THE ACCURACY OF A METHOD FOR THE BERGER DYNAMIC PLATE EQUATION

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Abstract

The paper deals with the boundary value problem for a nonlinear integro-differential equation modeling the dynamic state of the Berger rectangular plate. To approximate the solution with respect to the spatial variables, the Galerkin methid is used, the error of which is estimated.

Key words and phrases: Berger plate equation, Galerkin method, error estimate.

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1 Statement of the problem

By using the approach due to Berger [1] it was shown by Wah [4] that the vibration of rectangular plates $\Omega = \{(x, y) \mid 0 < x < a,\ 0 < y < b\}$ with large amplitudes may be described by the nonlinear differential equation

$$\frac{\partial^2 w}{\partial t^2} + \alpha\Delta^2 w - \beta\left[\int_\Omega \left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) dx\, dy\right]\Delta w = 0, \qquad (1)$$
$$(x, y) \in \Omega, \quad 0 < t \le T,$$

in which $w(x, y, t)$ is lateral deflection and α and β are some positive constants.

Consider equation (1) under the following initial boundary conditions

$$\begin{aligned} &\frac{\partial^p}{\partial t^p} w(x, y, 0) = w^p(x, y), \quad p = 0, 1, \qquad w(x, y, t)|_{\partial\Omega} = 0, \\ &\frac{\partial^2}{\partial x^2} w(x, y, t)|_{\partial\Omega_1} = 0, \quad \frac{\partial^2}{\partial y^2} w(x, y, t)|_{\partial\Omega_2} = 0, \end{aligned} \qquad (2)$$

where $w^0(x,y)$ and $w^1(x,y)$ are the given functions, $\partial\Omega$ is the boundary of the domain Ω and $\partial\Omega_1 = \{(x,y) \in \partial\Omega \mid x = 0 \vee x = a\}$, $\partial\Omega_2 = \{(x,y) \in \partial\Omega \mid y = 0 \vee y = b\}$.

Note that in [2] the existence and uniqueness of a generalized solution of the Cauchy problem is proved for the equation

$$(I + hA)u'' + A^2 u + [\lambda + M(|A^{1/2}u|^2)]Au = f$$

a particular case of which is equation (1). Therefore, according to [2], if $f(x,y,t) \in L^2(0,T;L^2(0,\Omega))$, $w^p(x,y) \in \overset{\circ}{W}{}_2^{2-p}(\Omega)$, $p = 0,1$, then there is a unique function $w = w(x,y,t)$, $w \in L^\infty(0,T;\overset{\circ}{W}{}_2^2(0,\Omega))$, $\frac{\partial w}{\partial t} \in L^\infty(0,T;\overset{\circ}{W}{}_2^1(0,\Omega))$, such that $w(x,y,t)$ is a weak solution of problem (1), (2). We can write the solution in the form

$$w(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} w_{ij}(t) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}\,, \tag{3}$$

where the coefficients $w_{ij}(t)$ satisfy the system of equations

$$\begin{gathered} w''_{ij}(t) + \alpha\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right)^2 w_{ij}(t) \\ +\frac{1}{4}\,ab\beta\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi l}{b}\right)^2\right) w_{kl}^2(t)\right) \\ \times\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right) w_{ij}(t) = 0, \quad i = 1,2,\ldots, \quad j = 1,2,\ldots, \end{gathered} \tag{4}$$

with the initial conditions

$$\begin{gathered} \frac{d^p}{dt^p}\, w_{ij}(0) = a_{ij}^p, \quad p = 0,1, \\ i = 1,2,\ldots, \quad j = 1,2,\ldots, \end{gathered} \tag{5}$$

where

$$a_{ij}^p = \frac{4}{ab}\int_\Omega w^p(x,y) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}\, dx\, dy\,.$$

