \sin \frac{\pi x}{x_1} \sin \frac{\pi y}{y_1}.$$

In this case, for the function T, formula (30) implies

$$T = \frac{\gamma_{11} w_{11} \operatorname{sh}(\gamma_{11} z)}{k(1+\nu)[\operatorname{ch}(\gamma_{11} z_2) - a \operatorname{ch}(\gamma_{11} z_3)]} \sin \frac{\pi x}{x_1} \sin \frac{\pi y}{y_1} + \frac{\tau_{22}[\operatorname{ch}(\gamma_{22}(z - z_2)) - a \operatorname{ch}(\gamma_{22}(z - z_3))]}{\operatorname{ch}(\gamma_{22} z_2) - a \operatorname{ch}(\gamma_{22} z_3)} \sin \frac{2\pi x}{x_1} \sin \frac{2\pi y}{y_1}.$$

Now let us consider the case where the anti-symmetry conditions (25a) and (26a) are given on the two neighbouring faces of the parallelepiped $x = 0$ and $y = 0$, and the symmetry conditions (25b) and (26b) are given on the faces $x = x_1$ and $y = y_1$. The other conditions and the aim of the problem are the same as in the preceding problem.

In that case, the functions $\tau(x, y)$ and $g(x, y)$ satisfy the following consistency conditions

$$\tau(0, y) = \tau(x, 0) = 0, \quad \partial_x \tau(x, y)\big|_{x=x_1} = 0, \quad \partial_y \tau(x, y)\big|_{y=y_1} = 0,$$

$$g(0,y)=g(x,0)=0, \quad \partial_x g(x,y)\big|_{x=x_1}=0, \quad \partial_y g(x,y)\big|_{y=y_1}=0.$$

Therefore functions $\tau(x,y)$ and $g(x,y)$ can be represented by Fourier series as

$$\tau(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tau_{mn}\,\sin\frac{(2m-1)\pi x}{2x_1}\,\sin\frac{(2n-1)\pi y}{2y_1}\,,$$
$$g(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}g_{mn}\,\sin\frac{(2m-1)\pi x}{2x_1}\,\sin\frac{(2n-1)\pi y}{2y_1}\,.$$

Let the Fourier coefficients τ_{mn} and g_{mn} satisfy conditions (16) where

$$\gamma_{mn}=\frac{\pi}{2}\sqrt{\left(\frac{2m-1}{x_1}\right)^2+\left(\frac{2n-1}{y_1}\right)^2}\,.$$

If conditions (20) are satisfied, then the solution of the considered problem is represented as

$$\begin{aligned} T=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}&\frac{1}{\mathrm{ch}(\gamma_{mn}z_2)-a\,\mathrm{ch}(\gamma_{mn}z_3)} \\ &\times\Big\{\left[\mathrm{ch}(\gamma_{mn}(z-z_2))-a\,\mathrm{ch}(\gamma_{mn}(z-z_3))\right]\tau_{mn} \\ &+\frac{\gamma_{mn}}{k(1+\nu)}\,\mathrm{sh}(\gamma_{mn}z)g_{mn}\Big\}\sin\frac{(2m-1)\pi x}{2x_1}\,\sin\frac{(2n-1)\pi y}{2y_1}\,. \end{aligned} \quad (31)$$

The substitution of z_1 instead of z in formula (31) gives the desired value of T on the upper face $z=z_1$ of the considered parallelepiped.

If for example we assume that $\tau(x,y)$ and $g(x,y)$ have the form

$$\tau(x,y)=\tau_{11}\sin\frac{\pi x}{x_1}\sin\frac{\pi y}{y_1} \quad \text{and} \quad g(x,y)=0,$$

then formula (31) implies for the function T the following elementary expression

$$T=\frac{\mathrm{ch}(\gamma_{11}(z-z_2))-a\,\mathrm{ch}(\gamma_{11}(z-z_3))}{\mathrm{ch}(\gamma_{11}z_2)-a\,\mathrm{ch}(\gamma_{11}z_3)}\,\tau_{11}\sin\frac{\pi x}{x_1}\sin\frac{\pi y}{y_1}\,.$$

The solutions of other non-classical problems of thermoelasticity are constructed in an absolutely analogous manner.