2 Galerkin method

Let us perform approximation of the solution of problem (1), (2) with respect to the variables x and y. For this, we use the Galerkin method. A solution will be sought in the form of the series

$$w_{mn}(x,y,t) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}^{mn}(t)\sin\frac{i\pi x}{a}\,\sin\frac{j\pi y}{b}\,, \tag{6}$$

where the coefficients $w_{ij}^{mn}(t)$ are the solution of the system of differential equations

$$\begin{gathered} w_{ij}^{mn\prime\prime}(t) + \alpha\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right)^2 w_{ij}^{mn}(t) \\ +\frac{1}{4}\,ab\beta\left(\sum_{k=1}^{m}\sum_{l=1}^{n}\left(\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi l}{b}\right)^2\right) w_{kl}^{mn2}(t)\right) \\ \times\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right) w_{ij}^{mn}(t) = 0, \\ i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n, \end{gathered} \tag{7}$$

with the initial conditions

$$\begin{gathered} \frac{d^p}{dt^p}\, w_{ij}^{mn}(0) = a_{ij}^p, \quad p = 0,1, \\ i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n. \end{gathered} \tag{8}$$

3 Method's error

Now our aim is to estimate the error of the Galerkin method. We apply the technique developed in [3] for a one-dimensional problem. Let us give the definition of the error. By the coefficients of decomposition (3) we form the function

$$\pi_{mn}w(x,y,t) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}(t)\sin\frac{i\pi x}{a}\,\sin\frac{j\pi y}{b}\,. \tag{9}$$

By the error of the Galerkin method we understood the difference between the functions $w_{mn}(x,y,t)$ and $\pi_{mn}w(x,y,t)$

$$\delta_{mn}(x,y,t) = w_{mn}(x,y,t) - \pi_{mn}w(x,y,t). \tag{10}$$

From (10), (6) and (9) follows

$$\delta_{mn}(x,y,t) = \sum_{i=1}^{m}\sum_{j=1}^{n} \delta_{ij}^{mn}(t)\sin\frac{i\pi x}{a}\,\sin\frac{j\pi y}{b}\,,$$

where

$$\delta_{ij}^{mn}(t) = w_{ij}^{mn}(t) - w_{ij}(t). \tag{11}$$

Let us derive the equations for $\delta_{ij}^{mn}(t)$. In system (4) and the initial conditions (5) we consider the equations and conditions which correspond to $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We subtract them from the corresponding equations of system (7) and conditions (8). As a result, applying (11), we get

$$\begin{gathered}
\delta_{ij}^{mn\prime\prime}(t) + \alpha\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right)^2 \delta_{ij}^{mn}(t) + \sigma_1\nu_1 - \sigma_2\nu_2 \\
= \rho_{mn}(t)\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right) w_{ij}(t), \\
i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n,
\end{gathered} \tag{12}$$

with the initial conditions

$$\begin{gathered}
\delta_{ij}^{mn}(0) = 0, \quad \delta_{ij}^{mn\prime}(0) = 0 \\
i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n,
\end{gathered} \tag{13}$$

where

$$\begin{aligned}
\sigma_1 &= \frac{1}{4}ab\beta\sum_{k=1}^{m}\sum_{l=1}^{n}\left(\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi l}{b}\right)^2\right) w_{kl}^{mn2}(t), \\
\sigma_2 &= \frac{1}{4}ab\beta\sum_{k=1}^{m}\sum_{l=1}^{n}\left(\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi l}{b}\right)^2\right) w_{kl}^{2}(t), \\
\nu_1 &= \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right) w_{ij}^{mn}(t), \\
\nu_2 &= \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right) w_{ij}(t), \\
\rho_{mn} &= \frac{1}{4}ab\beta\Bigg(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi l}{b}\right)^2\right) w_{kl}^{2}(t) \\
&\quad - \sum_{k=1}^{m}\sum_{l=1}^{n}\left(\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi l}{b}\right)^2\right) w_{kl}^{2}(t)\Bigg).
\end{aligned} \tag{14}$$

Taking into consideration (12)–(14) and equality

$$\sigma_1\nu_1 - \sigma_2\nu_2 = \frac{1}{2}\left((\sigma_1 - \sigma_2)(\nu_1 + \nu_2) + (\sigma_1 + \sigma_2)(\nu_1 - \nu_2)\right),$$

we obtain

$$
\begin{aligned}
&\delta_{ij}^{mn\prime\prime}(t) + \alpha\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right)^2 \delta_{ij}^{mn}(t) \\
&+\frac{1}{8}\,ab\beta\Bigg[\left(\sum_{k=1}^{m}\sum_{l=1}^{n}\left(\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi l}{b}\right)^2\right)(w_{kl}^{mn}(t)+w_{kl}(t))\delta_{kl}^{mn}(t)\right) \\
&\qquad\times\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right)(w_{ij}^{mn}(t)+w_{ij}(t)) \\
&+\left(\sum_{k=1}^{m}\sum_{l=1}^{n}\left(\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi l}{b}\right)^2\right)(w_{kl}^{mn2}(t)+w_{kl}^{2}(t))\right) \\
&\qquad\times\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right)\delta_{ij}^{mn}(t)\Bigg] \\
&= \rho_{mn}(t)\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right) w_{ij}(t),
\end{aligned}
\tag{15}
$$

$$
\begin{gathered}
\delta_{ij}^{mn}(0)=0, \quad \delta_{ij}^{mn\prime}(0)=0, \\
i=1,2,\ldots,m, \quad j=1,2,\ldots,n.
\end{gathered}
\tag{16}
$$

System (15) and conditions (16) are the starting point of the investigation of the problem of method accuracy estimation. We will need several a priori estimates.

Multiply (4) by $2w_{ij}'(t)$ and sum the obtained expression over $i = 1,2,\ldots,$ $j = 1,2,\ldots$. The result is written as $\Phi'(t) = 0$, $0 < t \le T$, where

$$
\begin{aligned}
\Phi(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} w_{ij}^{\prime 2}(t) + \alpha\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right)^2 w_{ij}^2(t) \\
+\frac{1}{8}\,ab\beta\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right) w_{ij}^2(t)\right)^2.
\end{aligned}
$$

After some transformations we obtain

Lemma 1. *The estimate*

$$
\begin{gathered}
\frac{1}{\alpha}(l-1)\sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}^{\prime 2}(t) + \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right)^l w_{ij}^2(t) \le \tau_{l-1}, \\
l=1,2, \quad m=1,2,\ldots, \quad n=1,2,\ldots, \quad 0<t\le T,
\end{gathered}
$$

where

$$\tau_0 = \frac{8\alpha}{ab\beta}\left(\frac{\pi}{\max(a,b)}\right)^2\left(\left(1+\frac{ab\beta}{8\alpha^2}\left(\frac{\max(a,b)}{\pi}\right)^4\Phi(0)\right)^{\frac{1}{2}}-1\right),$$

$$\tau_1 = \frac{1}{\alpha}\,\Phi(0),$$

is valid.

In a similar manner, using system (7) and the function

$$\Phi_{mn}(t) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}^{mn\prime 2}(t) + \alpha\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\frac{\pi i}{a}\right)^2+\left(\frac{\pi j}{b}\right)^2\right)^2 w_{ij}^{mn2}(t)$$
$$+\frac{1}{8}\,ab\beta\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\frac{\pi i}{a}\right)^2+\left(\frac{\pi j}{b}\right)^2\right)w_{ij}^{mn2}(t)\right)^2$$

we come to

Lemma 2. *The inequality*

$$\frac{1}{\alpha}(l-1)\sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}^{mn\prime 2}(t) + \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\frac{\pi i}{a}\right)^2+\left(\frac{\pi j}{b}\right)^2\right)^l w_{ij}^{mn2}(t) \le \tau_{l+1},$$

$$l=1,2, \quad m=1,2,..., \quad n=1,2,..., \qquad 0<t\le T,$$

where

$$\tau_2 = \frac{8\alpha}{ab\beta}\left(\frac{\pi}{\max(a,b)}\right)^2\left(\left(1+\frac{ab\beta}{8\alpha^2}\left(\frac{\max(a,b)}{\pi}\right)^4\Phi_{mn}(0)\right)^{\frac{1}{2}}-1\right),$$

$$\tau_3 = \frac{1}{\alpha}\,\Phi_{mn}(0),$$

is valid.