6 Solution of problems in the circular cylindrical coordinate system.

In the circular cylindrical system of coordinates r, φ, z, let an elastic body occupy the domain $\Omega = \{r_0 < r < r_1,\ 0 < \varphi < \varphi_1 \le 2\pi,\ 0 < z < z_1\}$ and be in the state of thermoelastic equilibrium (see Fig. 3).

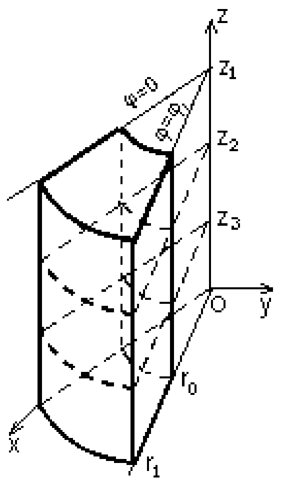

Fig. 2

Fig.3. The investigated thermoelastic circular cylindrical body.

Let us consider problem (7), (5), (1a), (2a), (3), (4a), (6). When conditions (16) and (20) are fulfilled, the solution of this problem follows from (23).

Thus we have

$$T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\mathrm{ch}(\gamma_{mn} z_2) - a\,\mathrm{ch}(\gamma_{mn} z_3)}$$
$$\times \left\{ \left[\mathrm{ch}(\gamma_{mn}(z - z_2)) - a\,\mathrm{ch}(\gamma_{mn}(z - z_3))\right] \tau_{mn} + \frac{\gamma_{mn}}{k(1+\nu)}\ \mathrm{sh}(\gamma_{mn} z) g_{mn} \right\}$$
$$\times \sin \frac{\pi m \varphi}{\varphi_1}\ \left[Y_{\widetilde{m}}(\gamma_{mn} r_0) J_{\widetilde{m}}(\gamma_{mn} r) - J_{\widetilde{m}}(\gamma_{mn} r_0) Y_{\widetilde{m}}(\gamma_{mn} r)\right], \quad (32)$$

where $\widetilde{m} = \frac{\pi m}{\varphi_1}$; $J_{\widetilde{m}}(\gamma_{mn} r)$, $Y_{\widetilde{m}}(\gamma_{mn} r)$ are Bessel functions of first and second kind, respectively; γ_{mn} is a solution of the following transcendental equation

$$Y_{\widetilde{m}}(\gamma_m r_0) J_{\widetilde{m}}(\gamma_m r_1) - J_{\widetilde{m}}(\gamma_m r_0) Y_{\widetilde{m}}(\gamma_m r_1) = 0. \quad (33)$$

When $\varphi_1 = 2\pi$, i.e. when the domain is a hollow cylinder ($r_0 \ne 0$), we have two expressions like (32) with the integer values of $\widetilde{m}$, which differ only in trigonometric functions (this is obvious by virtue of (17)).

As an illustration let us consider the following concrete example.

Assume that in the above problem we have $\varphi_1 = \frac{2\pi}{3}$ and functions $\tau(r,\varphi)$ and $g(r,\varphi)$ are written as

$$\tau(r,\varphi) = \tau_{11}\sin\frac{3\varphi}{2}\left[Y_{3/2}(\gamma_{11}r_0)J_{3/2}(\gamma_{11}r) - J_{3/2}(\gamma_{11}r_0)Y_{3/2}(\gamma_{11}r)\right],$$

$$g(r,\varphi) = 0,$$

then a solution of the problem will have the form

$$T = \frac{\mathrm{ch}(\gamma_{11}(z-z_2)) - a\,\mathrm{ch}(\gamma_{11}(z-z_3))}{\mathrm{ch}(\gamma_{11}z_2) - a\,\mathrm{ch}(\gamma_{11}z_3)} \times \tau_{11}\sin\frac{3\varphi}{2}\left[Y_{3/2}(\gamma_{11}r_0)J_{3/2}(\gamma_{11}r) - J_{3/2}(\gamma_{11}r_0)Y_{3/2}(\gamma_{11}r)\right]. \quad (34)$$