We denote by $\sum\limits_{i}\sum\limits_{j}$ the operation of summation over the indexes i and j which take the values

$$i=1,2,\dots,m, \quad j=n+1,n+2,\dots,$$
$$i=m+1,m+2,\dots, \quad j=1,2,\dots,n,$$
$$i=m+1,m+2,\dots, \quad j=n+1,n+2,\dots\,.$$

Using system (4), we prove

Lemma 3. *The estimate*

$$\rho_{mn}(t)\le\varepsilon_{mn},$$
$$m=1,2,\dots, \quad n=1,2,\dots, \quad 0<t\le T,$$

where

$$\varepsilon_{mn} = \frac{1}{8\alpha} ab\beta e^{\frac{1}{4\sqrt{\alpha}} ab\beta\tau_0 T} \left(\frac{\max(a,b)}{\pi}\right)^2$$
$$\times \Bigg(\sum_i \sum_j {a^1_{ij}}^2 + \alpha \sum_i \sum_j \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right)^2 {a^0_{ij}}^2 \Bigg),$$

is fulfilled.

Let us formulate the main result. Multiplying equation (15) by $2\delta^{mn\prime}_{ij}(t)$ and summing the obtained expression over $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, we obtain

$$F'_{mn}(t) = \frac{1}{8} ab\beta \Bigg[-2 \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right) (w^{mn}_{ij}(t) + w_{ij}(t)) \delta^{mn}_{ij}(t)$$
$$\times \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right) (w^{mn}_{ij}(t) + w_{ij}(t)) \delta^{mn\prime}_{ij}(t)$$
$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right) (w^{mn2}_{ij}(t) + w^2_{ij}(t))'$$
$$\times \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right) \delta^{mn2}_{ij}(t) \Bigg]$$
$$+ 2\rho_{mn}(t) \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right) w_{ij}(t) \delta^{mn\prime}_{ij}(t),$$

where

$$F_{mn}(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} \delta^{mn\prime 2}_{ij}(t) + \alpha \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right)^2 \delta^{mn2}_{ij}(t)$$
$$+ \frac{1}{8} ab\beta \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right) (w^{mn2}_{ij}(t) + w^2_{ij}(t))$$
$$\times \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 \right) \delta^{mn2}_{ij}(t).$$

We perform some transformations and apply Lemmas 1–3 together with the Gronwall inequality. Let $\|\cdot\|$ be the norm in the space $L_2(\Omega)$. We denote by Δ the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. As a result we come to

Theorem. *The inequality*

$$\left(\|\delta_{mnt}(x,y,t)\|^2 + \alpha \|\Delta \delta_{mn}(x,y,t)\|^2 \right)^{\frac{1}{2}} \leq c_0 \varepsilon_{mn},$$

where

$$c_0 = \frac{1}{2}\, T \left(ab\tau_1 \left(\frac{1}{2} + \frac{1}{\sqrt{2}} \right) \right)^{\frac{1}{2}} e^{c_1},$$

$$c_1 = \frac{1}{2} + \frac{1}{32\sqrt{\alpha}}\, ab\beta T \sum_{k=1}^{2} c_{k+1} \left(\frac{\max(a,b)}{\pi} \right)^k,$$

$$c_2 = 2(\sqrt{\tau_0} + \sqrt{\tau_2})(\sqrt{\tau_1} + \sqrt{\tau_3}), \quad c_3 = (1 - \sqrt{2})(\tau_1 + \tau_3),$$

is fulfilled for the error of the Galerkin method.

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