If we use the formulas [14]

$$J_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} - \cos x\right)$$

and

$$Y_{3/2}(x) = -\left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} + \cos x\right),$$

then the obtained solution (34) is expressed through elementary functions.

Let us now consider the problem which differs from the one considered at the beginning of this section only in that the anti-symmetry condition (2a) on the face $\varphi = \varphi_1$ is replaced by the symmetry condition (2b), i.e.

$$\partial_\varphi T = 0, \quad v = 0, \quad \sigma_{\varphi r} = 0, \quad \sigma_{\varphi z} = 0 \quad \text{for} \quad \varphi = \varphi_1.$$

A solution of the problem (we omit the algebraic details of its derivation) has the form

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\mathrm{ch}(\gamma_{mn}z_2) - a\,\mathrm{ch}(\gamma_{mn}z_3)} \times \left\{\left[\mathrm{ch}(\gamma_{mn}(z-z_2)) - a\,\mathrm{ch}(\gamma_{mn}(z-z_3))\right]\tau_{mn} + \frac{\gamma_{mn}}{k(1+\nu)}\,\mathrm{sh}(\gamma_{mn}z)g_{mn}\right\} \times \sin\frac{(2m-1)\pi\varphi}{2\varphi_1}\left[Y_{\widetilde{m}}(\gamma_{mn}r_0)J_{\widetilde{m}}(\gamma_{mn}r) - J_{\widetilde{m}}(\gamma_{mn}r_0)Y_{\widetilde{m}}(\gamma_{mn}r)\right], \quad (35)$$

where $\widetilde{m} = \frac{(2m-1)\pi}{2\varphi_1}$ and γ_{mn} is again a solution of equation (33).

It can be easily verified that if in the considered problem $\varphi_1 = \frac{\pi}{2k-1}$, $k \in N$, then in view of the formulas [9]

$$J_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\, x^{n+\frac{1}{2}} \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x},$$
$$Y_{n+\frac{1}{2}}(x) = (-1)^{n+1}\sqrt{\frac{2}{\pi}}\, x^{n+\frac{1}{2}} \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x},$$

we obtain the solution in elementary functions.

The results obtained in this paper enable us to solve a number of applied problems that arise in mechanical engineering. Let us consider the following two examples. Both examples concern the interaction of an elastic cylindrical body C whose end-face lies γ_1 at a given distance δ from the boundary plane Γ of the other body D (Γ is a part of the boundary surface of the body D) having some temperature T. It is assumed that this temperature brings about a given temperature distribution on the end-face γ_1.

Now we formulate the following two problems.

1. Using the cylinder C, it is required to transfer the torque to the body D. For this, the end-face γ_1 should get deformed so that the friction coefficient between the planes γ_1 and Γ would obey the necessary conditions. To this end, we make an appropriate choice of the temperature on the other end-face γ_0 of the cylinder C (we remind that we may have an access to the surface Γ_1 of the body D only by means of the cylinder C). It is assumed that conditions (1.b) and (2.b) are fulfilled on the lateral surface of the cylinder.

2. The second problem consists in choosing a temperature on the end-face γ_0 (the same conditions as in the first problem are fulfilled on the lateral surface of C)such that the distance between the planes γ_0 and Γ would become equal to $\delta_0 > \delta$.

Note that both problems are easily solved using the results obtained in the present paper.

7 Solution of problems for multi-layer bodies

Nonclassical problems of thermoelasticity can be considered for multilayer bodies along the coordinate z.

Assume that in the generalized cylindrical system of coordinates ξ, φ, z we have an elastic multilayer body along z which is in the state of thermoelastic equilibrium and occupies the domain Ω_z. Ω_z is the union of domains $\Omega_{z^{(1)}} = \{\xi_0 < \xi < \xi_1,\ \varphi_0 < \varphi < \varphi_1,\ 0 < z < z^{(1)}\}$, $\Omega_{z^{(2)}} = \{\xi_0 < \xi < \xi_1,$ $\varphi_0 < \varphi < \varphi_1,\ z^{(1)} < z < z^{(2)}\}, \ldots, \Omega_{z^{(l)}} = \{\xi_0 < \xi < \xi_1,\ \varphi_0 < \varphi < \varphi_1,$

$z^{(l-1)} < z < z^{(l)} = z_1\}$ which contact with one another on the planes $z = z^{(j)}$ where $j = 1, 2, \ldots, l-1$ and l is the number of layers. Each layer has its own elastic and thermal characteristics.

Conditions (1)–(4) are satisfied on the domain boundaries. On the contact planes $z = z^{(j)}$ $(j = 1, 2, \ldots, l-1)$ the following conditions are given:

$$\begin{aligned}
&T_j = T_{j+1}, \quad \beta_j \partial_z T_j = \beta_{j+1} \partial_z T_{j+1}, \\
&u_j = u_{j+1}, \quad v_j = v_{j+1}, \quad w_j = w_{j+1}, \\
&\sigma_{z\xi}^{(j)} = \sigma_{z\xi}^{(j+1)}, \quad \sigma_{z\varphi}^{(j)} = \sigma_{z\varphi}^{(j+1)}, \quad \sigma_{zz}^{(j)} = \sigma_{zz}^{(j+1)},
\end{aligned}$$

or

$$\begin{aligned}
&T_j = T_{j+1}, \quad \beta_j \partial_z T_j = \beta_{j+1} \partial_z T_{j+1}, \quad \sigma_{zz}^{(j)} = \sigma_{zz}^{(j+1)}, \ w_j = w_{j+1}, \\
&\sigma_{z\xi}^{(j)} = 0, \quad \sigma_{z\xi}^{(j+1)} = 0, \quad \sigma_{z\varphi}^{(j)} = 0, \quad \sigma_{z\varphi}^{(j+1)} = 0, \quad \text{etc.}
\end{aligned}$$

where β_j is the heat conductivity coefficient of the j-th layer.

The problem consists in defining a change in the temperature T on the boundary $z = z_1$ such that condition (6) would be satisfied. It is assumed that $z_2 \neq z^{(j)}$, $z_3 \neq z^{(j)}$, $j = 1, 2, \ldots, l-1$.

Based on the results of [11], it is not difficult to prove the following statement.

If, in addition to conditions (16) and (20), conditions

$$k_j(1 + \nu_j) = k_{j+1}(1 + \nu_{j+1}), \quad j = 1, 2, \ldots, l-1, \tag{36}$$

are also fulfilled, where k_j is the linear thermal expansion coefficient of the j-th layer, ν_j is Poisson's ratio of the j-th layer, then the considered problem has a unique solution T that coincides with the solution of an analogous problem for a homogeneous (one-layer) elastic body.

Note that due to the fulfillment of conditions (36), the nonclassical contact boundary value problem for a multilayer body has turned out identical to the nonclassical boundary value problem considered above. Moreover, the elasticity moduli of different layers are different in the expressions for stresses $\sigma_{\xi\xi}$, $\sigma_{\varphi\varphi}$¡ $\sigma_{\xi\varphi}$ (see (12)) because in conditions (36) there are no elasticity moduli. The obtained results can be easily extended to piecewise homogeneous multilayer transversal isotropic bodies as well.

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